Math 121 Exam 2 Fall 2024 Answer Key

1(a) $$\int_0^e x^3 \ln x \, dx = \lim_{t\to 0^+} \int_t^e x^3 \ln x \, dx = \lim_{t\to 0^+} \frac{x^4}{4}\cdot \ln x \Big|_t^e - \frac{1}{4}\int_t^e x^3\, dx$$

IBP

$$\boxed{\begin{array}{ll} u = \ln x & dv = x^3\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^4}{4} \end{array}}$$

$$= \lim_{t\to 0^+} \frac{x^4}{4}\cdot \ln x \Big|_t^e - \frac{x^4}{16}\Big|_t^e$$

$$= \lim_{t\to 0^+} \frac{e^4}{4}\cdot \underbrace{\ln e}_{1} - \underbrace{\frac{t^4}{4}\cdot \ln t}_{0^+\cdot(-\infty) \to 0} - \left(\frac{e^4}{16} - \underbrace{\frac{t^4}{16}}_{0}\right)$$

see ☆

$$= \frac{e^4}{4} - \frac{e^4}{16} = \frac{4e^4 - e^4}{16} = \frac{3e^4}{16} \quad \text{Converges}$$

$$☆ \quad \lim_{t\to 0^+} \underbrace{t^4 \cdot \ln t}_{0\cdot(-\infty)} = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t^4}} \overset{\frac{-\infty}{\infty}}{\underset{\text{L'H}}{=}} \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-4}{t^5}} = \lim_{t\to 0^+} \frac{-t^4}{4} = 0$$

Flip

$t^{-4} \to -4t^{-5}$

$-\frac{t^5}{4}$

Key Note: ln 0 is undefined, so must "sneak attack" 0 using Limit $\to 0^+$

1(b) $$\int_{-\infty}^{-9} \frac{7}{x^2+4x+53}\,dx = \lim_{t\to -\infty} \int_t^{-9} \frac{7}{x^2+4x+53}\,dx = \lim_{t\to -\infty} \int_t^{-9} \frac{7}{(x+2)^2+49}\,dx$$

Complete the Square

Scratch
$(x+2)^2 = x^2+4x+4$ $+49$

$$= \lim_{t\to -\infty} \int_{t+2}^{-7} \frac{7}{u^2+49}\,du = \lim_{t\to -\infty} \cancel{7}\left(\frac{1}{\cancel{7}}\right) \arctan\left(\frac{u}{7}\right)\Big|_{t+2}^{-7}$$

$$\boxed{\begin{array}{l} u = x+2 \\ du = dx \end{array}}$$

$$\boxed{\begin{array}{l} x = t \Rightarrow u = t+2 \\ x = -9 \Rightarrow u = -7 \end{array}}$$

$$= \lim_{t\to -\infty} \underbrace{\arctan\left(\frac{-7}{7}\right)}_{-\frac{\pi}{4}} - \underbrace{\arctan\left(\frac{t+2}{7}\right)}_{-\frac{\pi}{2}}$$

-1, $-\infty$

$$= -\frac{\pi}{4} + \underbrace{\frac{\pi}{2}}_{\frac{2\pi}{4}} = \frac{\pi}{4} \quad \text{Converges}$$

1(c) $\displaystyle\int_{-5}^{-4} \frac{7-x}{x^2+4x-5}\,dx = \int_{-5}^{-4} \frac{7-x}{(x-1)(x+5)}\,dx = \lim_{t\to -5^+} \int_t^{-4} \frac{7-x}{(x-1)(x+5)}\,dx$

PFD

$(x-1)(x+5)\quad \dfrac{7-x}{(x-1)(x+5)} = \dfrac{A}{x-1} + \dfrac{B}{x+5}\quad (x-1)(x+5)$

$7-x = A(x+5)+B(x-1)$
$= Ax+5A+Bx-B$
$= (A+B)x+(5A-B)$

Conditions

- $A+B=-1 \Rightarrow A=-1-B$
- $5A-B=7 \qquad 5(-1-B)-B=7$

$-5-5B-B=7$
$-6B=12$
$B=-2$

$A=-1-(-2)$
$=-1+2=1$

$= \displaystyle\lim_{t\to -5^+} \int_t^{-4} \frac{1}{x-1} - \frac{2}{x+5}\,dx$

$= \displaystyle\lim_{t\to -5^+} \ln|x-1| - 2\ln|x+5| \Big|_t^{-4}$

$= \displaystyle\lim_{t\to -5^+} \underbrace{\ln|-5|}_{\text{finite}}^{\ln 5} - \underbrace{2\ln|1|}_{\text{finite}}^{0} - \left(\underbrace{\ln|t-1|}_{\text{finite}}^{\ln 6} - 2\ln|t+5|\right)$ $\quad t+5 \to 0^+,\ t\to -5^+,\ -\infty$

$= -\infty$ Diverges

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^7} \longrightarrow$ study Related Function $f(x) = \dfrac{\ln x}{x^7}$

$\displaystyle\int_1^{\infty} \frac{\ln x}{x^7}\,dx = \lim_{t\to\infty} \int_1^t (\ln x)\cdot x^{-7}\,dx \overset{\text{IBP}}{=} \lim_{t\to\infty} -\frac{\ln x}{6x^6}\Big|_1^t + \frac{1}{6}\int_1^t x^{-7}\,dx$ $\qquad \dfrac{x^{-6}}{-6}$

$u=\ln x$	$dv = x^{-7}dx$
$du = \frac{1}{x}dx$	$v = \frac{x^{-6}}{-6} = -\frac{1}{6x^6}$

$= \displaystyle\lim_{t\to\infty} -\frac{\ln x}{6x^6}\Big|_1^t - \frac{1}{36x^6}\Big|_1^t$

$= \displaystyle\lim_{t\to\infty} \underbrace{-\frac{\ln t}{6t^6}}_{\text{L'H here}}^{\frac{\infty}{\infty}} + \frac{\cancel{\ln 1}}{6}^{0} - \left(\frac{1}{\cancel{36t^6}} - \frac{1}{36}\right)$

$\overset{\text{L'H}}{=} \displaystyle\lim_{t\to\infty} -\frac{\frac{1}{t}}{36t^5} + \frac{1}{36}$ $\qquad \dfrac{1}{36t^5}$

$= \displaystyle\lim_{t\to\infty} \frac{-1}{\cancel{36t^6}} + \frac{1}{36} = \frac{1}{36}$ Integral Converges

$\Rightarrow \boxed{\text{Series Converges by Integral Test}}$

3(a) $\sum_{n=2}^{\infty} \underbrace{\frac{e^{3n}}{7\ln n}}_{a_n}$ → Ratio Test OR nTDT

Option 1. Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{e^{3(n+1)}}{7\ln(n+1)}}{\frac{e^{3n}}{7\ln n}}\right| = \lim_{n\to\infty}\left(\frac{e^{3n+3}}{\cancel{e^{3n}}}\right)\left(\frac{\cancel{7}\ln n}{\cancel{7}\ln(n+1)}\right) = e^3 > 1$$

$\frac{7\ln n}{e^{3n}}$ + term

$\cancel{e^{3n}} \cdot e^3$ ☆ $\frac{\infty}{\infty}$ 1

⇒ $\boxed{\text{Series Diverges by Ratio Test}}$

☆ $$\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty}\frac{\ln x}{\ln(x+1)} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{n\to\infty}\frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty}\frac{x+1}{x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{1}{1} = 1$$

$x+1$

OR ↳ $= \lim_{x\to\infty} 1 + \cancel{\frac{1}{x}}^{\,0} = 1$

OR

Option 2: $\boxed{\text{Diverges by nTDT}}$

$$\lim_{n\to\infty}\frac{e^{3n}}{7\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{e^{3x}}{7\ln x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{3e^{3x}}{\frac{7}{x}} = \lim_{x\to\infty}\frac{3xe^{3x}}{7} = \infty \neq 0$$

$\frac{x}{7}$ ∞

3(b) $\sum_{n=1}^{\infty}\frac{1}{(3n+7)!}$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(3(n+1)+7)!}}{\frac{1}{(3n+7)!}}\right| = \lim_{n\to\infty}\frac{\cancel{(3n+7)!}}{(3n+10)!} = \lim_{n\to\infty}\frac{1}{(3n+10)(3n+9)(3n+8)} = 0 < 1$$

$(3n+10)(3n+9)(3n+8)\cancel{(3n+7)!}$

0 ∞

⇒ Absolutely Convergent by the Ratio Test

⇒ Since the Original Series is already the Absolute Series, then

Absolute Convergence = $\boxed{\text{Convergence}}$

OR The Original Series Converges by ACT since Absolute Convergence implies Convergence

Factor out

3(c) $-\frac{3}{\sqrt{1}}-\frac{3}{\sqrt{2}}-\frac{3}{\sqrt{3}}-\frac{3}{\sqrt{4}}-\frac{3}{\sqrt{5}}-\frac{3}{\sqrt{6}}\cdots = -3\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ Note: not Alternating here

Constant Multiple of the Divergent p-Series

with $p=\frac{1}{2}<1$ is $\boxed{\text{Divergent}}$

3(d) $\sum_{n=1}^{\infty}\frac{\ln 3}{n^7}+\frac{(-3)^n}{7^{2n+1}} = \sum_{n=1}^{\infty}\frac{\ln 3}{n^7} + \sum_{n=1}^{\infty}\frac{(-3)^n}{7^{2n+1}}$

$r=-\frac{3}{7^2}$

$\ln 3\sum_{n=1}^{\infty}\frac{1}{n^7} + -\frac{3}{7^3}+\frac{3^2}{7^5}-\frac{3^3}{7^7}+\ldots$

Constant Multiple of Convergent p-Series $p=7>1$ is Convergent

Converges by Geometric Series Test with $|r|=\left|-\frac{3}{7^2}\right|=\frac{3}{49}<1$

Original Series Converges because the Sum of Two Convergent Series is Convergent

3(e) $\sum_{n=1}^{\infty}\frac{(-1)^n 3}{n^7}$ → AST or ACT

Option 1: AST

1. choose $b_n=\frac{3}{n^7}>0$

2. $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{3}{n^7}=0$

3. Terms Decreasing

$b_{n+1}=\frac{3}{(n+1)^7}\le\frac{3}{n^7}=b_n$

Series $\boxed{\text{Converges}}$ by the Alternating Series Test

OR

Option 2: ACT

O.S. $\sum_{n=1}^{\infty}\frac{(-1)^n 3}{n^7} \xrightarrow{A.S.} \sum_{n=1}^{\infty}\frac{3}{n^7}=3\sum_{n=1}^{\infty}\frac{1}{n^7}$

Absolute Series Converges because a Constant Multiple of a Convergent p-Series $p=7>1$ is Convergent

ACT

Original Series Converges by the Absolute Convergence Test

3(f) $\sum_{n=2}^{\infty}\left(1-\frac{7}{n^3}\right)^{n^3}$ Diverges by nTDT because

1^{∞}

$$\lim_{n\to\infty}\left(1-\frac{7}{n^3}\right)^{n^3}=\lim_{x\to\infty}\left(1-\frac{7}{x^3}\right)^{x^3}=e^{\lim_{x\to\infty}\ln\left(\left(1-\frac{7}{x^3}\right)^{x^3}\right)}=e^{\lim_{x\to\infty}x^3\ln\left(1-\frac{7}{x^3}\right)}$$

$$=e^{\lim_{x\to\infty}\frac{\ln\left(1-\frac{7}{x^3}\right)}{\frac{1}{x^3}}}\overset{0/0}{\underset{L'H}{=}}e^{\lim_{x\to\infty}\frac{\frac{1}{1-\cancel{\frac{7}{x^3}}^{0}}\cdot\left(\cancel{\frac{21}{x^4}}\right)}{\cancel{-\frac{3}{x^4}}}}=e^{-7}\neq 0$$

$x^{-3}\to -3x^{-4}$ $\quad -7x^{-3}\to +21x^{-4}$ $\quad 1 \quad -7$

4. $\sum_{n=1}^{\infty}\frac{(-1)^n}{(3n+7)^7}$ $\xrightarrow{A.S.}$ $\sum_{n=1}^{\infty}\frac{1}{(3n+7)^7}\approx\sum_{n=1}^{\infty}\frac{1}{n^7}$ Converges p-Series $p=7>1$

Bound Terms

$$\frac{1}{(3n+7)^7}\le\frac{1}{n^7}$$

$\Rightarrow$ Absolute Series also Converges by the Comparison Test

ACT $\Leftarrow$

Original Series Converges by the Absolute Convergence Test

Absolutely Convergent example

5(a) $\sum_{n=1}^{\infty}(-1)^n\frac{n^3+7}{n^7+3}\longrightarrow\sum_{n=1}^{\infty}\frac{n^3+7}{n^7+3}\approx\sum_{n=1}^{\infty}\frac{n^3}{n^7}=\sum_{n=1}^{\infty}\frac{1}{n^4}$ Converges p-Series $p=4>1$

LCT Limit

$$\lim_{n\to\infty}\frac{\frac{n^3+7}{n^7+3}}{\frac{1}{n^4}}=\lim_{n\to\infty}\frac{n^7+7n^4}{n^7+3}\frac{\frac{1}{n^7}}{\frac{1}{n^7}}=\lim_{n\to\infty}\frac{1+\cancel{\frac{7}{n^3}}^{0}}{1+\cancel{\frac{3}{n^7}}^{0}}=1$$ Finite Non-Zero

n^4

$\Rightarrow$ Absolute Series also Converges by Limit Comparison Test

$\Rightarrow$ Original Series is Absolutely Convergent (by Definition)

$$5(b)\quad \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)!\, 3^n}{n^7 (n!)\, n^n} \qquad \text{Ratio Test}$$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}} (2(n+1)+1)!\, 3^{n+1}}{(n+1)^7 (n+1)!\,(n+1)^{n+1}}}{\dfrac{\cancel{(-1)^n} (2n+1)!\, 3^n}{n^7 (n!) \cdot n^n}} \right|$$

$$(2n+3)(2n+2)\cancel{(2n+1)!} \qquad \cancel{3^n} \cdot 3 \qquad \left(\frac{n}{n+1}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)^7 \qquad \left(\frac{1}{1+\frac{1}{n}\to 0}\right)^7 \to 1$$

see above

$$= \lim_{n\to\infty} \left(\frac{(2n+3)!}{\cancel{(2n+1)!}}\right)\left(\frac{3^{n+1}}{\cancel{3^n}}\right)\left(\frac{n^7}{(n+1)^7}\right)\left(\frac{\cancel{n!}}{(n+1)!}\right)\left(\frac{n^n}{(n+1)^{n+1}}\right) \to 1$$

$$(n+1)\cancel{n!} \qquad (n+1)^n \cdot (n+1)$$

$$= \lim_{n\to\infty} 3 \left(\frac{2n+3}{n+1}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{\overset{2\cancel{(n+1)}}{2n+2}}{\cancel{n+1}}\right)\left(\frac{n^n}{(n+1)^n}\right) \to \frac{1}{e} \qquad = \frac{12}{e} > 1$$

$$\frac{2+\frac{3}{n}\to 0}{1+\frac{1}{n}\to 0} \to 2$$

$\Rightarrow$ $\boxed{\text{Series Diverges by Ratio Test}}$

$$5(c)\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{3n+7} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{3n+7} \approx \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{Diverges p-Series } p=1$$

AST

$$\lim_{n\to\infty} \frac{\frac{1}{3n+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n+7}\,\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{3+\frac{7}{n}\to 0} \to \frac{1}{3} = \frac{1}{3} \quad \text{Finite Non-Zero}$$

1. Pick $b_n = \dfrac{1}{3n+7} > 0$

2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3n+7} = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{\underbrace{3(n+1)+7}_{3n+10}} \le \frac{1}{3n+7} = b_n$$

$\Rightarrow$ Absolute Series also Diverges by the Limit Comparison Test

Original Series Converges by the Alternating Series Test

Finally, the Original Series is Conditionally Convergent by Definition

OR show Related Function

$$f(x) = \frac{1}{3x+7} \quad \text{has} \quad f'(x) = \frac{-3}{(3x+7)^2} < 0$$

Bonus: First, show that the Series $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln(\ln n))\cdot 5^n (n!)^3 (2n)!}{n^{2n}\cdot (3n)!}$ Converges by the Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(\ln(\ln(n+1)))\cdot 5^{n+1}((n+1)!)^3(2(n+1))!}{(n+1)^{2(n+1)}\,(3(n+1))!}}{\dfrac{(\ln(\ln n))\cdot 5^n(n!)^3(2n)!}{n^{2n}\,(3n)!}}\right|$$

Flip + Multiply

See below ☆

$$= \lim_{n\to\infty}\underbrace{\left(\frac{\ln(\ln(n+1))}{\ln(\ln n)}\right)}_{1}\left(\frac{\overbrace{5^{n+1}}^{\cancel{5^n}\cdot 5}}{\cancel{5^n}}\right)\left(\frac{\overbrace{((n+1)!)^3}^{(n+1)^3\cancel{(n!)^3}}}{\cancel{(n!)^3}}\right)\left(\frac{\overbrace{(2n+2)!}^{(2n+2)(2n+1)\cancel{(2n)!}}}{\cancel{(2n)!}}\right)\left(\frac{n^{2n}}{\underbrace{(n+1)^{2n+2}}_{(n+1)^{2n}(n+1)^2}}\right)\left(\frac{\cancel{(3n)!}}{\underbrace{(3n+3)!}_{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}}\right)$$

$$= \lim_{n\to\infty} \frac{5\,\cancel{(n+1)}^{3}\,\overbrace{(2n+2)}^{2\cancel{(n+1)}}(2n+1)}{\cancel{(n+1)^2}\,\underbrace{(3n+3)}_{3\cancel{(n+1)}}(3n+2)(3n+1)}\left(\underbrace{\frac{n^n}{(n+1)^n}}_{\frac{1}{e}}\right)^{2} \quad \frac{1}{e^2}$$

$$= \lim_{n\to\infty} \frac{10}{3e^2}\left(\frac{n+1}{3n+2}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{2n+1}{3n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$$= \lim_{n\to\infty} \frac{10}{3e^2}\underbrace{\left(\frac{1+\frac{1}{n}}{3+\frac{2}{n}}\right)}_{\frac{1}{3}}\underbrace{\left(\frac{2+\frac{1}{n}}{3+\frac{1}{n}}\right)}_{\frac{2}{3}} = \frac{20}{27e^2} < 1$$

$\Rightarrow$ The Series is (Absolutely) Convergent by the Ratio Test

☆ $$\lim_{x\to\infty}\frac{\ln(\ln(x+1))}{\ln(\ln x)} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{\frac{1}{\ln(x+1)}\cdot\frac{1}{(x+1)}}{\frac{1}{\ln x}\cdot\frac{1}{x}} = \lim_{x\to\infty}\underbrace{\frac{\ln x}{\ln(x+1)}}_{1}\cdot\underbrace{\frac{x}{x+1}}_{1} = 1$$

See below

☆ $$\lim_{x\to\infty}\frac{\ln x}{\ln(x+1)} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty}\frac{x+1}{x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{1}{1} = 1$$

☆ $$\lim_{x\to\infty}\frac{x}{x+1} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{1}{1} = 1$$

Bonus (continued)

$\Rightarrow$ the Sequence terms must Approach 0 as $n \to \infty$ because

otherwise if $\lim\limits_{n\to\infty} a_n \neq 0$, then the Series would Diverge by nTDT

terms

which would contradict what we proved above... the Series Converges.

That is, since $\sum\limits_{n=1}^{\infty} \frac{\ln(\ln n)\, 5^n (n!)^3 (2n)!}{n^{2n}\,(3n)!}$ Converges then

$$\lim_{n\to\infty} \frac{(\ln(\ln n))\, 5^n \cdot (n!)^3 (2n)!}{n^{2n}\,(3n)!} = 0 \Rightarrow \text{Sequence Converges.}$$