Exam #1 Review Packet Answer Key

Derivatives

1. $f(x) = \arcsin x \cdot \arctan x + \arctan(\sin(\ln x))$

$$f'(x) = \boxed{\arcsin x \cdot \left[\frac{1}{1+x^2}\right] + \arctan x \cdot \left[\frac{1}{\sqrt{1-x^2}}\right] + \frac{1}{1+(\sin(\ln x))^2} \cdot \cos(\ln x) \cdot \frac{1}{x}}$$

2. $f(x) = \dfrac{\sinh(x^2-2)}{x+\sin^{-1}x}$

$$f'(x) = \boxed{\frac{(x+\sin^{-1}x)\cdot\cosh(x^2-2)(2x) - \sinh(x^2-2)\left[1+\frac{1}{\sqrt{1-x^2}}\right]}{(x+\sin^{-1}x)^2}}$$

3. $f(x) = \sin\left(e^{\arcsin e^x}\right)$

$$f'(x) = \boxed{\cos x\left(e^{\arcsin e^x}\right) \cdot e^{\arcsin e^x} \cdot \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x}$$

4. $f(x) = \ln\left(1-\arcsin\left(\frac{7}{x}\right)\right)$

$$f'(x) = \boxed{\frac{1}{1-\arcsin(7/x)} \cdot \left[-\frac{1}{\sqrt{1-(7/x)^2}}\right] \cdot \left(\frac{-7}{x^2}\right)}$$

5. $f(x) = \ln(1-x)$

$$f'(x) = \frac{-1}{1-x} = -(1-x)^{-1}$$

$$f''(x) = \boxed{\cancel{-}\,\cancel{-}+(1-x)^{-2}(-1)} \overset{\text{or}}{=} \boxed{\frac{-1}{(1-x)^2}}$$

6. $f(x) = \arctan(3x)$

$$f'(x) = \frac{1}{1+(3x)^2}(3) = 3(1+9x^2)^{-1}$$

$$f''(x) = \boxed{-3(1+9x^2)^{-2}(18x)} \overset{\text{or}}{=} \boxed{\frac{-54x}{(1+9x^2)^2}}$$

7. $f(x) = \arcsin(4x)$

$$f'(x) = \frac{1}{\sqrt{1-(4x)^2}} \cdot 4 = 4(1-16x^2)^{-1/2}$$

$$f''(x) = \boxed{-\cancel{4}\left(\cancel{\tfrac{1}{2}}\right)(1-16x^2)^{-3/2}(-32x)} \overset{\text{or}}{=} \boxed{\frac{64x}{(1-16x^2)^{3/2}}}$$

Proofs

8. Let $y = \arcsin x$

Invert $\sin y = x$

Differentiate $\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$

$$\cos y \cdot \frac{dy}{dx} = 1$$

Solve $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\underset{\substack{+ \\ \text{on } -\pi/2 \le y \le \pi/2}}{\sqrt{1-\sin^2 y}}} = \frac{1}{\sqrt{1-(\underset{x}{\sin y})^2}} = \frac{1}{\sqrt{1-x^2}} \quad \square$$

9. Let $y = \arctan x$

Invert $\tan y = x$

Differentiate $\frac{d}{dx}[\tan y] = \frac{d}{dx}[x]$

$$\sec^2 y \cdot \frac{dy}{dx} = 1$$

Solve $\quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{(1+(\tan y)^2} = \frac{1}{1+x^2} \quad \square$

x

ignore ≥ Fall 2022

~~10. Let $y = \sinh^{-1} x$~~

~~Invert $\sinh y = x$~~

~~Differentiate $\frac{d}{dx}[\sinh y] = \frac{d}{dx}[x]$~~

~~$\cosh y \cdot \frac{dy}{dx} = 1$~~

~~Identity~~

~~$\cosh^2 y - \sin^2 y = 1$~~

~~$\cosh y = \sqrt{1+\sinh^2 y}$~~

~~$+$~~

~~Solve $\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+(\sinh y)^2}} = \frac{1}{\sqrt{1+x^2}} \quad \square$~~

~~x~~

11. Let $y = \arcsin(5x)$

Invert $\sin y = 5x$

Differentiate $\frac{d}{dx}[\sin y] = \frac{d}{dx}[5x]$

$$\cos y \cdot \frac{dy}{dx} = 5$$

Solve $\frac{dy}{dx} = \frac{5}{\cos y} = \frac{5}{\sqrt{1-\sin^2 y}} = \frac{5}{\sqrt{1-(\sin y)^2}} = \frac{5}{\sqrt{1-(5x)^2}} = \frac{5}{\sqrt{1-25x^2}}$

$\square$

12. Let $y = \ln x$

Invert $e^y = x$

Differentiate $\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$

$$e^y \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x} \quad \square$

13. Let $y = \arctan(3x)$

Invert $\tan y = 3x$

Differentiate $\frac{d}{dx}\left[\tan y\right] = \frac{d}{dx}\left[3x\right]$

$$\sec^2 y \cdot \frac{dy}{dx} = 3$$

Solve $\frac{dy}{dx} = \frac{3}{\sec^2 y} = \frac{3}{1+\tan^2 y} = \frac{3}{1+(\tan y)^2} = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$ $\square$

14. $\int \frac{1}{9+x^2}\,dx = \int \frac{1}{9\left[1+x^2/9\right]}\,dx = \frac{1}{9}\int \frac{1}{1+x^2/9}\,dx$ create perfect square

$\boxed{u = \frac{x}{3} \quad du = \frac{1}{3}dx \quad 3du = dx}$

$$= \frac{1}{9}\int \frac{1}{1+\left(\frac{x}{3}\right)^2}\,dx = \frac{3}{9}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\arctan u + C$$

$$= \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C \quad \square$$

15. $\int \frac{1}{\sqrt{16-x^2}}\,dx = \int \frac{1}{\sqrt{16(1-x^2/16)}}\,dx = \int \frac{1}{\sqrt{16}\sqrt{1-x^2/16}}\,dx$ create perfect square

$\boxed{u = x/4 \quad du = 1/4\,dx}$

$$= 1/4\int \frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin u + C$$

$$= \arcsin\left(x/4\right) + C \quad \square$$

Limits

16. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to\infty} \ln\left[(1+\frac{1}{x})^x\right]} = e^{\lim_{x\to\infty} x \ln(1+\frac{1}{x})} = e^{\lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}}$

$\infty \cdot 0$ $\quad$ $\frac{0}{0}$

$\underset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}} = e^1 = \boxed{e}$

1 $\quad$ 0

17. $\lim_{x\to 0} \frac{7xe^x - \arctan(7x)}{\sin x + \ln(1-x)} \underset{L'H}{=} \lim_{x\to 0} \frac{7xe^x + 7e^x - \frac{1}{1+(7x)^2}\cdot 7}{\cos x + \frac{1}{1-x}(-1)}$ $\quad$ $\frac{0}{0}$

$\to -7(1+49x^2)^{-1}$ $\quad$ $\frac{0}{0}$

$\to -(1-x)^{-1}$

$\underset{L'H}{=} \lim_{x\to 0} \frac{7xe^x + 7e^x + 7e^x + 7(1+49x^2)^{-2}(98x)}{-\sin x + (1-x)(-1)}$

Simplify

$= \lim_{x\to 0} \frac{7xe^x + 14e^x + \frac{686x}{(1+49x^2)^2}}{-\sin x - \frac{1}{(1-x)^2}} = \frac{14}{-1} = \boxed{-14}$

0 $\quad$ 14 $\quad$ 0 $\quad$ 0 $\quad$ -1

18. $\lim_{x\to 0} \frac{\cos(4x) - 1 - \arctan(4x) + 4x}{\ln(1-x) + \arcsin x}$ $\quad$ $\frac{0}{0}$

$-4(1+16x^2)^{-1}$

$\underset{L'H}{=} \lim_{x\to 0} \frac{-4\sin(4x) - \frac{4}{1+(4x)^2} + 4}{-\frac{1}{1-x} + \frac{1}{\sqrt{1-x^2}}}$ $\quad$ $\frac{0}{0}$

$-(1-x)^{-1}$ $\quad$ $(1-x^2)^{-1/2}$

$$\underset{L'H}{=} \lim_{x\to 0} \frac{-16\cos(4x) + 4(1+16x^2)^{-2}(32x)}{-(1-x)^{-2}(-1) - \cancel{\frac{1}{2}}(1-x^2)^{-3/2}(\cancel{-2}x)}$$

simplify

$$= \lim_{x\to 0} \frac{\overset{-16}{\cancel{-16\cos(4x)}} + \overset{\to 0}{\cancel{\frac{128x}{(1+16x^2)^2}}}}{\underset{-1}{\cancel{\frac{-1}{(1-x)^2}}} + \overset{\to 0}{\cancel{\frac{x}{(1-x^2)^{3/2}}}}} = \frac{-16}{-1} = \boxed{16}$$

19. $$\lim_{x\to 0} \frac{1-e^{-4x}-\arctan(4x)}{x^2} \quad 0/0 \quad \underset{L'H}{=} \lim_{x\to 0} \frac{+4e^{-4x} - \frac{4}{1+(4x)^2}}{2x} \quad 0/0$$

$-4(1+16x^2)^{-1}$

$$\underset{L'H}{=} \lim_{x\to 0} \frac{-16e^{-4x} + 4(1+16x^2)^{-2}(32x)}{2} \underset{\text{simplify}}{=} \lim_{x\to 0} \frac{\overset{-16}{\cancel{-16e^{-4x}}} + \overset{\to 0}{\cancel{\frac{128x}{(1+16x^2)^2}}}}{2} = \frac{-16}{2} = \boxed{-8}$$

20. $$\lim_{x\to 0^+} x^3 \cdot \ln x \quad 0\cdot(-\infty) \quad = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^3}} \quad -\infty/\infty \quad \underset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-3}{x^4}} \left(\frac{x^4}{-3}\right) = \lim_{x\to 0^+} \overset{\to 0}{\cancel{\frac{-x^3}{3}}} = \boxed{0}$$

flip x^{-3}

21. $$\lim_{x\to 0^+} x\ln\left(\frac{1}{x}\right) \quad 0\cdot\infty \quad = \lim_{x\to 0^+} \frac{\ln(1/x)}{1/x} \quad \infty/\infty \quad \underset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1/x}\cdot\cancel{(-1/x^2)}}{\cancel{-1/x^2}} = \lim_{x\to 0^+} x = \boxed{0}$$

Flip, $x \to 0$

22. $$\lim_{x\to\infty} x^{1/x^2} \quad \infty^0 \quad = e^{\lim_{x\to\infty} \ln\left[x^{1/x^2}\right]} = e^{\lim_{x\to\infty} \frac{1}{x^2}\ln x} = e^{\lim_{x\to\infty} \frac{\ln x}{x^2}} \quad \infty/\infty$$

$$\underset{L'H}{=} e^{\lim_{x\to\infty} \frac{1/x}{2x}} \cdot \frac{1}{2x} = e^{\lim_{x\to\infty} \overset{\to 0}{\cancel{\frac{1}{2x^2}}}} \underset{\to\infty}{} = e^0 = \boxed{1}$$

23. $\lim_{x\to 0^+} (1-2x)^{1/x} \quad 1^\infty$

$$= e^{\lim_{x\to 0^+} \ln\left[(1-2x)^{1/x}\right]} = e^{\lim_{x\to 0^+} \frac{1}{x}\ln(1-2x)}$$

$$= e^{\lim_{x\to 0^+} \frac{\ln(1-2x)}{x}} \ \ 0/0 \ \underset{L'H}{=} e^{\lim_{x\to 0^+} \frac{\frac{1}{1-2x}(-2)}{1}} = \boxed{e^{-2}}$$

24. $\lim_{x\to\infty} (x^3+1)^{\frac{1}{\ln x}} \quad \infty^0$

$$= e^{\lim_{x\to\infty} \ln\left[(x^3+1)^{1/\ln x}\right]} = e^{\lim_{x\to\infty} \frac{1}{\ln x}\cdot \ln(x^3+1)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln(x^3+1)}{\ln x}} \ \ \infty/\infty \ \underset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{x^3+1}(3x^2)}{1/x}} \cdot \frac{x}{1} = e^{\lim_{x\to\infty} \frac{3x^3}{x^3+1}} \ \ \infty/\infty$$

$$\underset{L'H}{=} e^{\lim_{x\to\infty} \frac{9\cancel{x^2}}{3\cancel{x^2}}} = \boxed{e^3}$$

25. $\lim_{x\to\infty} x^2 \sin\left(\frac{1}{x^2}\right) \quad \infty\cdot 0$ Flip

$$= \lim_{x\to\infty} \frac{\sin(1/x^2)}{1/x^2} \ \ 0/0 \ \underset{L'H}{=} \lim_{x\to\infty} \frac{\cos(1/x^2)\cancel{\left(\frac{-2}{x^3}\right)}}{\cancel{\frac{-2}{x^3}}} = \boxed{1}$$

$\cos 0 = 1$

26. $\lim_{x\to 0^+} \sqrt{x}\ln x \quad 0\cdot(-\infty)$ Flip

$$= \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}} \ \ -\infty/\infty \ \underset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^{3/2}}} \cdot (-2x^{3/2}) = \lim_{x\to 0^+} -2\sqrt{x} = \boxed{0}$$

$1/\sqrt{x} \to x^{-1/2}$

27. $\lim_{x\to\infty} \left(1-\frac{3}{x}\right)^{4x} \quad 1^\infty$

$$= e^{\lim_{x\to\infty} \ln\left[\left(1-\frac{3}{x}\right)^{4x}\right]} = e^{\lim_{x\to\infty} 4x\ln\left(1-\frac{3}{x}\right)}$$

easier to leave 4 here.

$\infty\cdot 0$ Flip

$$= e^{\lim_{x\to\infty} \frac{4\ln(1-3/x)}{1/x}} \ \ 0/0 \ = e^{\lim_{x\to\infty} \frac{4\left(\frac{1}{1-3/x}\right)\cdot\left(\frac{3}{\cancel{x^2}}\right)}{-1/\cancel{x^2}}} = \boxed{e^{-12}}$$

28. $\lim_{x\to\infty}\left(1+\ln\left(1-\frac{6}{x^2}\right)\right)^{x^2}$ 1^∞ $= e^{\lim_{x\to\infty}\ln\left[\left(1+\ln\left(1-\frac{6}{x^2}\right)\right)^{x^2}\right]}$

$$= e^{\lim_{x\to\infty} x^2 \ln\left(1+\ln\left(1-\frac{6}{x^2}\right)\right)} = e^{\lim_{x\to\infty}\frac{\ln\left(1+\ln\left(1-\frac{6}{x^2}\right)\right)}{\frac{1}{x^2}}}$$

$\infty \cdot 0$ Flip $\frac{0}{0}$

$$\underset{L'H}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{1+\ln\left(1-\frac{6}{x^2}\right)}\cdot\frac{1}{1-\frac{6}{x^2}}\left(\frac{12}{x^3}\right)}{-\frac{2}{x^3}}} = e^{\frac{12}{-2}} = \boxed{e^{-6}}$$

29. $\lim_{x\to\infty}\left(1-\arctan\left(\frac{5}{x^4}\right)\right)^{3x^4}$ 1^∞ $= e^{\lim_{x\to\infty}\ln\left[\left(1-\arctan\left(\frac{5}{x^4}\right)\right)^{3x^4}\right]}$

easier to leave 3 here

$$= e^{\lim_{x\to\infty} 3x^4\ln\left(1-\arctan\left(\frac{5}{x^4}\right)\right)} = e^{\lim_{x\to\infty}\frac{3\ln\left(1-\arctan\left(\frac{5}{x^4}\right)\right)}{\frac{1}{x^4}}}$$

$\infty \cdot 0$ Flip $\frac{0}{0}$

$$\underset{L'H}{=} e^{\lim_{x\to\infty}\frac{\frac{3}{1-\arctan\left(\frac{5}{x^4}\right)}\cdot\left(-\frac{1}{1+\left(\frac{5}{x^4}\right)^2}\right)\cdot\left(\frac{-20}{x^5}\right)}{-\frac{4}{x^5}}} = \boxed{e^{-15}}$$

CHALLENGE!

30. $\lim_{x\to\infty}\left(\sqrt{1-\frac{5}{x^2}}-\sin\left(\frac{1}{x^2}\right)\right)^{5x^2}$ 1^∞ $= e^{\lim_{x\to\infty}\ln\left[\left(\sqrt{1-\frac{5}{x^2}}-\sin\left(\frac{1}{x^2}\right)\right)^{5x^2}\right]}$

leave

$$= e^{\lim_{x\to\infty} 5x^2\ln\left(\sqrt{1-\frac{5}{x^2}}-\sin\left(\frac{1}{x^2}\right)\right)} = e^{\lim_{x\to\infty}\frac{5\ln\left(\sqrt{1-\frac{5}{x^2}}-\sin\left(\frac{1}{x^2}\right)\right)}{\frac{1}{x^2}}}$$

$\infty \cdot 0$ Flip $\frac{0}{0}$

$$\overset{L'H}{=} e^{\lim_{x\to\infty} \dfrac{5\left(\dfrac{1}{\sqrt{1-\frac{5}{x^2}} - \sin\left(\frac{1}{x^2}\right)}\right)\cdot\left[\dfrac{1}{2\sqrt{1-\frac{5}{x^2}}}\left(\dfrac{10}{x^3}\right) - \cos\left(\frac{1}{x^2}\right)\left(\dfrac{-2}{x^3}\right)\right]}{-2/x^3}}$$

$\frac{1}{2}$ $\quad$ -5 $\quad$ $\left(\frac{-x^3}{2}\right)$

$$= e^{5\left(\frac{-5}{2} - 1\right)} = \boxed{e^{\frac{-35}{2}}}$$

WOAH!

31. $$\lim_{x\to\infty}\left(e^{1/x^8} - \frac{8}{x^8}\right)^{x^8} = e^{\lim_{x\to\infty} \ln\left[\left(e^{1/x^8} - \frac{8}{x^8}\right)^{x^8}\right]}$$

1^∞

$\infty \cdot 0$

$$= e^{\lim_{x\to\infty} x^8 \cdot \ln\left(e^{1/x^8} - \frac{8}{x^8}\right)} = e^{\lim_{x\to\infty} \dfrac{\ln\left(e^{1/x^8} - \frac{8}{x^8}\right)}{1/x^8}}$$

Flip $\qquad$ $0/0$

$$\overset{L'H}{=} e^{\lim_{x\to\infty} \dfrac{\dfrac{1}{e^{1/x^8} - 8/x^8}\left[e^{1/x^8}\left(-\frac{8}{x^9}\right) + \frac{64}{x^9}\right]}{-8/x^9}}$$

-8 $\quad$ $\left(\frac{-x^9}{8}\right)$

$$= e^{\lim_{x\to\infty} \dfrac{1}{e^{1/x^8} - 8/x^8}\left[e^{1/x^8} - 8\right]} = e^{1-8} = \boxed{e^{-7}}$$

32. $$\lim_{x\to\infty}\left(\frac{x}{x+3}\right)^x = e^{\lim_{x\to\infty} \ln\left[\left(\frac{x}{x+3}\right)^x\right]} = e^{\lim_{x\to\infty} x \ln\left(\frac{x}{x+3}\right)}$$

1^∞ $\quad$ $\infty \cdot 0$ $\quad$ Flip

Quotient Rule in Chain Rule

$$= e^{\lim_{x\to\infty} \dfrac{\ln\left(\frac{x}{x+3}\right)}{1/x}} \overset{L'H}{=} e^{\lim_{x\to\infty} \dfrac{\dfrac{1}{\left(\frac{x}{x+3}\right)}\left[\dfrac{(x+3)(1) - x(1)}{(x+3)^2}\right]}{-1/x^2}}$$

$0/0$

$$= e^{\lim_{x\to\infty} \frac{\cancel{\frac{x+3}{x}}\left(\frac{3}{(x+3)^{\cancel{2}}}\right)}{-1/x^2}} \quad (-x^{\cancel{2}}) \quad = e^{\lim_{x\to\infty} \frac{-3x}{x+3}} \quad -\infty/\infty$$

$$\underset{L'H}{=} e^{\lim_{x\to\infty} \frac{-3}{1}} = \boxed{e^{-3}}$$

Integrals

33. $\int (e^x+x)^2\,dx = \int e^{2x}+2xe^x+x^2\,dx = \frac{e^{2x}}{2} + 2\int \overset{P\ E}{xe^x}\,dx + \frac{x^3}{3} + C$ IBP

IBP

$u=x$	$dv=e^x dx$
$du=dx$	$v=e^x$

$$= \frac{e^{2x}}{2} + 2\left[xe^x - \cancel{\int e^x\,dx}^{\,e^x}\right] + \frac{x^3}{3} + C$$

$$= \boxed{\frac{e^{2x}}{2} + 2xe^x - 2e^x + \frac{x^3}{3} + C}$$

34. $\int x\sin^2 x\,dx \quad = x\left(\frac{x}{2} - \frac{\sin(2x)}{4}\right) - \int \frac{x}{2} - \frac{\sin(2x)}{4}\,dx$

IBP

$u=x$	$dv=\sin^2 x\,dx$
$du=dx$	$v=\int \sin^2 x\,dx$ ← See Below
	$=\frac{x}{2}-\frac{\sin(2x)}{4}$

$$= \frac{x^2}{2} - \frac{x\sin(2x)}{4} - \frac{x^2}{4} + \frac{1}{4}\int \sin(2x)\,dx$$

↑ u-sub?!

$$= \underbrace{\frac{x^2}{2}} - \frac{x\sin(4x)}{4} \underbrace{-\frac{x^2}{4}} - \frac{1}{8}\cos(2x) + C$$

(*) $\int \underset{EVEN}{\sin^2 x}\,dx = \int \frac{1-\cos(2x)}{2}\,dx$ Half-Angle.

$$= \frac{1}{2}\int 1-\cos(2x)\,dx$$

$$\frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C$$

$$= \boxed{\frac{x^2}{4} - \frac{x\sin(4x)}{4} - \frac{1}{8}\cos(2x) + C}$$

Note: you could instead substitute for $\sin^2 x$ right away using Half-Angle and then IBP. Either is fine.

35. $\int \frac{1}{\sqrt{25-x^2}}\,dx = \boxed{\arcsin\left(\frac{x}{5}\right) + C}$ Quick a-rule

OR, prove the a-rule with Algebra + u-sub.

36. $\displaystyle\int \frac{1}{x^2+25}\,dx = \boxed{\frac{1}{5}\arctan\left(\frac{x}{5}\right)+C}$

Quick a-rule

OR, prove the a-rule with Algebra + u-sub

37. $\displaystyle\int \frac{1}{x\sqrt{9-(\ln x)^2}}\,dx = \int \frac{1}{\sqrt{9-w^2}}\,dw = \arcsin\left(\frac{w}{3}\right)+C = \boxed{\arcsin\left(\frac{\ln x}{3}\right)+C}$

$$\boxed{\begin{aligned} w &= \ln x \\ dw &= \frac{1}{x}\,dx \end{aligned}}$$

38. $\displaystyle\int x\arcsin x\,dx \overset{IBP}{=} \frac{x^2}{2}\arcsin x - \frac{1}{2}\overbrace{\int \frac{x^2}{\sqrt{1-x^2}}\,dx}^{\text{Trig. sub}}$

Don't Forget

IBP

$$\boxed{\begin{aligned} u &= \arcsin x & dv &= x\,dx \\ du &= \frac{1}{\sqrt{1-x^2}}\,dx & v &= \frac{x^2}{2} \end{aligned}}$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot \cancel{\cos\theta}\,d\theta$$

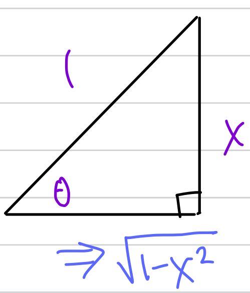
$\sqrt{1-\sin^2\theta} \to \sqrt{\cos^2\theta} \to \cancel{\cos\theta}$

cancel

Trig. Sub.

$$\boxed{\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\,d\theta \end{aligned}}$$

$\theta = \arcsin x$



$\Rightarrow \cos\theta = \frac{A}{H} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

Even Power

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta\,d\theta$$

Now Half-Angle Identity

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1-\cos(2\theta)\,d\theta$$

Now Double Angle Identity

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{\cancel{\sin(2\theta)}}{2}\right] + C$$

$2\sin\theta\cos\theta$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\underbrace{\cancel{\theta}}_{\arcsin x} + \frac{1}{4}\underbrace{\cancel{\sin\theta}}_{x}\,\underbrace{\cancel{\cos\theta}}_{\sqrt{1-x^2}} + C$$

$$= \boxed{\frac{x^2}{2}\arcsin x - \frac{\arcsin x}{4} + \frac{x\sqrt{1-x^2}}{4} + C}$$

39. $\displaystyle\int \frac{1}{(4-x^2)^{3/2}}\,dx = \int \frac{1}{\left(\sqrt{4-x^2}\right)^3}\,dx = \int \frac{1}{\left(\sqrt{4-(2\sin\theta)^2}\right)^3}\cdot 2\cos\theta\, d\theta$

don't drop

$4-4\sin^2\theta$

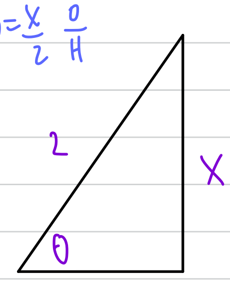
Trig. Sub.

$$\boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\, d\theta \end{aligned}}$$

$$= \int \frac{1}{\left(\sqrt{4(1-\sin^2\theta)}\right)^3}\cdot 2\cos\theta\, d\theta = \int \frac{1}{\left(\sqrt{4\cos^2\theta}\right)^3}\cdot 2\cos\theta\, d\theta$$

$4\cos^2\theta$ $2\cos\theta$

$\sin\theta = \frac{x}{2}$ $\frac{O}{H}$



$\Rightarrow \sqrt{4-x^2}$

$$= \int \frac{1}{2^3\cos^3\theta}\cdot 2\cos\theta\, d\theta = \frac{2}{8}\int \frac{\cos\theta}{\cos^3\theta}\, d\theta = \frac{1}{4}\int \frac{1}{\cos^2\theta}\, d\theta$$

$$\boxed{= \frac{1}{4}\int \sec^2\theta\, d\theta = \frac{1}{4}\tan\theta + C = \frac{1}{4}\left[\frac{x}{\sqrt{4-x^2}}\right] + C}$$

$\Rightarrow \tan\theta = \dfrac{x}{\sqrt{4-x^2}}$

40. $\displaystyle\int_1^e \ln x\, dx = \int_1^e \ln x \cdot 1\, dx = x\ln x\Big|_1^e - \int_1^e \cancel{x}\cdot \frac{1}{\cancel{x}}\, dx = x\ln x\Big|_1^e - x\Big|_1^e$

IBP

$$\boxed{\begin{aligned} u &= \ln x & dv &= 1\,dx \\ du &= \frac{1}{x}\,dx & v &= x \end{aligned}}$$

$$= e\ln e - \ln 1 - (e-1) = \cancel{e} - \cancel{e} + 1 = \boxed{1}$$

41. $\displaystyle\int \frac{\ln(2x^5)}{x^2}\,dx = \int \ln(2x^5)\, x^{-2}\, dx = -\frac{\ln(2x^5)}{x} + 5\int \frac{1}{x^2}\,dx$

x^{-2}

IBP

$$\boxed{\begin{aligned} u &= \ln(2x^5) & dv &= x^{-2} \\ du &= \frac{1}{2x^5}(10x^4) & v &= \frac{x^{-1}}{-1} \\ &= \frac{5}{x} & &= \frac{-1}{x} \end{aligned}}$$

$$= -\frac{\ln(2x^5)}{x} + 5\left(\frac{x^{-1}}{-1}\right) + C$$

$$= \boxed{-\frac{\ln(2x^5)}{x} - \frac{5}{x} + C}$$

42. $\displaystyle\int \cos^5 x\, dx = \int \cos^4 x \cdot \cos x\, dx = \int (\cos^2 x)^2 \cdot \cos x\, dx = \int (1-\sin^2 x)^2 \cdot \cos x\, dx$

ODD   ISOLATE   CONVERT   Finish with u-sub

$$\boxed{\begin{aligned} u &= \sin x \\ du &= \cos x\, dx \end{aligned}}$$

$$= \int (1-u^2)^2\, du = \int 1 - 2u^2 + u^4\, du = u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + C}$$

43. $\int \ln^2(x^{20})\,dx = \int (\ln(x^{20}))^2 \cdot 1\,dx = x\left[\ln(x^{20})\right]^2 - 40\int \ln(x^{20})\,dx$

IBP

$u = (\ln(x^{20}))^2 \quad dv = 1\,dx$

$du = 2(\ln(x^{20}))\frac{1}{x^{20}}\cdot 20x^{19}dx \quad v = x$

cancel

$$= x\left[\ln(x^{20})\right]^2 - 40\left[x\ln(x^{20}) - \int 20\,dx\right]$$

$$= \boxed{x\left[\ln(x^{20})\right]^2 - 40x\ln(x^{20}) + 800x + C}$$

IBP

$u = \ln(x^{20}) \quad dv = 1dx$

$du = \frac{1}{x^{20}}(20x^{19}) \quad v = x$

Note: OR you could use Log Algebra to Simplify First, then do a Double IBP.

OR $\int \left[\ln(x^{20})\right]^2 dx = \int \left[20\ln x\right]^2 dx = 400\int (\ln x)^2 dx = \ldots$

44. $\int \sin^5 \cos^2 x\,dx = \int \sin^4 x \cdot \cos^2 x \cdot \sin x\,dx = \int (\sin^2 x)^2 \cdot \cos^2 x \cdot \sin x dx$

ODD ISOLATE CONVERT

$$= \int (1-\cos^2 x)^2 \cdot \cos^2 x \cdot \sin x dx = -\int (1-u^2)^2 \cdot u^2\,du = -\int (1-2u^2+u^4)u^2\,du$$

FINISH U-SUB

$u = \cos x$
$du = -\sin x dx$
$-du = \sin x dx$

$$= -\int u^2 - 2u^4 + u^6\,du = -\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C$$

$$= \boxed{-\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C}$$

NOTE: Don't Mix Half-Angle Identity in ODD Power Case

45. $\int \sin^2 x \cos^3 x dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos x\,dx = \int \sin^2 x\,(1-\sin^2 x)\cdot \cos x dx$

leave ISOLATE CONVERT

$u = \sin x$
$du = \cos x dx$

$$= \int u^2(1-u^2)\,du = \int u^2 - u^4 du = \frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}$$

FINISH USUB

46. $\displaystyle\int_{e^{\sqrt{3}}}^{e^3} \frac{1}{x(9+(\ln x)^2)}\,dx = \int \frac{1}{9+u^2}\,du = \frac{1}{3}\arctan\left(\frac{u}{3}\right)\Big|_{\sqrt{3}}^{3}$

$\boxed{\begin{array}{l} u=\ln x \\ du=\frac{1}{x}dx \end{array}}$ $\boxed{\begin{array}{l} x=e^{\sqrt{3}} \Rightarrow u=\cancel{\ln e}^{\sqrt{3}}=\sqrt{3} \\ x=e^3 \Rightarrow u=\cancel{\ln e}^{3}=3 \end{array}}$

$= \frac{1}{3}\left[\underbrace{\arctan\left(\cancel{\tfrac{3}{3}}^{1}\right)}_{\pi/4} - \underbrace{\arctan\left(\cancel{\tfrac{\sqrt{3}}{3}}^{\frac{1}{\sqrt{3}}}\right)}_{\pi/6}\right]$

$= \frac{1}{3}\left[\frac{\pi}{4}-\frac{\pi}{6}\right] = \frac{1}{3}\left[\frac{3\pi}{12}-\frac{2\pi}{12}\right] = \frac{1}{3}\left(\frac{\pi}{12}\right) = \boxed{\frac{\pi}{36}}$

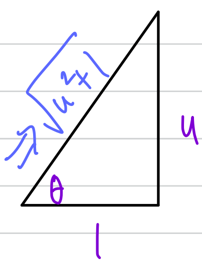
47. $\displaystyle\int_1^e \frac{1}{x(1+(\ln x)^2)^{3/2}}\,dx = \int_{x=1}^{x=e} \frac{1}{(1+u^2)^{3/2}}\,du = \int_{x=1}^{x=e} \frac{1}{(\sqrt{1+u^2})^3}\,du = \int_{x=1}^{x=e} \frac{1}{(\underbrace{\sqrt{1+\tan^2\theta}}_{\sqrt{\sec^2\theta}\ =\ \sec\theta})^3}\cdot \sec^2\theta\,d\theta$

$\boxed{\begin{array}{l} u=\ln x \\ du=\frac{1}{x}dx \end{array}}$

or change limits?

Trig Sub

$\boxed{\begin{array}{l} u=\tan\theta \\ du=\sec^2\theta\,d\theta \end{array}}$



$= \displaystyle\int_{x=1}^{x=e} \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \int_{x=1}^{x=e} \frac{1}{\sec\theta}\,d\theta = \int_{x=1}^{x=e} \cos\theta\,d\theta = \sin\theta\Big|_{x=1}^{x=e}$

$= \displaystyle\frac{u}{\sqrt{u^2+1}}\Big|_{x=1}^{x=e} = \frac{\ln x}{\sqrt{1+(\ln x)^2}}\Big|_1^e = \frac{\cancel{\ln e}^{1}}{\sqrt{1+(\cancel{\ln e}^{1})^2}} - \frac{\cancel{\ln 1}^{0}}{\sqrt{1+(\ln 1)^2}}$

$= \displaystyle\frac{1}{\sqrt{1+1}} = \boxed{\frac{1}{\sqrt{2}}} = \boxed{\frac{\sqrt{2}}{2}}$

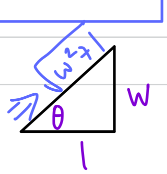
48. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(1+\sin^2 x)^{7/2}}\,dx = \int_0^1 \frac{1}{(\sqrt{1+w^2})^7}\,dw = \int_{w=0}^{w=1} \frac{1}{(\underbrace{\sqrt{1+\tan^2\theta}}_{\sqrt{\sec^2\theta}\ =\ \sec\theta})^{7}}\cdot\sec^2\theta\,d\theta$

don't drop

$\boxed{\begin{array}{l} w=\sin x \\ dw=\cos x\,dx \end{array}}$ $\boxed{\begin{array}{l} x=0 \Rightarrow w=\sin 0=0 \\ x=\pi/2 \Rightarrow w=\sin\pi/2=1 \end{array}}$

Trig Sub

$\boxed{\begin{array}{l} w=\tan\theta \\ dw=\sec^2\theta\,d\theta \end{array}}$



$= \displaystyle\int_{w=0}^{w=1} \frac{\sec^2\theta}{\sec^7\theta}\,d\theta = \int_{w=0}^{w=1} \frac{1}{\sec^5\theta}\,d\theta = \int_{w=0}^{w=1} \cos^5\theta\,d\theta$

$= \displaystyle\int_{w=0}^{w=1} \cos^4\theta\cdot\cos\theta\,d\theta = \int_{w=0}^{w=1} (\cos^2\theta)^2\cos\theta\,d\theta = \int_{w=0}^{w=1} (1-\sin^2\theta)^2\cos\theta\,d\theta$

another u-sub

$$\boxed{\begin{aligned} u &= \sin\theta \\ du &= \cos\theta\, d\theta \end{aligned}}$$

$$= \int_{w=0}^{w=1} (1-u^2)^2\,du = \int_{w=0}^{w=1} 1-2u^2+u^4\,du = u - \frac{2u^3}{3} + \frac{u^5}{5}\Bigg|_{w=0}^{w=1}$$

$$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{\sin^5\theta}{5}\Bigg|_{w=0}^{w=1} = \frac{w}{\sqrt{w^2+1}} - \frac{2}{3}\left[\frac{w}{\sqrt{w^2+1}}\right]^3 + \frac{1}{5}\left[\frac{w}{\sqrt{w^2+1}}\right]^5\Bigg|_0^1$$

$$= \frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5 = \frac{1}{\sqrt{2}} - \frac{\cancel{2}}{3}\left(\frac{1}{\cancel{2}\sqrt{2}}\right) + \frac{1}{5}\left(\frac{1}{4\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}} = \frac{60}{60\sqrt{2}} - \frac{20}{60\sqrt{2}} + \frac{3}{60\sqrt{2}} = \boxed{\frac{43}{60\sqrt{2}}}$$

wow!

49. $$\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \arcsin\left(\frac{x}{2}\right)\Bigg|_1^{\sqrt{3}} = \underbrace{\arcsin\left(\frac{\sqrt{3}}{2}\right)}_{\pi/3} - \underbrace{\arcsin\left(\frac{1}{2}\right)}_{\pi/6} = \pi/3 - \pi/6 = \boxed{\pi/6}$$

50. $$\int \frac{1}{(x^2+4)^2}\,dx = \int \frac{1}{(\underbrace{4\tan^2\theta+4}_{\substack{4(\tan^2\theta+1)\\4\sec^2\theta}})^2}\cdot 2\sec^2\theta\, d\theta = \int \frac{1}{\underbrace{(4\sec^2\theta)^2}_{16\sec^4\theta}}\cdot 2\sec^2\theta\,d\theta = \frac{2}{16}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

Trig. Sub.

$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\,d\theta \end{aligned}}$$

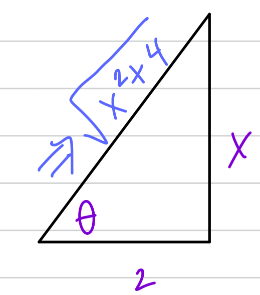
$$= \frac{1}{8}\int \frac{1}{\sec^2\theta}\,d\theta = \frac{1}{8}\int \underbrace{\cos^2\theta\,d\theta}_{\text{Half-Angle}} = \frac{1}{8}\int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$\tan\theta = \frac{x}{2}$

$\Rightarrow \theta = \arctan\left(\frac{x}{2}\right)$

$$= \frac{1}{16}\int 1+\cos(2\theta)\,d\theta = \frac{1}{16}\left[\theta + \frac{\overbrace{\sin(2\theta)}^{\cancel{2}\sin\theta\cos\theta}}{\cancel{2}}\right] + C = \frac{1}{16}\left[\theta + \sin\theta\cdot\cos\theta\right] + C$$

Double Angle



$$= \frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}}\right] + C$$

$\Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}}$

$$= \boxed{\frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right] + C}$$

$\Rightarrow \cos\theta = \frac{2}{\sqrt{x^2+4}}$

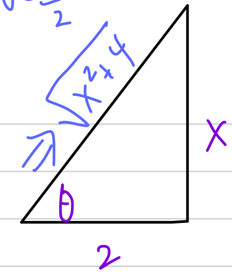
51. $$\int \frac{1}{(x^2+4)^{7/2}}\,dx = \int \frac{1}{(\sqrt{x^2+4})^7}\,dx = \int \frac{1}{(\sqrt{\underbrace{4\tan^2\theta+4}_{\substack{4(\tan^2\theta+1)\\4\sec^2\theta}}})^7}\cdot 2\sec^2\theta\,d\theta$$

don't drop

Trig. Sub

$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\,d\theta \end{aligned}}$$

$\tan\theta = \frac{x}{2}$



$\Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}}$

$$= \int \frac{1}{\left(\sqrt{4\sec^2\theta}\right)^7} \cdot 2\sec^2\theta\, d\theta = \frac{2}{2^7} \int \frac{\sec^2\theta}{\sec^7\theta}\, d\theta = \frac{1}{2^6} \int \frac{1}{\sec^5\theta}\, d\theta$$

$(2\sec\theta)^7$

$$= \frac{1}{64} \int \cos^5\theta\, d\theta = \frac{1}{64} \int \cos^4\theta \cdot \cos\theta\, d\theta = \frac{1}{64} \int (\cos^2\theta)^2 \cos\theta\, d\theta$$

ODD. ISOLATE CONVERT

$\boxed{\begin{array}{l} w = \sin\theta \\ dw = \cos\theta\, d\theta \end{array}}$

$$= \frac{1}{64} \int (1-\sin^2\theta)^2 \cos\theta\, d\theta = \frac{1}{64} \int (1-w^2)^2\, dw = \frac{1}{64} \int 1 - 2w^2 + w^4\, dw$$

FINISH U-SUB

$$= \frac{1}{64}\left[w - \frac{2w^3}{3} + \frac{w^5}{5}\right] + C = \frac{1}{64}\left[\sin\theta - \frac{2\sin^3\theta}{3} + \frac{\sin^5\theta}{5}\right] + C$$

$$= \boxed{\frac{1}{64}\left[\frac{x}{\sqrt{x^2+4}} - \frac{2}{3}\left(\frac{x}{\sqrt{x^2+4}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{x^2+4}}\right)^5\right] + C}$$

or $= \frac{x^3}{(x^2+4)^{3/2}}$ or $= \frac{x^5}{(x^2+4)^{5/2}}$

52. $$\int x^4 \arcsin x\, dx = \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{x^5}{\sqrt{1-x^2}}\, dx$$

don't drop.

IBP

$\boxed{\begin{array}{ll} u = \arcsin x & dv = x^4\, dx \\ du = \frac{1}{\sqrt{1-x^2}}\, dx & v = \frac{x^5}{5} \end{array}}$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{\sin^5\theta}{\sqrt{1-\sin^2\theta}}\, \cancel{\cos\theta}\, d\theta$$

$\sqrt{\cos^2\theta}$

$\cancel{\cos\theta}$

Trig. Sub.

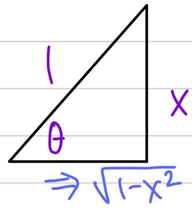
$\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{array}}$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \sin^5\theta\, d\theta$$

ODD Power

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \sin^4\theta \cdot \sin\theta\, d\theta$$

$(\sin^2\theta)^2$



$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int (1-\cos^2\theta)^2 \cdot \sin\theta\, d\theta$$

$\boxed{\begin{array}{l} w = \cos\theta \\ dw = -\sin\theta\, d\theta \\ -dw = \sin\theta\, d\theta \end{array}}$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\int (1-w^2)^2\, dw$$

$1 - 2w^2 + w^4$

52 (continued)

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[w - \frac{2w^3}{3} + \frac{w^5}{5}\right] + C$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\cos\theta - \frac{2}{3}\cos^3\theta + \frac{\cos^5\theta}{5}\right] + C$$

$$= \boxed{\frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\sqrt{1-x^2} - \frac{2}{3}\left(\sqrt{1-x^2}\right)^3 + \frac{1}{5}\left(\sqrt{1-x^2}\right)^5\right] + C}$$

53 $$\int x\arctan x\,dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2 +1-1}{1+x^2}\,dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx$$

slip-in|slip-out

split-split

$$\boxed{\begin{array}{ll} u = \arctan x & dv = x\,dx \\ du = \frac{1}{1+x^2}dx & v = \frac{x^2}{2} \end{array}}$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \cancel{\frac{x^2+1}{x^2+1}}^{\,1} - \frac{1}{1+x^2}\,dx$$

$$= \boxed{\frac{x^2}{2}\arctan x - \frac{1}{2}\left(x - \arctan x\right) + C}$$

distribute

54. $$\int_0^1 x\tan^{-1}(x^2)\,dx = \frac{1}{2}\int_0^1 \tan^{-1}w \cdot 1\,dw = \frac{1}{2}\left[w\tan^{-1}w\Big|_0^1 - \int_0^1 \frac{w}{1+w^2}\,dw\right]$$

$$\boxed{\begin{array}{l} w = x^2 \\ dw = 2x\,dx \\ \frac{1}{2}dw = x\,dx \end{array}}
\quad
\boxed{\begin{array}{l} x=0 \Rightarrow w=0 \\ x=1 \Rightarrow w=1 \end{array}}$$

Same by chance.

IBP

$$\boxed{\begin{array}{ll} u = \tan^{-1}w & dv = 1\,dw \\ dw = \frac{1}{1+w^2}dw & v = w \end{array}}$$

$$\boxed{\begin{array}{l} p = 1+w^2 \\ dp = 2w\,dw \\ \frac{1}{2}dp = w\,dw \end{array}}
\quad
\boxed{\begin{array}{l} w=0 \Rightarrow p=1 \\ w=1 \Rightarrow p=2 \end{array}}$$

$$= \frac{1}{2}\left[w\tan^{-1}w\Big|_0^1 - \frac{1}{2}\int_1^2 \frac{1}{p}\,dp\right]$$

$$= \frac{1}{2}\left[w\tan^{-1}w\Big|_0^1 - \frac{1}{2}\ln|p|\Big|_1^2\right]$$

$$= \frac{1}{2}\left[\cancelto{\pi/4}{\tan^{-1}1} - \cancelto{0}{\tan^{-1}0} - \frac{1}{2}\left(\ln 2 - \cancelto{0}{\ln 1}\right)\right]$$

$$= \boxed{\frac{\pi}{8} - \frac{\ln 2}{4}}$$

Note: OR, just go straight for IBP right away $u = \arctan(x^2)$ $dv = x\,dx$ ⋮

55. $\displaystyle\int \frac{x^2}{x^6+1}\,dx = \int \frac{x^2}{(x^3)^2+1}\,dx = \frac{1}{3}\int \frac{1}{w^2+1}\,dw = \frac{1}{3}\arctan w + C$

$= \boxed{\frac{1}{3}\arctan(x^3)+C}$

$$\boxed{\begin{aligned} w &= x^3 \\ dw &= 3x^2\,dx \\ \tfrac{1}{3}\,dw &= x^2\,dx \end{aligned}}$$

56. $\displaystyle\int_1^{e^2} x\ln\sqrt{x}\,dx = \frac{x^2}{2}\ln\sqrt{x}\Big|_1^{e^2} - \frac{1}{4}\int_1^{e^2}\frac{x^{\cancel{2}}}{\cancel{x}}\,dx = \frac{x^2}{2}\ln\sqrt{x}\Big|_1^{e^2} - \frac{x^2}{8}\Big|_1^{e^2}$

IBP

$$\boxed{\begin{aligned} u &= \ln\sqrt{x} \quad dv = x\,dx \\ du &= \frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\,dx \quad v = \frac{x^2}{2} \\ &= \frac{1}{2x}\,dx \end{aligned}}$$

$= \frac{e^4}{2}\cancel{\ln\sqrt{e^2}} - \frac{1}{2}\cancel{\ln\sqrt{1}} - \frac{1}{8}\left(e^4-1\right)$ ($\ln\sqrt{e^2}\to \ln e \to 1$; $\ln\sqrt{1} \to \ln 1 \to 0$)

$= \frac{e^4}{2} - \frac{e^4}{8} + \frac{1}{8} = \frac{3e^4}{8} + \frac{1}{8} = \boxed{\frac{3e^4+1}{8}}$

Note: you can also simplify first using Log Algebra, then use IBP.

OR// $\displaystyle\int_1^{e^2} x\ln\sqrt{x}\,dx = \int_1^{e^2} x\ln[x^{1/2}]\,dx = \frac{1}{2}\int_1^{e^2} x\ln x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\ln x\Big|_1^{e^2} - \frac{1}{2}\int_1^{e^2} x\,dx\right]$

IBP

$$\boxed{\begin{aligned} u &= \ln x \quad dv = x\,dx \\ du &= \tfrac{1}{x}\,dx \quad v = \frac{x^2}{2} \end{aligned}}$$

$= \frac{1}{2}\left[\frac{x^2}{2}\ln x\Big|_1^{e^2} - \frac{x^2}{4}\Big|_1^{e^2}\right] = \frac{x^2}{4}\ln x\Big|_1^{e^2} - \frac{x^2}{8}\Big|_1^{e^2}$

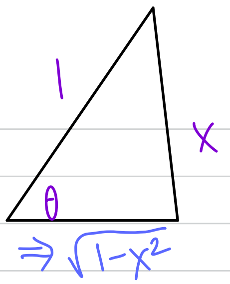
$= \frac{e^4}{4}\left(\cancel{\ln e^2} - \cancel{\ln 1}\right) - \frac{1}{8}\left(e^4 - 1\right) = \frac{e^4}{2} - \frac{e^4}{8} + \frac{1}{8} = \boxed{\frac{3e^4+1}{8}}$ Match

($\ln e^2 \to 2$; $\ln 1 \to 0$)

57. $\displaystyle\int \frac{x^2}{(1-x^2)^{3/2}}\,dx = \int \frac{x^2}{\left(\sqrt{1-x^2}\right)^3}\,dx = \int \frac{\sin^2\theta}{\left(\sqrt{1-\sin^2\theta}\right)^3}\cdot\cos\theta\,d\theta = \int \frac{\sin^2\theta}{\cos^{\cancel{3}}\theta}\,\cancel{\cos\theta}\,d\theta$

$\sqrt{\cos^2\theta}$
$(\cos\theta)^3$

$$\boxed{\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\,d\theta \end{aligned}}$$ $\Rightarrow \theta = \arcsin x$



$$= \int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta = \int \tan^2\theta\, d\theta = \int \sec^2\theta - 1\, d\theta = \tan\theta - \theta + C$$

Identity

$$= \boxed{\frac{x}{\sqrt{1-x^2}} - \arcsin x + C}$$

$\Rightarrow \tan\theta = \frac{x}{\sqrt{1-x^2}}$

58. $$\int_1^e (\ln x)^{\cancel{2}\,1}\, dx = x(\ln x)^2\Big|_1^e - 2\int_1^e \cancel{\ln x}^{\,1}\, dx = x(\ln x)^2\Big|_1^e - 2\left[x\ln x\Big|_1^e - \int_1^e \cancel{1}^{\,x} dx\right]$$

warning: $(\ln x)^2 \neq 2\ln x$ No!

$$= x(\ln x)^2\Big|_1^e - 2x\ln x\Big|_1^e + 2x\Big|_1^e$$

or pull all limits off...

IBP

$$\boxed{\begin{array}{ll} u = (\ln x)^2 & dv = 1dx \\ du = 2\ln x \cdot \frac{1}{\cancel{x}}\,dx & v = \cancel{x} \end{array}}$$

$$= e(\cancel{\ln e}^{\,1})^2 - (\cancel{\ln 1}^{\,0})^2 - 2(e\cancel{\ln e}^{\,1} - \cancel{\ln 1}^{\,0}) + 2(e-1)$$

$$= e - \cancel{2e} + \cancel{2e} - 2 = \boxed{e - 2}$$

IBP

$$\boxed{\begin{array}{ll} u = \ln x & dv = 1dx \\ du = \frac{1}{\cancel{x}}\,dx & v = \cancel{x} \end{array}}$$

Note: you can also split $\int_1^e \ln x \cdot \ln x\, dx$ but still double IBP.

59. $$\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} + \frac{1}{x^2+9}\,dx = \arcsin\left(\frac{x}{2}\right) + \frac{1}{3}\arctan\left(\frac{x}{3}\right)\Big|_0^{\sqrt{3}}$$

a-rules

$$= \cancel{\arcsin\frac{\sqrt{3}}{2}}^{\,\pi/3} - \cancel{\arcsin 0}^{\,0} + \frac{1}{3}\left[\cancel{\arctan\left(\frac{\sqrt{3}}{3}\right)}^{\,1/\sqrt{3}}_{\,\pi/6} - \cancel{\arctan 0}^{\,0}\right]$$

$$= \frac{\pi}{3} + \frac{\pi}{18} = \frac{6\pi}{18} + \frac{\pi}{18} = \boxed{\frac{7\pi}{18}}$$

60. $$\int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int \frac{16\sin^2\theta}{\sqrt{16-16\sin^2\theta}} \cdot \cancel{4\cos\theta}\, d\theta = 16\int \sin^2\theta\, d\theta = 16\int \frac{1-\cos(2\theta)}{2}\, d\theta$$

Half-Angle

EVEN

$\sqrt{16(1-\sin^2\theta)}$

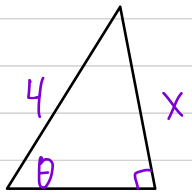
$\sqrt{16\cos^2\theta}$

$\cancel{4\cos\theta}$

Trig-Sub.

$$\boxed{\begin{array}{l} x = 4\sin\theta \\ dx = 4\cos\theta\, d\theta \end{array}}$$

$\sin\theta = \frac{x}{4} \Rightarrow \theta = \arcsin\left(\frac{x}{4}\right)$



$\Rightarrow \sqrt{16-x^2}$

$\Rightarrow \cos\theta = \frac{\sqrt{16-x^2}}{4}$

$$= 8\int 1-\cos(2\theta)\,d\theta = 8\left[\theta - \frac{\overset{\cancel{2}\sin\theta\cos\theta}{\sin(2\theta)}}{\cancel{2}}\right]+C$$

$$\boxed{= 8\left[\arcsin\left(\frac{x}{4}\right) - \left(\frac{x}{4}\right)\left(\frac{\sqrt{16-x^2}}{4}\right)\right]+C}$$

OR $$= 8\arcsin\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C$$

61. $$\int x^3\sqrt{9-x^2}\,dx = \int 27\sin^3\theta\cdot\sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,d\theta = 243\int \sin^3\theta\cos^2\theta\,d\theta$$

Trig.Sub

$$\boxed{\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta\,d\theta \end{aligned}}$$

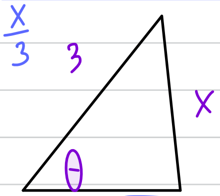
$\sqrt{9(1-\sin^2\theta)}$

$\sqrt{9\cos^2\theta}$

$3\cos\theta$

ODD POWER.

$\sin\theta = \frac{x}{3}$



$\Rightarrow \sqrt{9-x^2}$

$\Rightarrow \cos\theta = \frac{\sqrt{9-x^2}}{3}$

or 3^5

$$= 243\int \underbrace{\sin^2\theta}_{\text{CONVERT}}\cos^2\theta\cdot\sin\theta\,d\theta = 243\int(1-\cos^2\theta)\cdot\cos^2\theta\cdot\sin\theta\,d\theta$$

$$= -243\int(1-w^2)\,w^2\,dw = -243\int w^2 - w^4\,dw = -243\left[\frac{w^3}{3} - \frac{w^5}{5}\right]+C$$

$$= -243\left[\frac{\cos^3\theta}{3} - \frac{\cos^5\theta}{5}\right]+C = -243\left[\frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3 - \frac{1}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5\right]+C$$

$$\boxed{\begin{aligned} w &= \cos\theta \\ dw &= -\sin\theta\,d\theta \\ -dw &= \sin\theta\,d\theta \end{aligned}}$$

$$= \boxed{-3\left(9-x^2\right)^{3/2} + \frac{1}{5}\left(9-x^2\right)^{5/2} + C}$$

note:
$243 = 3^5$

Note: OR Inverted Substitution also works
OR IBP works

62. $$\int\frac{x^2}{x^2+3}\,dx = \int\frac{x^2+3-3}{x^2+3}\,dx = \int\frac{\cancel{x^2+3}}{\cancel{x^2+3}} - \frac{3}{x^2+3}\,dx$$

slip-in | slip-out   split | split

$$= x - \frac{3}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)+C$$

$$= \boxed{x - \sqrt{3}\arctan\left(\frac{x}{\sqrt{3}}\right)+C}$$

Note: Trig. Sub also works

63. $\int_{-3}^{3} \sqrt{9-x^2}\,dx = \int_{x=-3}^{x=3} \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta\,d\theta = 9\int_{x=-3}^{x=3} \cos^2\theta\,d\theta$

Trig.Sub.

$\sqrt{9(1-\sin^2\theta)}$

$\sqrt{9\cos^2\theta}$

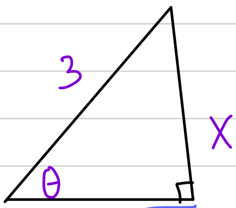
$3\cos\theta$

Half-Angle

$\boxed{\begin{array}{l} x = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{array}}$

$\sin\theta = \frac{x}{3} \Rightarrow \theta = \arcsin\left(\frac{x}{3}\right)$

$= 9\int_{x=-3}^{x=3} \frac{1+\cos(2\theta)}{2}\,d\theta = \frac{9}{2}\int_{x=-3}^{x=3} 1+\cos(2\theta)\,d\theta$



$\Rightarrow \sqrt{9-x^2}$

$2\sin\theta\cos\theta$

$= \frac{9}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]\Big|_{x=-3}^{x=3} = \frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)\frac{\sqrt{9-x^2}}{3}\right]\Big|_{-3}^{3}$

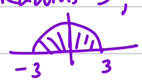
$\Rightarrow \cos\theta = \frac{\sqrt{9-x^2}}{3}$

$= \frac{9}{2}\left[\arcsin\left(\frac{3}{3}\right) + \frac{3\sqrt{9-9}}{9} - \left(\arcsin\left(\frac{-3}{3}\right) - \left(\frac{-3}{3}\right)\frac{\sqrt{9-9}}{3}\right)\right]$

$\pi/2$, 1, 0, $-\pi/2$, -1, 0

$= \frac{9}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \boxed{\frac{9\pi}{2}}$

Makes Sense: Area of half circle of Radius 3, $x^2+y^2=9$



$\pi r^2 \to 9\pi$ Half $\to \frac{9\pi}{2}$

64. $\int_1^e \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x\Big|_1^e - \frac{2}{3}\int_1^e \frac{x^{3/2}}{x}\,dx = \frac{2}{3}x^{3/2}\ln x\Big|_1^e - \frac{4}{9}x^{3/2}\Big|_1^e$

$x^{1/2}$

IBP

$\boxed{\begin{array}{ll} u = \ln x & dv = x^{1/2}dx \\ du = \frac{1}{x}dx & v = \frac{2}{3}x^{3/2} \end{array}}$

$= \frac{2}{3}\left[e^{3/2}\cdot\ln e - \ln 1\right] - \frac{4}{9}\left[e^{3/2} - 1\right]$

$= \frac{2}{3}e^{3/2} - \frac{4}{9}e^{3/2} + \frac{4}{9} = \boxed{\frac{2}{9}e^{3/2} + \frac{4}{9}}$

$\frac{6}{9}e^{3/2}$ ✓

65. $\int \frac{x+3}{\sqrt{4-x^2}}\,dx = \int \frac{x}{\sqrt{4-x^2}} + 3\int\frac{1}{\sqrt{4-x^2}}\,dx$

split/split u-sub. a-rule.

$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du + 3\arcsin\left(\frac{x}{2}\right) + C$

$u^{-1/2}$

$\boxed{\begin{array}{l} u = 4-x^2 \\ du = -2x\,dx \\ -\frac{1}{2}du = x\,dx \end{array}}$

$= -\frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) + 3\arcsin\left(\frac{x}{2}\right) + C$

$$\boxed{= -\sqrt{4-x^2} + 3\arcsin\left(\frac{x}{2}\right) + C}$$

OR Trig. Sub.

$$\int \frac{x+3}{\sqrt{4-x^2}}\,dx = \int \frac{2\sin\theta+3}{\cancel{\sqrt{4-4\sin^2\theta}}}\,\cancel{2\cos\theta}\,d\theta = \int 2\sin\theta+3\,d\theta = -2\cos\theta+3\theta+C$$

$\sqrt{4(1-\sin^2\theta)}$

$\sqrt{4\cos^2\theta}$

$\cancel{2\cos\theta}$

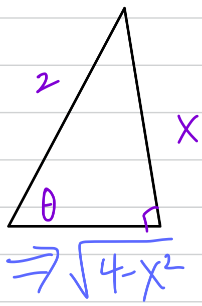
Trig. Sub

$$\boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta \end{aligned}}$$

$\sin\theta = \frac{x}{2} \Rightarrow \theta = \arcsin\left(\frac{x}{2}\right)$

$$= -\cancel{2}\left(\frac{\sqrt{4-x^2}}{\cancel{2}}\right) + 3\arcsin\left(\frac{x}{2}\right) + C$$

$$\boxed{= -\sqrt{4-x^2} + 3\arcsin\left(\frac{x}{2}\right) + C}$$ Match!



CHALLENGE

67. $$\int (\arcsin x)^2\,dx = x(\arcsin x)^2 - 2\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx$$

1

IBP

$$\boxed{\begin{aligned} u &= (\arcsin x)^2 & dv &= 1\,dx \\ du &= 2\arcsin x\cdot\frac{1}{\sqrt{1-x^2}}\,dx & v &= x \end{aligned}}$$

$$= x(\arcsin x)^2 - 2\int (\sin w)\cdot w\,dw$$

T P

NEW IBP

$$\boxed{\begin{aligned} w &= \arcsin x \overset{\text{invert}}{\Rightarrow} x = \sin w \\ dw &= \frac{1}{\sqrt{1-x^2}}\,dx \end{aligned}}$$

$$= x(\arcsin x)^2 - 2\left[-w\cos w + \int \cos w\,dw\right]$$

$\sin w$

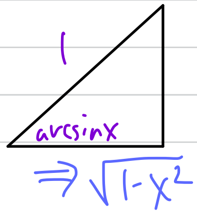
$$= x(\arcsin x)^2 + 2w\cos w - 2\sin w + C$$

$$= x(\arcsin x)^2 + 2(\arcsin x)\cos(\arcsin x) - 2\sin(\arcsin x) + C$$

$\sqrt{1-x^2}$ x

$$\boxed{\begin{aligned} u &= w & dw &= \sin w\,dw \\ du &= dw & w &= -\cos w \end{aligned}}$$

$$\boxed{= x(\arcsin x)^2 + 2\arcsin x\sqrt{1-x^2} - 2x + C}$$



$\Rightarrow \cos(\arcsin x) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$