Exam 2 Spring 26 Answer Key

1(a) $\displaystyle\int_0^e \ln x\,dx = \lim_{t\to0^+}\int_t^e \ln x\cdot 1\,dx = \lim_{t\to0^+} x\ln x\Big|_t^e - \int_t^e 1\,dx$

$u=\ln x \quad dv = 1\,dx$
$du=\frac{1}{x}dx \quad v=x$

$$= \lim_{t\to0^+} x\cdot\ln x\Big|_t^e - x\Big|_t^e$$

$$= \lim_{t\to0^+} e\cdot\underbrace{\ln e}_{1} - \underbrace{t\cdot\ln t}_{0\cdot(-\infty)\to 0} - (e-\underbrace{t}_{\to 0})$$

$$= e - e = \boxed{0} \quad \text{Converges}$$

$$\lim_{t\to0^+} \underbrace{t\cdot\ln t}_{0\cdot(-\infty)} = \lim_{t\to0^+}\frac{\ln t}{1/t}\ \frac{-\infty}{\infty} \underset{L'H}{=} \lim_{t\to0^+}\frac{\frac{1}{t}}{\frac{-1}{t^2}} = \lim_{t\to0^+} -t = 0$$

$t^{-1} \to -t^{-2}$

$\frac{-t^2}{1}$

1(b) $\displaystyle\int_0^{e^4}\frac{1}{x[16+(\ln x)^2]}dx = \lim_{t\to0^+}\int_t^{e^4}\frac{1}{x[16+(\ln x)^2]}dx = \lim_{t\to0^+}\int_{\ln t}^4 \frac{1}{16+u^2}du$

$u=\ln x$
$du=\frac{1}{x}dx$

$x=t \Rightarrow u=\ln t$
$x=e^4 \Rightarrow u=\ln e^4=4$

$$= \lim_{t\to0^+}\frac{1}{4}\arctan\left(\frac{u}{4}\right)\Big|_{\ln t}^4$$

$$= \lim_{t\to0^+}\frac{1}{4}\left(\underbrace{\arctan\left(\frac{4}{4}\right)}_{\frac{\pi}{4}} - \underbrace{\arctan\left(\frac{\ln t}{4}\right)}_{-\frac{\pi}{2}}\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{4}+\frac{\pi}{2}\right) = \frac{1}{4}\left(\frac{3\pi}{4}\right) = \boxed{\frac{3\pi}{16}} \quad \text{Converges}$$

1(c) $\displaystyle\int_3^4\frac{x+1}{x^2-4x+3}dx = \lim_{t\to3^+}\int_t^4\frac{x+1}{(x-3)(x-1)}dx \underset{PFD}{=} \lim_{t\to3^+}\int_t^4 \frac{2}{x-3}-\frac{1}{x-1}\,dx$

$x^2-4x+3 = (x-3)(x-1)$

Partial Fractions given for free this time

$$= \lim_{t\to3^+} 2\ln|x-3| - \ln|x-1|\Big|_t^4$$

$$= \lim_{t\to3^+} \underbrace{2\ln|1|}_{\text{Finite}} - \underbrace{\ln 3}_{\text{Finite}} - \left(\underbrace{2\ln|t-3|}_{-\infty} - \underbrace{\ln|t-1|}_{\text{Finite}}\right)$$

$$= -(-\infty) = \boxed{+\infty} \quad \text{Diverges}$$

Important: Make sure to simplify all finite values

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+4n+7} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{n^2+4n+7}$ Integral Test   Related Function $f(x)=\frac{1}{x^2+4x+7}$

Not Equal

$$\int_1^{\infty} \frac{1}{x^2+4x+7}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4x+7}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{(x+2)^2+3}\,dx$$

$(x+2)^2$
$= x^2+4x+4$
+3 more to 7

$\boxed{\begin{array}{l} u=x+2 \\ du=dx \end{array}}$

$\boxed{\begin{array}{l} x=1 \Rightarrow u=3 \\ x=t \Rightarrow u=t+2 \end{array}}$

$$= \lim_{t\to\infty} \int_3^{t+2} \frac{1}{u^2+3}\,du = \lim_{t\to\infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_3^{t+2}$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t+2}{\sqrt{3}}\right) - \arctan\left(\frac{3}{\sqrt{3}}\right)\right)$$

$\frac{\pi}{2}$ ∞ $\frac{\pi}{3}$ $\sqrt{3}$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \frac{\pi}{6\sqrt{3}}$$ Integral Converges

$\frac{3\pi}{6} - \frac{2\pi}{6}$

Important
Two Separate Conclusions

$\Rightarrow$ Absolute Series Converges by the Integral Test

$\xLeftarrow{ACT}$ Original Series Converges by the Absolute Convergence Test

Create a Series

3. Converges by Comparison Test. Need "Smaller than Converge" Comparison

Examples: $\sum_{n=1}^{\infty} \frac{n^2}{n^9+5}$ $\sum_{n=1}^{\infty} \frac{1}{9^n+5}$ $\sum_{n=1}^{\infty} \frac{3^n}{7^n+6}$ $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^5+4}$ $\sum_{n=1}^{\infty} \frac{\cos^2 n}{8^n+1}$

Sample. $\sum_{n=1}^{\infty} \frac{1}{n^6+9} \approx \sum_{n=1}^{\infty} \frac{1}{n^6}$ Converges p-Series $p=6>1$

Bound Terms
$\frac{1}{n^6+9} \le \frac{1}{n^6}$

$\Rightarrow$ Original Series also Converges by Comparison Test

4(a) $-6-\frac{6}{2}-\frac{6}{3}-\frac{6}{4}-\frac{6}{5}\ldots = -6\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots\right)$

$$= -6\sum_{n=1}^{\infty}\frac{1}{n}$$

Constant Multiple of the Divergent (Harmonic) p-Series $p=1$ is $\boxed{\text{Divergent}}$

4(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(2026)!}{n^6}+\frac{(-2)^n}{6^{2n}} \overset{\text{split}}{=} \sum_{n=1}^{\infty}\frac{(2026)!}{n^6}+\sum_{n=1}^{\infty}\frac{(-2)^n}{6^{2n}}$

$6^{2n}=(6^2)^n=36^n$

Constant

$$=(2026)!\sum_{n=1}^{\infty}\frac{1}{n^6}+\sum_{n=1}^{\infty}\frac{(-2)^n}{36^n}$$

$r=\frac{-2}{36}=-\frac{1}{18}$

$\frac{-2}{36}+\frac{(-2)^2}{(36)^2}+\frac{(-2)^3}{(36)^3}+\ldots$

Constant Multiple of a Convergent p-Series $p=6>1$ is Convergent

Convergent by GST with $|r|=\left|\frac{-2}{36}\right|=\frac{1}{18}<1$

Sum of Two Convergent Series is $\boxed{\text{Convergent}}$

4(c) $\displaystyle\sum_{n=2}^{\infty}\frac{n^6}{\ln(2n+1)}$ $\rightarrow$ $\boxed{\text{Diverges by nTDT}}$ because

bigger numerator as $n\to\infty$

$$\lim_{n\to\infty}\frac{n^6}{\ln(2n+1)} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{x^6}{\ln(2x+1)} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=}} \lim_{x\to\infty}\frac{6x^5}{\frac{2}{2x+1}} = \lim_{x\to\infty} = \frac{6x^5\cdot(2x+1)}{2} = \infty \neq 0$$

$\frac{2x+1}{2}$ ∞ ∞

5(a) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n^2+6}{n^6+2}\right) \xrightarrow{A.S.} \sum_{n=1}^{\infty}\frac{n^2+6}{n^6+2} \approx \sum_{n=1}^{\infty}\frac{n^2}{n^6} = \sum_{n=1}^{\infty}\frac{1}{n^4}$ Convergent p-Series

$p=4>1$

don't need to study the original series

LCT Limit

$$\lim_{n\to\infty}\frac{\frac{n^2+6}{n^6+2}}{\frac{1}{n^4}} = \lim_{n\to\infty}\frac{n^6+6n^4}{n^6+2}\cdot\frac{\frac{1}{n^6}}{\frac{1}{n^6}} = \lim_{n\to\infty}\frac{1+\cancel{\frac{6}{n^2}}^{\,0}}{1+\cancel{\frac{2}{n^6}}^{\,0}} = 1$$ Finite Non-Zero

$\Rightarrow$ the Absolute Series also Converges by LCT

$\Rightarrow$ the Original Series is $\boxed{\text{Absolutely Convergent}}$ (by Definition)

5(b) $\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)!}{n^n(n)!}$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(2(n+1)+1)!}{(n+1)^{n+1}((n+1)!}}{\frac{\cancel{(-1)^n}(2n+1)!}{n^n\cdot n!}}\right| \qquad \frac{n^n\cdot n!}{(2n+1)!}$$

$$(2n+3)(2n+2)\cancel{(2n+1)!}$$

$$= \lim_{n\to\infty}\frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n(n+1)}}\cdot\frac{(2n+3)!}{\cancel{(2n+1)!}}\cdot\frac{\cancel{n!}}{\underbrace{(n+1)!}_{(n+1)\cancel{n!}}}$$

$$2\cancel{(n+1)}$$

$$= \lim_{n\to\infty}\cancel{\frac{n^n}{(n+1)^n}}^{\frac{1}{e}}\cdot\left(\frac{2n+3}{n+1}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{2n+2}{\cancel{n+1}}\right)$$

$$= \lim_{n\to\infty}\frac{2}{e}\left(\frac{2+\cancel{\frac{3}{n}}^{\,0}}{1+\cancel{\frac{1}{n}}^{\,0}}\right) = \frac{4}{e}>1$$

$e\approx 2.7$

$\boxed{\text{Original Series Diverges by Ratio Test}}$

5(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{6n+2} \xrightarrow{\text{1st A.S.}} \sum_{n=1}^{\infty} \frac{1}{6n+2} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Divergent (Harmonic) p-Series $p=1$

2nd AST

1. Isolate $b_n = \dfrac{1}{6n+2} > 0$

2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{6n+2} = 0$ (denominator $\to \infty$, fraction $\to 0$)

3. Terms Decreasing

$$b_{n+1} = \frac{1}{\underbrace{6(n+1)+2}_{6n+8}} \le \frac{1}{6n+2} = b_n$$

Original Series Converges by AST

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{1}{6n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{6n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{6+\cancel{\frac{2}{n}}^{\,0}} = \frac{1}{6}$$

Finite + Non-Zero

$\Rightarrow$ Absolute Series Diverges by LCT

Original Series is $\boxed{\text{Conditionally Convergent}}$ by Definition

Optional Bonus Create a Series

Sample. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n \cdot e^{2n} \cdot (2n)!\, 7^n}{\pi^{2n} (n!)^3}$

Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(n+1)^{n+1} e^{2(n+1)} (2(n+1))!\, 7^{n+1}}{\pi^{2(n+1)} ((n+1)!)^3}}{\dfrac{n^n e^{2n} (2n)!\, 7^n}{\pi^{2n} (n!)^3}} \right|$$

Flip + partner

$$= \lim_{n\to\infty} \frac{\overbrace{(n+1)^{n+1}}^{(n+1)^n\cdot(n+1)}}{n^n} \cdot \left(\frac{\overbrace{e^{2n+2}}^{\cancel{e^{2n}}\cdot e^2}}{\cancel{e^{2n}}}\right) \cdot \left(\frac{\overbrace{(2n+2)!}^{(2n+2)(2n+1)\cancel{(2n)!}}}{\cancel{(2n)!}}\right) \cdot \left(\frac{\overbrace{7^{n+1}}^{\cancel{7^n}\cdot 7}}{\cancel{7^n}}\right) \cdot \left(\frac{\cancel{\pi^{2n}}}{\underbrace{\pi^{2n+2}}_{\cancel{\pi^{2n}}\cdot\pi^2}}\right) \left(\frac{\cancel{(n!)^3}}{\underbrace{((n+1)!)^3}_{(n+1)^3\cancel{(n!)^3}}}\right)$$

$$= \lim_{n\to\infty} \frac{7e^2}{\pi^2}\cdot \cancel{\frac{(n+1)^n}{n^n}}^{\,e} \cdot \left(\cancel{\frac{n+1}{n+1}}\right)\left(\frac{2n+2}{\cancel{n+1}}\right)\left(\frac{2n+1}{n+1}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$$2(n+1)$$

$$= \lim_{n\to\infty} \frac{14e^3}{\pi^2}\cdot\left(\frac{2+\frac{1}{n}}{1+\frac{1}{n}}\right)^{2} = \frac{28e^3}{\pi^2} > 1$$

$e \approx 2.7$ $\quad \pi \approx 3.14$

$e^3 > 8$ $\quad \pi^2 > 9$

Diverges by Ratio Test

OR $$\sum_{n=1}^{\infty} \frac{(2n)!\, e^{4n}\, 7^n}{n^n \cdot n! \cdot \pi^{2n}}$$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(2(n+1))!\, e^{4(n+1)}\, 7^{n+1}}{(n+1)^{n+1}\,(n+1)!\,\pi^{2(n+1)}}}{\dfrac{(2n)!\, e^{4n}\, 7^n}{n^n \cdot n! \cdot \pi^{2n}}}\right|$$

flip + partner

$$= \lim_{n\to\infty}\left(\frac{(2n+2)!}{(2n)!}\right)\left(\frac{e^{4n+4}}{e^{4n}}\right)\cdot \frac{n^n}{(n+1)^{n+1}} \cdot \left(\frac{7^{n+1}}{7^n}\right)\cdot\left(\frac{\pi^{2n}}{\pi^{2n+2}}\right)\left(\frac{n!}{(n+1)!}\right)$$

$(2n+2)(2n+1)\cancel{(2n)!}$ $\quad \cancel{e^{4n}}\cdot e^4$ $\quad \cancel{7^n}\cdot 7$

$(n+1)^n\cdot(n+1)$ $\quad \cancel{\pi^{2n}}\,\pi^2$ $\quad (n+1)\,\cancel{n!}$

$2(n+1)$

$$= \lim_{n\to\infty} \frac{7e^4}{\pi^2}\left(\frac{2n+2}{\cancel{n+1}}\right)\left(\frac{2n+1}{n+1}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)\cancel{\frac{n^n}{(n+1)^n}}^{\,\frac{1}{e}}$$

$$= \lim_{n\to\infty} \frac{14\cancel{e^4}^{\,e^3}}{\pi^2\cdot \cancel{e}}\cdot\left(\frac{2+\frac{1}{n}}{1+\frac{1}{n}}\right)^{2} = \frac{28e^3}{\pi^2} > 1$$

Diverges by Ratio Test

OR $$\sum_{n=1}^{\infty} \frac{e^{4n}\, 28^n \cdot 4^n (n!)^3}{n^n (2n)!\, \pi^{2n}}$$