Math 121 Midterm Exam #3 December 4, 2015

Answer Key

1. [15 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty} \frac{(-1)^n\ (n+1)\ (2x-3)^n}{n^2\ 6^n}$$

Use Ratio Test.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(n+2)(2x-3)^{n+1}}{(n+1)^2 6^{n+1}}}{\dfrac{(-1)^n(n+1)(2x-3)^n}{n^2 6^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2x-3)^{n+1}}{(2x-3)^n}\right| \cdot \left(\frac{n+2}{n+1}\right) \cdot \left(\frac{n}{n+1}\right)^2 \cdot \frac{6^n}{6^{n+1}} = \frac{|2x-3|}{6}$$

The Ratio Test gives convergence for x when $\frac{|2x-3|}{6} < 1$ or $|2x-3| < 6$.

That is $-6 < 2x-3 < 6 \Longrightarrow -3 < 2x < 9 \Longrightarrow -\frac{3}{2} < x < \frac{9}{2}$

Endpoints:

•$x = \frac{9}{2}$ The original series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)\left(2\left(\frac{9}{2}\right)-3\right)^n}{n^2\ 6^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)6^n}{n^2\ 6^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n^2}$ which is convergent by AST:

1. $b_n = \frac{n+1}{n^2} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n+1}{n^2}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}}{1} = 0$

3. $b_{n+1} < b_n$ because $f(x) = \frac{x+1}{x^2}$ has derivative $f'(x) = -\frac{x^2+2x}{x^4} < 0$ for $x > 0$, so the terms are decreasing.

•$x = -\frac{3}{2}$ The original series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)\left(2\left(-\frac{3}{2}\right)-3\right)^n}{n^2\ 6^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)(-6)^n}{n^2\ 6^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}(n+1)}{n^2} = \sum_{n=1}^{\infty} \frac{n+1}{n^2} \approx \sum_{n=1}^{\infty} \frac{1}{n}$

TWO OPTIONS:

CT: Bound the terms $\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent Harmonic Series, $p = 1$. Finally, the (larger) O.S. is also divergent by CT.

OR LCT: $\lim_{n\to\infty} \frac{\frac{n+1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1$ which is *finite* and *non-zero*. Therefore, $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ is also divergent by LCT.

Finally, Interval of Convergence $\boxed{I = \left(-\frac{3}{2}, \frac{9}{2}\right]}$ with Radius of Convergence $\boxed{R = 3}$.

2. [12 Points] Find the **MacLaurin series** representation for each of the following functions. **State** the Radius of Convergence for each series. Your answer should be in sigma notation $\sum_{n=0}^{\infty}$.

(a) $f(x) = \frac{x^6}{1+6x} = \frac{x^6}{1-(-6x)} = x^6 \sum_{n=0}^{\infty} (-6x)^n = x^6 \sum_{n=0}^{\infty} (-1)^n 6^n x^n = \boxed{\sum_{n=0}^{\infty} (-1)^n 6^n x^{n+6}}$

Here need $|-6x| < 1$ or $|x| < \frac{1}{6}$, so $\boxed{R = \frac{1}{6}}$.

(b) $f(x) = x^7 \arctan(7x)$

First, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

Next, $\arctan(7x) = \sum_{n=0}^{\infty} (-1)^n \frac{(7x)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{7^{2n+1} x^{2n+1}}{2n+1}$

Finally, $x^7 \arctan(7x) = x^7 \sum_{n=0}^{\infty} (-1)^n \frac{7^{2n+1} x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{7^{2n+1} x^{2n+8}}{2n+1}$

Here need $|7x| < 1$ or $|x| < \frac{1}{7}$, so $\boxed{R = \frac{1}{7}}$.

(c) $f(x) = \sinh x$ (think about the defintion of $\sinh x$)

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right)$$

$$= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right)$$

$$= \frac{1}{2}\left(2x + 2\left(\frac{x^3}{3!}\right) + 2\left(\frac{x^5}{5!}\right) + \ldots\right)$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$ Here $R = \infty$ because $\sinh x$ is built of exponentials.

OR you can use the chart method to find MacLaurin Series.

$f(x) = \sinh x \qquad f(0) = \sinh 0 = 0$

$f'(x) = \cosh x \qquad f'(0) = \cosh 0 = 1$

$f''(x) = \sinh x \qquad f''(0) = \sinh 0 = 0$

$f'''(x) = \cosh x \qquad f'''(0) = \cosh 0 = 1$

$\ldots \qquad \ldots$

Finally, MacLaurin Series $= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

$$= 0 + x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \ldots$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$ Match!

Note, all of the even powered derivatives equal 0, so we are left with only the odd powered terms.

3. [15 Points]

(a) Use the MacLaurin Series representation for $f(x) = x^3 \cos\left(x^3\right)$ to

Estimate $\int_0^1 x^3 \cos\left(x^3\right)\, dx$ with error less than $\frac{1}{100}$.

Justify in words that your error is indeed less than $\frac{1}{100}$.

First $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Second, $\cos\left(x^3\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$

Finally, $x^3 \cos\left(x^3\right) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n)!}}$

Now,

$$\int_0^1 x^3\cos\left(x^3\right)\ dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n)!}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n)!(6n+4)}\Bigg|_0^1$$

$$= \frac{x^4}{1\cdot 4} - \frac{x^{10}}{2!\cdot 10} + \frac{x^{16}}{4!\cdot 16} - \ldots\Bigg|_0^1 = \frac{x^4}{4} - \frac{x^{10}}{20} + \frac{x^{16}}{384} - \ldots\Bigg|_0^1$$

$$= \frac{1}{4} - \frac{1}{20} + \frac{1}{384} - \ldots - (0 - 0 + 0 - \ldots)$$

$$\approx \frac{1}{4} - \frac{1}{20} = \frac{5}{20} - \frac{1}{20} = \frac{4}{20} = \boxed{\frac{1}{5}} \quad \leftarrow \quad \textbf{estimate}$$

Using the Alternating Series Estimation Theorem (ASET), we can approximate the actual sum with only the first two terms, and the error from the actual sum will be *at most* the absolute value of the next (first neglected) term, $\frac{1}{384}$. Here $\frac{1}{384} < \frac{1}{100}$ as desired.

(b) Estimate $\frac{1}{\sqrt{e}}$ with error less than $\frac{1}{100}$. Justify in words that your error is indeed less than $\frac{1}{100}$.

Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$$\frac{1}{\sqrt{e}} = e^{-\frac{1}{2}} = 1 - \frac{1}{2} + \frac{\left(-\frac{1}{2}\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)^3}{3!} + \frac{\left(-\frac{1}{2}\right)^4}{4!} + \ldots = 1 - \frac{1}{2} + \frac{\frac{1}{4}}{2!} - \frac{\frac{1}{8}}{3!} + \frac{\frac{1}{16}}{4!} + \ldots$$

$$= 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} + \ldots \approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} = \frac{48}{48} - \frac{24}{48} + \frac{6}{48} - \frac{1}{48} = \boxed{\frac{29}{48}} \leftarrow \text{estimate}$$

Using ASET we can estimate the full sum using only the first four terms with error *at most* (the first neglected term in absolute value) $\frac{1}{384} < \frac{1}{100}$ as desired.

4. [18 Points] Find the **sum** for each of the following series.

(a) $$\frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3}\,)^3} + \frac{1}{5(\sqrt{3}\,)^5} - \frac{1}{7(\sqrt{3}\,)^7} + \ldots = \arctan\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{\pi}{6}}$$

(b) $$\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{9^n\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{3^{2n}\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{3}\right)^{2n}}{(2n+1)!}\left(\frac{\frac{\pi}{3}}{\frac{\pi}{3}}\right)$$

$$= \frac{3}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \frac{3}{\pi}\sin\left(\frac{\pi}{3}\right) = \frac{3}{\pi}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{3\sqrt{3}}{2\pi}}$$

(c) $$\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 9)^n}{2^{n+1}\ n!} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 9)^n}{2^n\ n!} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\left(\frac{-\ln 9}{2}\right)^n}{n!}$$

$$= \frac{1}{2} e^{\left(\frac{-\ln 9}{2}\right)} = \frac{1}{2} e^{\ln\left(9^{-\frac{1}{2}}\right)} = \frac{1}{2}\left(9^{-\frac{1}{2}}\right) = \frac{1}{2}\left(\frac{1}{9^{\frac{1}{2}}}\right) = \frac{1}{2}\left(\frac{1}{3}\right) = \boxed{\frac{1}{6}}$$

$$\text{OR} = \ldots = \frac{1}{2} e^{\left(\frac{-\ln 9}{2}\right)} = \frac{1}{2}\left(\frac{1}{e^{\frac{\ln 9}{2}}}\right) = \frac{1}{2}\left(\frac{1}{e^{\ln\left(9^{\frac{1}{2}}\right)}}\right) = \ldots$$

(d) $\frac{1}{e} - \frac{1}{2(e)^2} + \frac{1}{3(e)^3} - \frac{1}{4(e)^4} + \frac{1}{5(e)^5} - \frac{1}{6(e)^6} + \ldots = \boxed{\ln\left(1 + \frac{1}{e}\right)} = \boxed{\ln\left(\frac{e+1}{e}\right)}$

$= \ln(e+1) - \ln e = \boxed{\ln(e+1) - 1}$

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n-1}}{3\ (2n)!} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n-1}}{(2n)!} \left(\frac{\pi}{\pi}\right) = \frac{1}{3\pi} \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(2n)!}$

$= \frac{1}{3\pi} \cos(\pi) = \frac{1}{3\pi}(-1) = \boxed{-\frac{1}{3\pi}}$

(f) $-\frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \ldots = \cos(\pi) - 1 = -1 - 1 = \boxed{-2}$

5. [20 Points] Volumes of Revolution

(a) Consider the region bounded by $y = e^x + 2$, $y = \sin x$, $x = 0$ and $x = \pi$. Rotate this region about the vertical line $\boxed{x = -2}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

See me for a sketch.

$$V = \int_0^{\pi} 2\pi \text{ radius height } dx = \boxed{2\pi \int_0^{\pi} (x+2)\left(e^x + 2 - \sin x\right)\ dx}$$

(b) Consider the region bounded by $y = \ln x$, $y = 1$, and $x = 4$. Rotate this region about the vertical line $\boxed{x = 5}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

See me for a sketch. The left endpoint is when $\ln x = 1$ which is when $x = e$.

$$V = \int_e^4 2\pi \text{ radius height } dx = \boxed{2\pi \int_e^4 (5-x)((\ln x) - 1)\ dx}$$

(c) Consider the region bounded by $y = \arctan x$, $y = 0$, $x = 0$ and $x = 1$. Rotate this region about the $\boxed{y\text{-axis}}$. **COMPUTE** the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

See me for a sketch.

$$V = \int_0^1 2\pi \text{ radius height } dx = 2\pi \int_0^1 x \ \arctan x \ dx \overset{(**)}{=} 2\pi \left[\frac{x^2}{2} \arctan x - \frac{1}{2}x + \frac{1}{2} \arctan x \right] \Bigg|_0^1$$

$$= \pi \left[x^2 \arctan x - x + \arctan x \right] \Bigg|_0^1 = \pi \left[(\arctan 1 - 1 + \arctan 1) - (0 \arctan 0 - 0 + \arctan 0) \right]$$

$$= \pi \left[\left(\frac{\pi}{4} - 1 + \frac{\pi}{4} \right) - (0 \arctan 0 - 0 + \arctan 0) \right] = \pi \left[\frac{\pi}{2} - 1 \right] = \boxed{\frac{\pi^2}{2} - \pi}$$

$$(**) \int_1^2 x \ \arctan x \ dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2 + 1 - 1}{1 + x^2} \ dx$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2 + 1}{1 + x^2} - \frac{1}{1 + x^2} \ dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int 1 - \frac{1}{1 + x^2} \ dx$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2}(x - \arctan x) + C = \boxed{\frac{x^2}{2} \arctan x - \frac{1}{2}x + \frac{1}{2} \arctan x + C}$$

$u = \arctan x$	$dv = x dx$
$du = \frac{1}{1 + x^2} dx$	$v = \frac{x^2}{2}$

OR if you don't like the "slip-in/slip out" technique, use a tangent trig. substitution instead to finish the second piece of the I.B.P. $\int \frac{x^2}{1 + x^2} \ dx$

$$\int \frac{x^2}{1 + x^2} \ dx = \int \frac{\tan^2 \theta}{1 + \tan^2 \theta} \sec^2 \theta \ d\theta = \int \frac{\tan^2 \theta}{\sec^2 \theta} \sec^2 \theta \ d\theta = \int \tan^2 \theta \ d\theta = \int \sec^2 \theta - 1 \ d\theta$$

$$= \tan \theta - \theta = x - \arctan x$$

Trig. Substitute

$x = \tan \theta$
$dx = \sec^2 \theta d\theta$

6. [20 Points] Parametric Equations

(a) Consider the Parametric Curve given by $x = (\arctan t) - t$ and $y = 2 \sinh^{-1} t$.

$$\text{Recall} \quad \frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}}$$

<u>COMPUTE</u> the **arclength** of this parametric curve for $0 \leq t \leq \sqrt{3}$.

First compute

$$\frac{dx}{dt} = \frac{1}{1 + t^2} - 1$$

$\frac{dy}{dt} = \frac{2}{\sqrt{1+t^2}}$

$L = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2} - 1\right)^2 + \left(\frac{2}{\sqrt{1+t^2}}\right)^2}\, dt$

$= \int_0^{\sqrt{3}} \sqrt{\frac{1}{(1+t^2)^2} - \frac{2}{1+t^2} + 1 + \frac{4}{1+t^2}}\, dt = \int_0^{\sqrt{3}} \sqrt{\frac{1}{(1+t^2)^2} + \frac{2}{1+t^2} + 1}\, dt$

$= \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2} + 1\right)^2}\, dt = \int_0^{\sqrt{3}} \frac{1}{1+t^2} + 1\, dt$

$= \arctan t + t\Big|_0^{\sqrt{3}} = \arctan\sqrt{3} + \sqrt{3} - (\arctan 0 + 0) = \frac{\pi}{3} + \sqrt{3} - 0 = \boxed{\frac{\pi}{3} + \sqrt{3}}$

(b) Consider a *different* Parametric Curve given by $x = t + \frac{1}{t}$ and $y = \ln(t^2)$.

COMPUTE the **surface area** obtained by rotating this curve about the y-axis, for $1 \le t \le 2$.

First compute

$\frac{dx}{dt} = 1 - \frac{1}{t^2}$

$\frac{dy}{dt} = \frac{1}{t^2}(2t) = \frac{2}{t}$

$\text{S.A.} = \int_1^2 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 2\pi \int_1^2 \left(t + \frac{1}{t}\right)\sqrt{\left(1 - \frac{1}{t^2}\right)^2 + \left(\frac{2}{t}\right)^2}\, dt$

$= 2\pi \int_1^2 \left(t + \frac{1}{t}\right)\sqrt{1 - \frac{2}{t^2} + \frac{1}{t^4} + \frac{4}{t^2}}\, dt = 2\pi \int_1^2 \left(t + \frac{1}{t}\right)\sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}}\, dt$

$= 2\pi \int_1^2 \left(t + \frac{1}{t}\right)\sqrt{\left(1 + \frac{1}{t^2}\right)^2}\, dt = 2\pi \int_1^2 \left(t + \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right) dt = 2\pi \int_1^2 t + \frac{2}{t} + \frac{1}{t^3}\, dt$

$= 2\pi \left(\frac{t^2}{2} + 2\ln|t| - \frac{1}{2t^2}\right)\Big|_1^2 = 2\pi\left(2 + 2\ln 2 - \frac{1}{8} - \left(\frac{1}{2} + 2\ln 1 - \frac{1}{2}\right)\right) = \boxed{2\pi\left(\frac{15}{8} + \ln 4\right)}$

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OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute $\sum_{n=0}^{\infty} \frac{n^3}{2^n}$

First $\displaystyle\sum_{n=0}^{\infty}\frac{n^3}{2^n}$ comes from $\displaystyle\sum_{n=0}^{\infty}n^3x^n$ with $x=\frac{1}{2}$

Recognize $\displaystyle\sum_{n=0}^{\infty}n^3x^n=\sum_{n=0}^{\infty}n^3x^{n-1}x=x\sum_{n=0}^{\infty}n^2\cdot nx^{n-1}=x\frac{d}{dx}\left(\sum_{n=0}^{\infty}n^2x^n\right)$

$$=x\frac{d}{dx}\left(\sum_{n=0}^{\infty}n\cdot nx^{n-1}x\right)=x\frac{d}{dx}\left(x\sum_{n=0}^{\infty}n\cdot nx^{n-1}\right)=x\frac{d}{dx}\left(x\frac{d}{dx}\left(\sum_{n=0}^{\infty}nx^n\right)\right)$$

$$=x\frac{d}{dx}\left(x\frac{d}{dx}\left(\sum_{n=0}^{\infty}nx^{n-1}x\right)\right)=x\frac{d}{dx}\left(x\frac{d}{dx}\left(x\sum_{n=0}^{\infty}nx^{n-1}\right)\right)$$

$$=x\frac{d}{dx}\left(x\frac{d}{dx}\left(x\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right)\right)\right)=x\frac{d}{dx}\left(x\frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{1}{1-x}\right)\right)\right)$$

$$=x\frac{d}{dx}\left(x\frac{d}{dx}\left(x\left(\frac{1}{(1-x)^2}\right)\right)\right)=x\frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)\right)$$

$$=x\frac{d}{dx}\left(x\left(\frac{(1-x)^2(1)-x(2(1-x)(-1))}{(1-x)^4}\right)\right)=x\frac{d}{dx}\left(x\left(\frac{(1-x)((1-x)+2x)}{(1-x)^4}\right)\right)$$

$$=x\frac{d}{dx}\left(x\left(\frac{1+x}{(1-x)^3}\right)\right)=x\frac{d}{dx}\left(\frac{x+x^2}{(1-x)^3}\right)=x\left(\frac{(1-x)^3(1+2x)-(x+x^2)3(1-x)^2(-1)}{(1-x)^6}\right)$$

$$=x\left(\frac{(1-x)^2((1-x)(1+2x)+3(x+x^2))}{(1-x)^6}\right)=x\left(\frac{(1-x)(1+2x)+3(x+x^2)}{(1-x)^4}\right)$$

$$=x\left(\frac{1-x+2x-2x^2+3x+3x^2}{(1-x)^4}\right)=x\left(\frac{1+4x+x^2}{(1-x)^4}\right)=\frac{x+4x^2+x^3}{(1-x)^4}$$

Finally, substituting $x=\frac{1}{2}$,

we get $\displaystyle\sum_{n=0}^{\infty}\frac{n^3}{2^n}=\frac{\frac{1}{2}+4\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3}{\left(1-\frac{1}{2}\right)^4}=\frac{\frac{1}{2}+1+\frac{1}{8}}{\frac{1}{16}}=8+16+2=\boxed{26}$

OPTIONAL BONUS #2

Determine the 27^{th} and 28^{th} derivatives for $f(x)=x^3\arctan(x^5)$ evaluated at $x=0$. You do *not* need to simplify your answers here.

$$x^3\arctan(x^5)=x^3\sum_{n=0}^{\infty}(-1)^n\frac{(x^5)^{2n+1}}{2n+1}=x^3\sum_{n=0}^{\infty}(-1)^n\frac{x^{10n+5}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{10n+8}}{2n+1}$$

$$=x^8-\frac{x^{18}}{3}+\frac{x^{28}}{5}-\frac{x^{38}}{7}+\ldots$$

In general, the MacLaurin Series for any f is given as

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots+\frac{f^{(8)}(0)}{8!}x^8+\frac{f^9(0)}{9!}x^9+\ldots$$

Match coefficients of like degreed terms:

$\dfrac{f^{(27)}(0)}{(27)!} = 0$ since there is no x^{27} term $\Rightarrow \boxed{f^{(27)}(0) = 0}$

Equating coefficients of the x^{28} term shows that $\dfrac{f^{(28)}(0)}{(28)!} = \dfrac{1}{5}$

and we solve $\boxed{f^{(28)}(0) = \dfrac{(28)!}{5}}$.