Exam 2 Spring 2022 Answer Key

1(a) $\displaystyle\int_{-4}^{-3} \frac{6}{x^2+2x-8}\,dx = \int_{-4}^{-3} \frac{6}{(x-2)(x+4)}\,dx = \lim_{t\to-4^+} \int_{t}^{-3} \frac{6}{(x-2)(x+4)}\,dx$

Given PFD

$\displaystyle = \lim_{t\to-4^+} \int_{t}^{-3} \frac{1}{x-2} - \frac{1}{x+4}\,dx = \lim_{t\to-4^+} \ln|x-2| - \ln|x+4| \Big|_{t}^{-3}$

$\displaystyle = \lim_{t\to-4^+} \underbrace{\cancelto{\ln 5}{\ln|-5|} - \cancelto{0}{\ln 1}}_{\text{Finite}} - \left(\underbrace{\cancelto{\ln 6}{\ln|t-2|}}_{\text{Finite}} - \cancelto{-\infty}{\ln|t+4|}\right) = -\infty$

Diverges

2(b) $\displaystyle\int_{-\infty}^{0} \frac{6}{x^2+2x+4}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{6}{x^2+2x+4}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{6}{(x+1)^2+3}\,dx$

Complete Square

Discriminant
b^2-4ac
$= 4-4(1)(4) = -12 < 0$

$u = x+1$
$du = dx$

$x = t \Rightarrow u = t+1$
$x = 0 \Rightarrow u = 1$

$\displaystyle = \lim_{t\to-\infty} 6\int_{t+1}^{1} \frac{1}{u^2+3}\,du$

$\displaystyle = \lim_{t\to-\infty} \frac{6}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Big|_{t+1}^{1}$

$\displaystyle = \lim_{t\to-\infty} \frac{6}{\sqrt{3}}\left(\cancelto{\frac{\pi}{6}}{\arctan\left(\frac{1}{\sqrt{3}}\right)} - \cancelto{-\frac{\pi}{2}}{\arctan\left(\cancelto{-\infty}{\frac{t+1}{\sqrt{3}}}\right)}\right)$

$\displaystyle = \frac{6}{\sqrt{3}}\left(\frac{\pi}{6} + \frac{\pi}{2}\right) = \frac{\cancel{6}}{\sqrt{3}}\left(\frac{4\pi}{\cancel{6}}\right) = \frac{4\pi}{\sqrt{3}}$

$\frac{3\pi}{6}$

Converges

1(c) $\displaystyle\int_{0}^{1} \ln x\,dx = \lim_{t\to0^+} \int_{t}^{1} \ln x\,dx = \lim_{t\to0^+} x\ln x - x \Big|_{t}^{1}$

$\displaystyle = \lim_{t\to0^+} \cancelto{0}{\ln 1} - 1 - \left(\cancelto{0}{\underbrace{t\ln t}_{0\cdot(-\infty)\ (*)}} - \cancelto{0}{t}\right) = -1$

Converges

(*) $\displaystyle\lim_{t\to0^+} \underbrace{t\cdot\ln t}_{0\cdot(-\infty)} = \lim_{t\to0^+} \underbrace{\frac{\ln t}{1/t}}_{\frac{-\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{t\to0^+} \frac{\frac{1}{t}}{\frac{-1}{t^2}} = \lim_{t\to0^+} -t = 0$

1(d) $\displaystyle\int_0^{e^5} \frac{1}{x(25+(\ln x)^2)}\,dx = \lim_{t\to 0^+} \int_t^{e^5} \frac{1}{x(25+(\ln x)^2)}\,dx$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$x = t \Rightarrow u = \ln t$

$x = e^5 \Rightarrow u = \ln e^5 = 5$

$$= \lim_{t\to 0^+} \int_{\ln t}^{5} \frac{1}{25+u^2}\,du$$

$$= \lim_{t\to 0^+} \frac{1}{5} \arctan\left(\frac{u}{5}\right)\Bigg|_{\ln t}^{5}$$

$$= \lim_{t\to 0^+} \frac{1}{5}\left(\arctan\left(\frac{5}{5}\right) - \arctan\left(\frac{\ln t}{5}\right)\right)$$

$\frac{\pi}{4}$, 1, $-\frac{\pi}{2}$, $-\infty$, 0^+

$$= \frac{1}{5}\left(\frac{\pi}{4}+\frac{\pi}{2}\right) = \frac{1}{5}\left(\frac{3\pi}{4}\right) = \frac{3\pi}{20}$$

2(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6^{2n}} + \frac{\ln(2022)}{n^6} = \sum_{n=1}^{\infty} \frac{1}{6^{2n}} + \sum_{n=1}^{\infty} \frac{\ln(2022)}{n^6}$

$$= \sum_{n=1}^{\infty} \frac{1}{(36)^n} + \underbrace{\ln(2022)}_{\text{Constant}} \sum_{n=1}^{\infty} \frac{1}{n^6}$$

Convergent Geometric Series with $|r| = \frac{1}{36} < 1$

Constant Multiple of Convergent p-Series $p=6>1$ is Convergent

Sum of 2 Convergent Series is Convergent

2(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{2022}$ Diverges by nTDT b/c

$$\lim_{n\to\infty} \frac{\arctan n}{2022} = \frac{\pi}{4044} \neq 0$$

$\frac{\pi}{2}$

2(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^6}{\ln(n+2022)}$ Diverges by nTDT b/c

$$\lim_{n\to\infty} \frac{n^6}{\ln(n+2022)} = \lim_{x\to\infty} \frac{x^6}{\ln(x+2022)} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=}} \lim_{x\to\infty} \frac{6x^5}{\frac{1}{x+2022}} = \lim_{x\to\infty} 6x^5(x+2022) = \infty \neq 0$$

∞

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6+2022} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{n^6+2022} \approx \sum_{n=1}^{\infty} \frac{1}{n^6}$ Converges p-Series $p=6>1$

<u>Bound Terms</u>

$$\frac{1}{n^6+2022} \le \frac{1}{n^6}$$

$\Rightarrow$ Absolute Series Converges by Comparison Test

$\Rightarrow$ Original Series Converges by Absolute Convergence Test

4(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2+6}{n^6+2} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{n^2+6}{n^6+2} \approx \sum_{n=1}^{\infty} \frac{n^2}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ Converges p-Series $p=4>1$

LCT

$$\lim_{n\to\infty} \frac{\frac{n^2+6}{n^6+2}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^6+6n^4}{n^6+2} \cdot \frac{\frac{1}{n^6}}{\frac{1}{n^6}} = \lim_{n\to\infty} \frac{1+\cancel{\frac{6}{n^2}}^{\,0}}{1+\cancel{\frac{2}{n^6}}^{\,0}} = 1$$

Finite Non-Zero

$\Rightarrow$ Absolute Series Converges by LCT

$\Rightarrow$ Original Series Absolutely Convergent by Definition

4(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 6^n \cdot n!}{n^6 \cdot n^n}$

Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}}\, 6^{n+1} (n+1)!}{(n+1)^6 (n+1)^{n+1}}}{\frac{\cancel{(-1)^n}\cdot 6^n\, n!}{n^6 \cdot n^n}} \right| = \lim_{n\to\infty} \frac{6^{n+1}}{6^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{n^6}{(n+1)^6} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$6^{n+1} = \cancel{6^n} \cdot 6 \qquad (n+1)! = \cancel{(n+1)}\,\cancel{n!} \qquad (n+1)^{n+1} = (n+1)^n \cancel{(n+1)}$

$$= \lim_{n\to\infty} 6 \cdot \left(\frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}\right)^6 \cdot \frac{n^n}{(n+1)^n} \to \frac{1}{e}$$

$$= \lim_{n\to\infty} 6 \cdot \left(\frac{1}{1+\frac{1}{n}}\right)^6 \cdot \frac{1}{e} = \frac{6}{e} > 1$$ Diverges by Ratio Test

4(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{6n+2022} \xrightarrow[\text{1st}]{A.S.} \sum_{n=1}^{\infty} \frac{1}{6n+2022} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges p-Series (Harmonic) $p=1$

2nd

1. $b_n = \frac{1}{6n+2022} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{6n+2022} = 0$

3. Terms Decreasing

$b_{n+1} = \frac{1}{\underbrace{6(n+1)+2022}_{6n+2028}} \leq \frac{1}{6n+2022} = b_n$

Original Series Converges by AST

LCT

$\lim_{n\to\infty} \frac{\frac{1}{6n+2022}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{6n+2022} \frac{1/n}{1/n}$

$= \lim_{n\to\infty} \frac{1}{6+\frac{2022}{n}} = \frac{1}{6}$ Finite Non-Zero

$\Rightarrow$ Absolute Series Diverges by LCT

Original Series Conditionally Convergent by Definition

Optional Bonus

$\left\{ \frac{(\ln n)\cdot 2^n \cdot (n!)^2}{n^{2n}\cdot (3n)!} \right\}$

First, Consider $\sum_{n=1}^{\infty} \frac{(\ln n)\cdot 2^n\cdot (n!)^2}{n^{2n}\cdot(3n)!}$ which Converges by Ratio Test (see below)

and therefore $\lim_{n\to\infty} \frac{\ln n \cdot 2^n (n!)^2}{n^{2n}(3n)!} = 0$ and the Sequence Converges because

otherwise <u>if</u> the terms did <u>not</u> approach 0, then the Series would Diverge by nTDT, which would Contradict the Convergence of the Series using RatioTest

RatioTest

See(*)

$\lim_{n\to\infty} \frac{\frac{\ln(n+1)\, 2^{n+1}\cdot((n+1)!)^2}{(n+1)^{2(n+1)}(3(n+1))!}}{\frac{\ln n \cdot 2^n (n!)^2}{n^{2n}(3n)!}} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \frac{2^{n+1}}{2^n} \cdot \frac{((n+1)!)^2}{(n!)^2} \cdot \frac{n^{2n}}{(n+1)^{2n+2}} \cdot \frac{(3n)!}{(3n+3)!}$

$\cancel{2^n}\cdot 2 \quad (n+1)^2\cdot(n!)^2$

$(n+1)^{2n}\cancel{(n+1)^2} \quad (3n+3)(3n+2)(3n+1)\cancel{(3n)!}$

$= \lim_{n\to\infty} 2\left(\frac{n^n}{(n+1)^n}\right)^2 \cdot \frac{1}{(3n+3)(3n+2)(3n+1)} = 0 < 1$ A.C. by R.T.

Made with Goodnotes

$$(*)\quad \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{1}{1} = 1$$