Exam 1 Spring 2026 Answer Key

1(a) $$\lim_{x\to 0} \frac{\ln(1+5x) - 5x}{\arcsin(3x) + e^{-3x} - 1} \quad \frac{0}{0}$$

$$\underset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{1+5x}\cdot 5 - 5}{\frac{1}{\sqrt{1-(3x)^2}}\cdot 3 - 3e^{-3x}} \quad \frac{0}{0}$$

$$\underset{\text{prep}}{=} \lim_{x\to 0} \frac{5(1+5x)^{-1} - 5}{3(1-9x^2)^{-1/2} - 3e^{-3x}}$$

$$\underset{\text{L'H}}{=} \lim_{x\to 0} \frac{-5(1+5x)^{-2}\cdot 5}{-\frac{3}{2}(1-9x^2)^{-3/2}(-18x) + 9e^{-3x}}$$

$$\underset{\text{rewrite}}{=} \lim_{x\to 0} \frac{\frac{-25}{(1+5x)^2} \to -25}{\frac{27x}{(1-9x^2)^{3/2}} \to 0 \; + \; 9e^{-3x} \to 9 \cdot 1}$$

$$= -\frac{25}{9}$$ Match

1(b) $$\lim_{x\to 0^+} x^3 \ln x \quad 0\cdot(-\infty) \; = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^3}} \quad \frac{-\infty}{\infty} \; \underset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-3}{x^4}} = \lim_{x\to 0^+} \frac{-x^3}{3} = 0$$ Match

$x^{-3} \hookrightarrow -3x^{-4}$, $\quad \frac{-x^4}{3}$

1(c) $$\lim_{x\to\infty} \left(1 - \arctan\left(\frac{3}{x^4}\right)\right)^{x^4} = e^{\lim_{x\to\infty} \ln\left(\left(1-\arctan\left(\frac{3}{x^4}\right)\right)^{x^4}\right)}$$

$$= e^{\lim_{x\to\infty} x^4 \cdot \ln\left(1 - \arctan\left(\frac{3}{x^4}\right)\right)} \quad \infty\cdot 0$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(1-\arctan\left(\frac{3}{x^4}\right)\right)}{\frac{1}{x^4}}} \quad \frac{0}{0}$$

$3x^{-4}$; $\quad x^{-4} \to -4x^{-5}$

$$\underset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{1-\arctan\left(\frac{3}{x^4}\right)} \cdot \frac{-1}{1+\left(\frac{3}{x^4}\right)^2} \cdot \left(\frac{-12}{x^5}\right)}{\frac{-4}{x^5}}}$$

$$= e^{1\cdot(-1)\cdot 3} = e^{-3}$$ Match

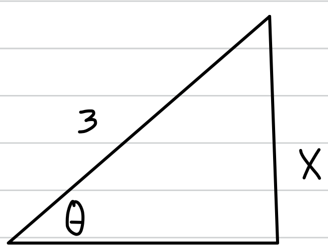
2. $$\int_{-3}^{3}\sqrt{9-x^2}\,dx = \int_{x=-3}^{x=3}\underbrace{\sqrt{9-9\sin^2\theta}}_{\substack{\sqrt{9(1-\sin^2\theta)} \\ \sqrt{9\cos^2\theta}}}^{3\cos\theta}\cdot 3\cos\theta\,d\theta = 9\int_{x=-3}^{x=3}\cos^2\theta\,d\theta$$

Trig. Sub $\boxed{\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta\,d\theta \end{aligned}}$

$\sin\theta = \frac{x}{3}$

$\theta = \arcsin\left(\frac{x}{3}\right)$

$$= 9\int_{x=-3}^{x=3}\frac{1+\cos(2\theta)}{2}\,d\theta = \frac{9}{2}\int_{x=-3}^{x=3} 1+\cos(2\theta)\,d\theta$$



$\Rightarrow \sqrt{9-x^2}$

$$= \frac{9}{2}\left(\theta + \frac{\overset{\cancel{2}\sin\theta\cos\theta}{\sin(2\theta)}}{\cancel{2}}\right)\Bigg|_{x=-3}^{x=3} = \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)\frac{\sqrt{9-x^2}}{3}\right)\Bigg|_{-3}^{3}$$

$$= \frac{9}{2}\left(\overset{\frac{\pi}{2}}{\cancel{\arcsin}}\overset{1}{\cancel{\left(\frac{3}{3}\right)}} + \left(\frac{3}{3}\right)\overset{0}{\cancel{\frac{\sqrt{9-9}}{2}}} - \left(\overset{-\frac{\pi}{2}}{\cancel{\arcsin}}\overset{-1}{\cancel{\left(\frac{-3}{3}\right)}} + \left(\frac{-3}{3}\right)\overset{0}{\cancel{\frac{\sqrt{9-9}}{3}}}\right)\right)$$

$$= \frac{9}{2}\left(\underbrace{\frac{\pi}{2} + \frac{\pi}{2}}_{\pi}\right) = \frac{9\pi}{2}$$ Match

3. $$\int_{0}^{\ln\sqrt{3}}\frac{e^{2x}}{3+e^{4x}}\,dx = \int_{0}^{\ln\sqrt{3}}\frac{e^{2x}}{3+(e^{2x})^2}\,dx = \frac{1}{2}\int_{1}^{3}\frac{1}{3+u^2}\,du = \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_{1}^{3}$$

$\boxed{\begin{aligned} u &= e^{2x} \\ du &= 2e^{2x}dx \\ \tfrac{1}{2}du &= e^{2x}dx \end{aligned}}$

$\boxed{\begin{aligned} x=0 &\Rightarrow u=e^0=1 \\ x=\ln\sqrt{3} &\Rightarrow u=e^{2\ln\sqrt{3}} \\ &= \cancel{e}^{\ln(\sqrt{3}^2)} \\ &= 3 \end{aligned}}$

$$= \frac{1}{2\sqrt{3}}\left(\overset{\frac{\pi}{3}}{\cancel{\arctan}}\left(\overset{\sqrt{3}}{\cancel{\frac{3}{\sqrt{3}}}}\right) - \overset{\frac{\pi}{6}}{\cancel{\arctan}}\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{1}{2\sqrt{3}}\left(\underbrace{\frac{\pi}{3} - \frac{\pi}{6}}_{\frac{\pi}{6}}\right) = \frac{\pi}{12\sqrt{3}}$$ Match

4. $$\int_{1}^{e} x^3\cdot\ln x\,dx = \frac{x^4}{4}\cdot\ln x\Bigg|_{1}^{e} - \frac{1}{4}\int_{1}^{e}\overset{x^3}{\cancel{\frac{x^4}{x}}}\,dx$$

$\boxed{\begin{aligned} u &= \ln x & dv &= x^3dx \\ du &= \frac{1}{x}dx & v &= \frac{x^4}{4} \end{aligned}}$

$$= \frac{x^4}{4}\cdot\ln x\Bigg|_{1}^{e} - \frac{x^4}{16}\Bigg|_{1}^{e}$$

$$= \frac{e^4}{4}\cdot\overset{1}{\cancel{\ln e}} - \frac{1}{4}\overset{0}{\cancel{\ln 1}} - \left(\frac{e^4}{16} - \frac{1}{16}\right)$$

$$= \underset{\frac{4e^4}{16}}{\frac{e^4}{4}} - \frac{e^4}{16} + \frac{1}{16} = \frac{3e^4}{16} + \frac{1}{16} = \frac{1+3e^4}{16}$$ Match

5. $\int x^4 \arcsin x\,dx = \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{x^5}{\sqrt{1-x^2}}\,dx = \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{\sin^5\theta}{\sqrt{1-\sin^2\theta}}\cdot \cancel{\cos\theta}\,d\theta$

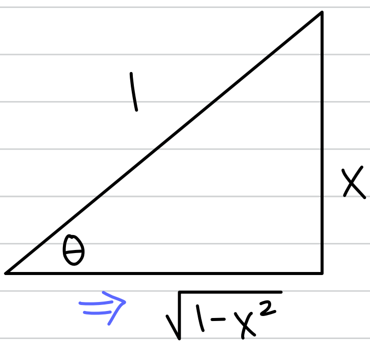
$\sqrt{\cos^2\theta}$

$\cancel{\cos\theta}$

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = x^4\,dx \\ du = \frac{1}{\sqrt{1-x^2}}\,dx & v = \frac{x^5}{5} \end{array}}$$

$$\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{array}}$$

$\theta = \arcsin x$



$\hookrightarrow \cos\theta = \sqrt{1-x^2}$

$$\boxed{\begin{array}{l} w = \cos\theta \\ dw = -\sin\theta\,d\theta \\ -dw = \sin\theta\,d\theta \end{array}}$$

$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \sin^5\theta\,d\theta$ ODD Power

$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \sin^4\theta\cdot\sin\theta\,d\theta$

$(\sin^2\theta)^2$

$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int (1-\cos^2\theta)^2\cdot\sin\theta\,d\theta$

$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\int (1-w^2)^2\,dw$

$1 - 2w^2 + w^4$

$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left(w - \frac{2w^3}{3} + \frac{w^5}{5}\right) + C$

$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left(\cos\theta - \frac{2}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right) + C$

$= \boxed{\frac{x^5}{5}\arcsin x + \frac{1}{5}\left(\sqrt{1-x^2} - \frac{2}{3}\left(\sqrt{1-x^2}\right)^3 + \frac{1}{5}\sqrt{1-x^2}\right) + C}$

OR $(1-x^2)^{3/2}$

6. $\displaystyle\int \frac{1}{(9+x^2)^{7/2}}\,dx = \int \frac{1}{\left(\sqrt{x^2+9}\right)^7}\,dx = \int \frac{1}{\left(\sqrt{9\tan^2\theta+9}\right)^{7}}\cdot 3\sec^2\theta\,d\theta$

Don't Drop

$\sqrt{9(\tan^2\theta+1)}$

$\sqrt{9\sec^2\theta}$

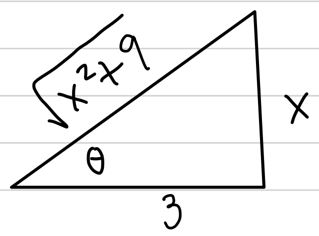
Must Use Trig. Sub here

$$\boxed{\begin{aligned} x &= 3\tan\theta \\ dx &= 3\sec^2\theta\,d\theta \end{aligned}}$$

$\tan\theta = \frac{x}{3}$

$$= \int \frac{1}{\left(\sqrt{9\sec^2\theta}\right)^7}\cdot 3\sec^2\theta\,d\theta$$

$(3\sec\theta)^7$



$$= \frac{3}{3^7}\int \frac{1}{\cancel{\sec^7\theta}}\,\cancel{\sec^2\theta}\,d\theta$$

$\sec^5\theta$

$\hookrightarrow \sin\theta = \dfrac{x}{\sqrt{x^2+9}}$

$$= \frac{1}{3^6}\int \cos^5\theta\,d\theta \qquad \text{ODD}$$

$\cos^4\theta\cdot\cos\theta$

$$= \frac{1}{729}\int \left(\cos^2\theta\right)^2\cdot\cos\theta\,d\theta$$

$$= \frac{1}{729}\int \left(1-\sin^2\theta\right)^2\cos\theta\,d\theta \qquad \text{u-sub}$$

$$\boxed{\begin{aligned} u &= \sin\theta \\ du &= \cos\theta\,d\theta \end{aligned}}$$

$$= \frac{1}{729}\int (1-u^2)^2\cdot du$$

$1-2u^2+u^4$

$$= \frac{1}{729}\left(u-\frac{2}{3}u^3+\frac{u^5}{5}\right)+C$$

$$= \frac{1}{729}\left(\sin\theta-\frac{2}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta\right)+C$$

$$= \boxed{\frac{1}{729}\left(\frac{x}{\sqrt{x^2+9}}-\frac{2}{3}\left(\frac{x}{\sqrt{x^2+9}}\right)^3+\frac{1}{5}\left(\frac{x}{\sqrt{x^2+9}}\right)^5\right)+C}$$

Optional Bonus

$$\int \frac{xe^x}{\sqrt{e^x-1}}\,dx = \int \frac{\ln(w+1)}{\sqrt{w}}\,du = \int \ln(w+1)\cdot w^{-1/2}\,du$$

$$\boxed{\begin{aligned} w &= e^x-1 \\ dw &= e^x\,dx \end{aligned}} \Rightarrow e^x = w+1 \Rightarrow x = \ln(w+1)$$

IBP

$$\boxed{\begin{aligned} u &= \ln(w+1) & dv &= w^{-\frac{1}{2}}\,dw \\ du &= \tfrac{1}{w+1}\,dw & v &= 2w^{1/2} \end{aligned}}$$

either u-sub 1st and IBP second or, vice versa

$$= 2\sqrt{w}\cdot\ln(w+1) - 2\int \frac{\overset{K}{\sqrt{w}}}{\underset{\underset{K}{(\sqrt{w})^2+1}}{w+1}}\,dw$$

Let

$$\boxed{\begin{array}{l} K=\sqrt{w} \\ dK=\frac{1}{2\sqrt{w}}\,dw \\ 2\sqrt{w}\,dK=dw \\ 2K\,dK=dw \end{array}}$$

$$= 2\sqrt{w}\ln(w+1) - 4\int \frac{\overset{K\cdot K}{K^2}}{K^2+1}\,dK$$

$$= 2\sqrt{w}\cdot\ln(w+1) - 4\int \frac{K^2+1-1}{K^2+1}\,dK$$

$$= 2\sqrt{w}\cdot\ln(w+1) - 4\int \overset{1}{\cancel{\frac{K^2+1}{K^2+1}}} - \frac{1}{K^2+1}\,dK$$

$$= 2\sqrt{w}\ln(w+1) - 4\left(K - \arctan K\right) + C$$

$$= 2\sqrt{w}\cdot\ln(w+1) - 4\left(\sqrt{w} - \arctan\sqrt{w}\right) + C$$

$$= 2\sqrt{e^x-1}\cdot\ln(e^x\cancel{-1}\cancel{+1}) - 4\sqrt{e^x-1} + 4\arctan\sqrt{e^x-1} + C$$

$$\cancel{\ln e}^{x}$$

$$= \boxed{2x\sqrt{e^x-1} - 4\sqrt{e^x-1} + 4\arctan\sqrt{e^x-1} + C}$$

OR

$$\int \frac{xe^x}{\sqrt{e^x-1}}\,dx = 2x\sqrt{e^x-1} - 2\int\sqrt{e^x-1}\,dx = 2x\sqrt{e^x-1} - 2\int\frac{\sqrt{w}}{w+1}\,dw$$

$$\boxed{\begin{array}{ll} u=x & dv=\frac{e^x}{\sqrt{e^x-1}}\,dx \;\star \\ du=dx & v=2\sqrt{e^x-1} \end{array}}$$

$$\boxed{\begin{array}{l} w=e^x-1 \\ dw=e^x\,dx \end{array}} \Rightarrow w+1=e^x$$

$$\frac{1}{e^x}\,dw=dx$$

$$\frac{1}{w+1}\,dw=dx$$

Same as above

⋮

$$= 2x\sqrt{e^x-1} - 4\left(\sqrt{w} - \arctan\sqrt{w}\right) + C$$

$$= 2x\sqrt{e^x-1} - 4\sqrt{e^x-1} + 4\arctan\sqrt{e^x-1} + C$$

$$\star\ \int\frac{e^x}{\sqrt{e^x-1}}\,dx = \int\frac{1}{\sqrt{u}}\,du$$

$$\boxed{\begin{array}{l} u=e^x-1 \\ du=e^x\,dx \end{array}}$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{e^x-1} + C$$