# Math 121 Final Exam Spring 2021

1. $$\lim_{x\to\infty}\left[1-\arcsin\left(\frac{3}{x^4}\right)-\sin\left(\frac{1}{x^4}\right)\right]^{x^4} \qquad 1^{\infty}$$

$$= e^{\lim_{x\to\infty}\ln\left[\left(1-\arcsin\left(\frac{3}{x^4}\right)-\sin\left(\frac{1}{x^4}\right)\right)^{x^4}\right]}$$

$\infty \cdot 0$

$$= e^{\lim_{x\to\infty} x^4 \ln\left(1-\arcsin\left(\frac{3}{x^4}\right)-\sin\left(\frac{1}{x^4}\right)\right)}$$

$$= e^{\lim_{x\to\infty}\frac{\ln\left(1-\arcsin\left(\frac{3}{x^4}\right)-\sin\left(\frac{1}{x^4}\right)\right)}{\frac{1}{x^4}}} \qquad 0/0$$

$x^{-4}$

$$\underset{\text{L'H}}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{1-\arcsin(3/x^4)-\sin(1/x^4)}\left[\frac{-1}{\sqrt{1-\left(\frac{3}{x^4}\right)^2}}\left(\frac{-12}{x^5}\right)-\cos\left(\frac{1}{x^4}\right)\left(\frac{-4}{x^5}\right)\right]}{-4/x^5}}$$

$1 \quad 0 \quad 3x^{-4} \quad 0 \quad -1 \quad 1 \quad 3 \quad 0 \quad -1 \quad 0 \quad \frac{x^5}{4}$

$$= e^{1(-3-1)} = e^{-4}$$

2(a) $$\int_0^{\sqrt{3}}\frac{1}{(1+x^2)^{7/2}}\,dx = \int_0^{\sqrt{3}}\frac{1}{\left(\sqrt{1+x^2}\right)^7}\,dx = \int_{x=0}^{x=\sqrt{3}}\frac{1}{\left(\sqrt{1+\tan^2\theta}\right)^7}\cdot \cancel{\sec^2\theta}\,d\theta$$

$\theta = \pi/3$ or $\theta = 0$

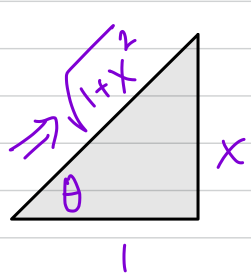
$\sqrt{\sec^2\theta}$

$(\cancel{\sec\theta})^7$

$\sec^5\theta$

Trig. Sub

$\boxed{x=\tan\theta \qquad dx=\sec^2\theta\,d\theta}$




$$= \int_{x=0}^{x=\sqrt{3}}\frac{1}{\sec^5\theta}\,d\theta = \int_{x=0}^{x=\sqrt{3}}\cos^5\theta\,d\theta = \int_{x=0}^{x=\sqrt{3}}\cos^4\theta\cdot\cos\theta\,d\theta$$

ODD Power $\qquad (\cos^2\theta)^2$

$\boxed{u=\sin\theta \qquad du=\cos\theta\,d\theta}$

$$= \int_{x=0}^{x=\sqrt{3}}(1-\sin^2\theta)^2\cdot\cos\theta\,d\theta = \int_{x=0}^{x=\sqrt{3}}(1-u^2)^2\,du = \int_{x=0}^{x=\sqrt{3}}1-2u^2+u^4\,du$$

$u \qquad du$

$$= u - \frac{2}{3}u^3 + \frac{u^5}{5}\Bigg|_{x=0}^{x=\sqrt{3}} = \sin\theta - \frac{2}{3}\sin^3\theta + \frac{\sin^5\theta}{5}\Bigg|_{x=0}^{x=\sqrt{3}}$$

$$= \frac{x}{\sqrt{1+x^2}} - \frac{2}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{1+x^2}}\right)^5\Bigg|_0^{\sqrt{3}}$$

$$= \left(\frac{\sqrt{3}}{\underbrace{\sqrt{4}}_{2}} - \frac{2}{3}\left(\frac{\sqrt{3}}{\underbrace{\sqrt{4}}_{2}}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{3}}{\underbrace{\sqrt{4}}_{2}}\right)^5\right) - \underbrace{(0 - 0 + 0)}_{0}$$

$$= \frac{\sqrt{3}}{2} - \cancel{\frac{2}{3}} \cdot \frac{\cancel{3}\sqrt{3}}{\cancel{8}\,4} + \frac{1}{5} \cdot \frac{9\sqrt{3}}{32} = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4} + \frac{9\sqrt{3}}{160}$$

$$= \frac{80\sqrt{3}}{160} - \frac{40\sqrt{3}}{160} + \frac{9\sqrt{3}}{160} = \frac{49\sqrt{3}}{160}$$

Match!

2(b) $$\int x^4 \arcsin x \, dx = \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{x^5}{\sqrt{1-x^2}}\,dx$$

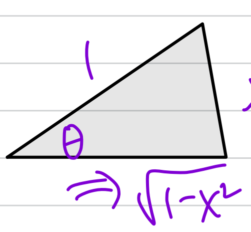
IBP

| | |
|---|---|
| $u = \arcsin x$ | $dv = x^4 dx$ |
| $du = \frac{1}{\sqrt{1-x^2}}$ | $v = \frac{x^5}{5}$ |

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{\sin^5\theta}{\underbrace{\sqrt{1-\sin^2\theta}}_{\sqrt{\cos^2\theta}\ =\ \cancel{\cos\theta}}}\cancel{\cos\theta}\,d\theta$$

Trig. Sub

$\boxed{x = \sin\theta \quad dx = \cos\theta\, d\theta}$




$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \sin^5\theta\, d\theta$$

ODD Power Again

$\sin^4\theta \ \sin\theta$

$(\sin^2\theta)^2$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int (1-\cos^2\theta)^2 \sin\theta\, d\theta$$

| |
|---|
| $u = \cos\theta$ |
| $du = -\sin\theta\, d\theta$ |
| $-du = \sin\theta\, d\theta$ |

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\int (1-u^2)^2\, du$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\int 1 - 2u^2 + u^4\, du$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\overset{\cos\theta}{u} - \frac{2}{3}u^3 + \frac{u^5}{5}\right] + C$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\overset{\sqrt{1-x^2}}{\cancel{\cos\theta}} - \frac{2}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right] + C$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\sqrt{1-x^2} - \frac{2}{3}\left(\sqrt{1-x^2}\right)^3 + \frac{1}{5}\left(\sqrt{1-x^2}\right)^5\right] + C$$

2(c)

$$\int_0^{\sqrt{3}} (x+3)\arctan x\,dx = \left(\frac{x^2}{2}+3x\right)\arctan x\Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{\frac{x^2}{2}+3x}{1+x^2}\,dx$$

split

IBP

| | |
|---|---|
| $u = \arctan x$ | $dv = x+3\,dx$ |
| $du = \frac{1}{1+x^2}\,dx$ | $v = \frac{x^2}{2}+3x$ |

$$= \left(\frac{x^2}{2}+3x\right)\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2+1-1}{1+x^2}\,dx - 3\int_0^{\sqrt{3}}\frac{x}{1+x^2}\,dx$$

slip-in | slip-out

split - split

$$-\frac{1}{2}\int_0^{\sqrt{3}} \frac{\cancel{x^2+1}}{\cancel{1+x^2}} - \frac{1}{1+x^2}\,dx$$

$$-\frac{1}{2}\left[x - \arctan x\right]\Big|_0^{\sqrt{3}}$$

$$= \left(\frac{x^2}{2}+3x\right)\arctan x\Big|_0^{\sqrt{3}} - \frac{x}{2} + \frac{\arctan x}{2}\Big|_0^{\sqrt{3}} - \frac{3}{2}\ln|1+x^2|\Big|_0^{\sqrt{3}}$$

or pull limits of integration off all the way to the end!

$$= \left(\frac{3}{2}+3\sqrt{3}\right)\overset{\pi/3}{\cancel{\arctan\sqrt{3}}} - 0 - \frac{\sqrt{3}}{2} + \frac{\overset{\pi/3}{\cancel{\arctan\sqrt{3}}}}{2} - \left(0 + \overset{0}{\cancel{\arctan 0}}\right) - \frac{3}{2}\left(\ln 4 - \overset{0}{\cancel{\ln 1}}\right)$$

$$= \frac{\cancel{3}}{2}\left(\frac{\pi}{\cancel{3}}\right) + \cancel{3}\sqrt{3}\left(\frac{\pi}{\cancel{3}}\right) - \frac{\sqrt{3}}{2} + \frac{\pi}{6} - \frac{3}{2}\ln 4$$

$$= \frac{\pi}{2} + \sqrt{3}\pi - \frac{\sqrt{3}}{2} + \frac{\pi}{6} - \ln\left[\overset{8}{\cancel{4^{3/2}}}\right]$$

$2\pi/3$

$$4^{3/2} = \left(\sqrt[2]{4}\right)^3 = 8$$

$$\frac{\pi}{2}+\frac{\pi}{6}$$

$$= \frac{3\pi}{6}+\frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$= \left(\frac{2}{3}+\sqrt{3}\right)\pi - \frac{\sqrt{3}}{2} - \ln 8$$

Match!

3(a) $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^2-6x+25}\,dx = \lim_{t\to-\infty} \int_t^{-1} \frac{1}{x^2-6x+25}\,dx$

Improper

Quadratic Irreducible

$b^2-4ac = 36-100 < 0$

$\ominus$

Complete Square

$(x-3)^2 + 16$

$(x-3)^2 = x^2-6x+9$

$$= \lim_{t\to-\infty} \int_t^{-1} \frac{1}{(x-3)^2+16}\,dx$$

$\boxed{\begin{aligned} u &= x-3 \\ du &= dx \end{aligned}}$

$\boxed{\begin{aligned} x=t &\Rightarrow u = t-3 \\ x=-1 &\Rightarrow u=-4 \end{aligned}}$

$$= \lim_{t\to-\infty} \int_{t-3}^{-4} \frac{1}{u^2+16}\,du$$

a-rule

$$= \lim_{t\to-\infty} \frac{1}{4}\arctan\left(\frac{u}{4}\right)\Big|_{t-3}^{-4}$$

$$= \lim_{t\to-\infty} \frac{1}{4}\left(\underbrace{\arctan\left(\frac{-4}{4}\right)}_{-\frac{\pi}{4}} - \underbrace{\arctan\left(\frac{t-3}{4}\right)}_{-\pi/2}\right)$$

$-1$, $-\infty$

$$= \frac{1}{4}\left(-\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{1}{4}\left(\frac{\pi}{4}\right) = \frac{\pi}{16}$$

Converges

3(b) $\displaystyle\int_{-1}^{6} \frac{15-x}{x^2-6x-7}\,dx = \int_{-1}^{6} \frac{15-x}{(x-7)(x+1)}\,dx = \lim_{t\to-1^+} \int_t^6 \frac{15-x}{(x-7)(x+1)}\,dx$

Factors

Improper

PFD

$$= \lim_{t\to-1^+} \int_t^6 \frac{1}{x-7} - \frac{2}{x+1}\,dx$$

PFD:

$$\cancel{(x-7)(x+1)}\left[\frac{15-x}{\cancel{(x-7)(x+1)}} = \frac{A}{x-7} + \frac{B}{x+1}\right](x-7)(x+1)$$

$$15-x = A(x+1) + B(x-7)$$
$$= Ax + A + Bx - 7B$$
$$= (A+B)x + (A-7B)$$

Conditions:

- $A+B = -1 \Rightarrow A = -B-1$
- $A - 7B = 15$

$-B-1-7B = 15$

$-8B = 16$

$B = -2 \Rightarrow A = 1$

$$= \lim_{t\to-1^+} \ln|x-7| - 2\ln|x+1|\Big|_t^6$$

$$= \lim_{t\to-1^+} \ln 1 - 2\ln 7 - \left(\ln|t-7| - 2\ln|t+1|\right)$$

$0$ ; Finite ; $-1$, $|-8|$, $8$, $\ln 8$ Finite ; $-1^+$, $0^+$, $-\infty$

$$= -\infty$$

Diverges

3(c) $\displaystyle\int_{0}^{e} \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to0^+}\int_t^e \ln x \cdot x^{-1/2}\,dx$

Improper

IBP

| | |
|---|---|
| $u=\ln x$ | $dv = x^{-1/2}\,dx$ |
| $du=\frac{1}{x}\,dx$ | $v = 2\sqrt{x}$ |

$$= \lim_{t\to0^+} 2\sqrt{x}\ln x\Big|_t^e - 2\int_t^e \frac{\sqrt{x}}{\cancel{x}}\,dx \qquad x^{-1/2}$$

$$= \lim_{t\to0^+} 2\sqrt{x}\ln x\Big|_t^e - 4\sqrt{x}\Big|_t^e$$

$$= \lim_{t\to0^+} 2\sqrt{e}\,\underbrace{\cancel{\ln e}}_{1} - \underbrace{2\sqrt{t}\ln t}_{0\cdot(-\infty)\to 0} - 4\sqrt{e} + 4\underbrace{\cancel{\sqrt{t}}}_{\to 0}$$

See ◯

$$= 2\sqrt{e} - 4\sqrt{e} = -2\sqrt{e}$$

Converges

◯ $\displaystyle\lim_{t\to0^+}\underbrace{\sqrt{t}\ln t}_{0\cdot(-\infty)} = \lim_{t\to0^+}\frac{\ln t}{1/\sqrt{t}} \overset{-\infty/\infty}{\underset{\text{L'H}}{=}} \lim_{t\to0^+}\frac{\frac{1}{t}}{\frac{-1}{2t^{3/2}}} \;\; (-2t^{3/2}) = \lim_{t\to0^+} -2\underbrace{\sqrt{t}}_{\to 0} = 0$

3(d) $\displaystyle\int_0^{1/2}\frac{1}{x\ln x}\,dx = \lim_{t\to0^+}\int_t^{1/2}\frac{1}{x\ln x}\,dx$

Improper

⋮ u-sub if needed

$$= \lim_{t\to0^+} \ln|\ln x|\Big|_t^{1/2}$$

$$= \lim_{t\to0^+} \underbrace{\ln|\ln \tfrac{1}{2}|}_{\text{Finite}} - \underbrace{\ln|\ln t|}_{\infty}$$

$\ln t \to -\infty$, $|-\infty| \to \infty$, $t \to 0^+$

$$= -\infty$$

Diverges

4(a) $\sum_{n=1}^{\infty} \cos^2\left(\frac{\pi n^6 + 2021}{6n^6+1}\right)$ Diverges by nTDT because

$$\lim_{n\to\infty} \cos^2\left(\frac{\pi n^6 + 2021}{6n^6+1}\right) = \left[\cos\left(\frac{\pi}{6}\right)\right]^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \neq 0$$

$\pi/6$ $\quad \frac{\sqrt{3}}{2}$

pass limit

4(b) $\sum_{n=1}^{\infty} \frac{(-1)^n \cos^2(\pi n^6 + 2021)}{6n^6+1} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{\cos^2(\pi n^6 + 2021)}{6n^6+1} \approx \sum \frac{1}{n^6}$

Converges p-Series $p = 6 > 1$

Bound Terms

$$\frac{\cos^2(\pi n^6 + 2021)}{6n^6+1} \le \frac{1}{6n^6+1} \le \frac{1}{6n^6} \le \frac{1}{n^6}$$

$\Rightarrow$ A.S. also Converges by C.T.

$\Rightarrow$ O.S. A.C. by Definition

4(c) $\sum_{n=1}^{\infty} \underbrace{\frac{\ln(2021)}{n^6}}_{\oplus \text{ terms}} = \ln(2021) \sum_{n=1}^{\infty} \frac{1}{n^6}$

constant

Note: O.S. = A.S.

Constant Multiple of Convergent p-Series $p = 6 > 1$ is Convergent

$\Rightarrow$ O.S. is also A.C. because the original given series is already the Same as the Absolute Series

4(d) $\sum_{n=1}^{\infty} \frac{n^6}{\ln(n+2021)}$ Diverges by nTDT because

$$\lim_{n\to\infty} \frac{n^6}{\ln(n+2021)} = \lim_{x\to\infty} \frac{x^6}{\ln(x+2021)} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{6x^5}{\frac{1}{x+2021}} = \lim_{x\to\infty} 6x^5(x+2021) = \infty \neq 0$$

5(a) $\frac{5}{3} - 1 + \frac{5}{7} - \frac{5}{9} + \frac{5}{11} - \dots = \frac{5}{3} - \frac{5}{5} + \frac{5}{7} - \frac{5}{9} + \frac{5}{11} - \dots$

$$= 5\left[\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \dots\right]$$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

$$= -5\left[-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots\right]$$

$$= -5\left[\cancel{(\arctan 1)}^{\pi/4} - 1\right]$$

$$= -5\left[\frac{\pi}{4} - 1\right] = 5 - \frac{5\pi}{4}$$

5(b) $\frac{1}{2} - \frac{1}{8} + \frac{1}{3\cdot 2^3} - \frac{1}{64} + \frac{1}{5\cdot 2^5} - \dots = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \frac{1}{5\cdot 2^5} - \dots$

$8 = 2^3 \qquad 64 = 4^3$

$$= \left(\frac{1}{2}\right) - \frac{(1/2)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{(1/2)^5}{5} - \dots$$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$$= \ln(1 + 1/2) = \ln(3/2)$$

5(c) $\sum_{n=0}^{\infty} \frac{(-3)^n - 2}{4^n} = \sum_{n=0}^{\infty} \frac{(-3)^n}{4^n} - \frac{2}{4^n} = \sum_{n=0}^{\infty} \frac{(-3)^n}{4^n} - 2\sum_{n=0}^{\infty} \frac{1}{4^n}$ Both Geometric.

Split-Split

$$= \sum_{n=0}^{\infty} \left(\frac{-3}{4}\right)^n - 2\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

or more generally

$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

$$= \frac{1}{1-\left(-\frac{3}{4}\right)} - 2\left(\frac{1}{1-1/4}\right)$$

$+ \quad 7/4 \qquad 3/4$

$$= 4/7 - 2(4/3) = 4/7 - 8/3 = \frac{12}{21} - \frac{56}{21} = -\frac{44}{21}$$

5(d) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(\sqrt{2})^{4n}(2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{\left((\sqrt{2})^{\cancel{2}}\right)^{2n}(2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{2n}(2n)!}$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $\quad = \pi \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n}}{(2n)!} = \pi \cancel{\cos(\pi/2)}^{0} = 0$

5(e) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2^4)\pi^{2n}}{2^{4n}(2n)!} = -16\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2^2)^{2n}(2n)!} = -16\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$

$$= -16\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = -16 \cancel{\cos(\pi/4)}^{\sqrt{2}/2} = -8\sqrt{2}$$

5(f) $-\pi + \frac{\pi^3}{3!} - \frac{\pi^5}{5!} + \frac{\pi^7}{7!} - \frac{\pi^9}{9!} + \dots = -\left[\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!} - \dots\right]$

$$= -\cancelto{0}{\sin\pi} = 0$$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

5(g) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(\ln 9)^n}{2^{n+1}\, n!} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(\ln 9)^n}{2^n \cdot n!} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{-\ln 9}{2}\right)^n}{n!}$

$$= -\frac{1}{2}e^{\left(\frac{-\ln 9}{2}\right)} = -\frac{1}{2}e^{-\frac{1}{2}\cdot\ln 9}$$

$$= -\frac{1}{2}\cancel{e^{\ln}}(9^{-1/2}) = -\frac{1}{2}\cdot\frac{1}{\cancelto{3}{\sqrt{9}}} = -\frac{1}{2}\cdot\frac{1}{3} = -\frac{1}{6}$$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)(5x-2)^n}{n^2\ 8^n}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(n+2)(5x-2)^{n+1}}{(n+1)^2\ 8^{n+1}}}{\frac{\cancel{(-1)^n}(n+1)(5x-2)^n}{n^2\cdot 8^n}}\right| = \lim_{n\to\infty}\cancelto{1}{\frac{n+2}{n+1}}\ \cancelto{|5x-2|}{\left|\frac{(5x-2)^{n+1}}{(5x-2)^n}\right|}\cdot\frac{n^2}{(n+1)^2}\cdot\frac{\cancel{8^n}}{\cancel{8^{n+1}}_{1/8}}$$

$\cancel{\left(\frac{n}{n+1}\right)^2} \to 1$

$$= \frac{|5x-2|}{8} < 1$$

Converges by R.T. when

$|5x-2| < 8$

$-8 < 5x-2 < 8$
$+2 \quad +2 \quad +2$
$-6 < 5x < 10$
$-6/5 < x < 2$

Manually Test Convergence at Endpoints (where $L=1$)

Take $x = -6/5$. Series becomes

$$\hookrightarrow \sum_{n=1}^{\infty}\frac{(-1)^n(n+1)\left(\cancel{5}\left(-\frac{6}{\cancel{5}}\right)-2\right)^n}{n^2\cdot 8^n} = \sum_{n=1}^{\infty}\frac{\cancelto{1}{(-1)^{2n}}(n+1)\cancel{8^n}}{n^2\cdot\cancel{8^n}}$$

$(-1)^n 8^n$, $(-8)^n$, Even

$$= \sum_{n=1}^{\infty}\frac{n+1}{n^2} \approx \sum_{n=0}\frac{1}{n}$$ Diverges, Harmonic, p-Series $p=1$

Bound Terms:

$$\frac{n+1}{n^2} \geq \frac{n}{n^2} = \frac{1}{n}$$ $\Rightarrow$ Series also Diverges by C.T.

(or use LCT Limit too)

Take $x=2$. Series becomes

$$\hookrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(5(2)-2)^n}{n^2 \cdot \cancel{8^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{n^2}$$

AST 1. $b_n = \frac{n+1}{n^2} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n+1}{n^2} = 0$ ✓

3. Terms Decreasing.

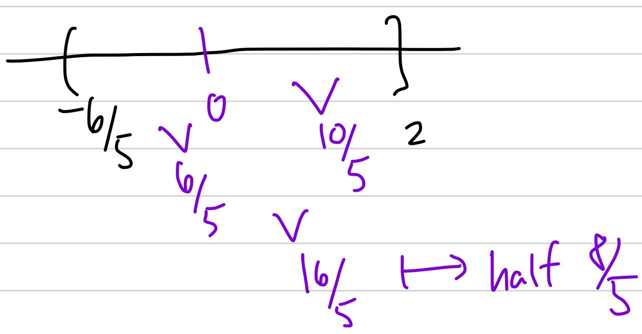
$$f(x) = \frac{x+1}{x^2} \text{ has } f'(x) = \frac{x^2(1)-(x+1)(2x)}{x^4} = \frac{-x^2-x}{x^4} < 0$$

$\ominus$ $\oplus$

$\Rightarrow$ Series Converges by A.S.T.

Finally, $I = \left[-\frac{6}{5}, 2\right]$

$R = \frac{8}{5}$




6(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n (\ln n)(x-7)^n}{(2n)!\, e^n \sqrt{n}}$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(n+1)^{n+1} \ln(n+1)(x-7)^{n+1}}{(2(n+1))!\, e^{n+1}\sqrt{n+1}}}{\dfrac{n^n (\ln n)(x-7)^n}{(2n)!\, e^n \sqrt{n}}} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{\ln(n+1)}{\ln n} \cdot \left|\frac{(x-7)^{n+1}}{(x-7)^n}\right| \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{e^n}{e^{n+1}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}}$$

$(n+1)^n (n+1)$ → e

1, $|x-7|$, $\sqrt{\frac{n}{n+1}}$ → 1

See ◯

$(2n+2)(2n+1)(2n)!$ · $1/e$

$2(n+1)$

$$= \lim_{n\to\infty} \frac{e}{e} \cdot \frac{|x-7|}{2(2n+1)} = 0 < 1$$

0, $\infty$

Converges by R.T. for all x.

$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \underset{L'H}{\overset{\infty/\infty}{=}} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} \underset{L'H}{\overset{\infty/\infty}{=}} \lim_{x\to\infty} \frac{1}{x} = 1$ ✓

Finally, $I = (-\infty, \infty)$

$R = \infty$

7. $\ln\left(\frac{3}{2}\right) = \ln\left(1+\frac{1}{2}\right) = \frac{1}{2} - \frac{\overset{1/4}{(1/2)^2}}{2} + \frac{\overset{1/8}{(1/2)^3}}{3} - \frac{\overset{1/16}{(1/2)^4}}{4} + \cdots$

$= \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \cdots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Estimate

$\approx \frac{1}{2} - \frac{1}{8} + \frac{1}{24} = \frac{12}{24} - \frac{3}{24} + \frac{1}{24} = \frac{10}{24} = \frac{5}{12}$

Using ASET we can Estimate the full Sum using only the first three terms with error at most $|\text{First Neglected Term}| = \frac{1}{64} < \frac{1}{50}$ as desired.

Fun Note: This is the Same Sum as above in 5(b).

8(a) $\ln(x+3) = \int \frac{1}{x+3}\,dx = \int \frac{1}{3+x}\,dx = \int \frac{1}{3(1+x/3)}\,dx = \int \frac{1}{3(1-(-x/3))}\,dx$

$= \int \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{-x}{3}\right)^n dx = \int \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}\,dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)} + C$ $\ln 3$

Expand

$n=0 \quad n=1 \quad n=2 \quad n=3$

$\ln(x+3) = \frac{x}{3} - \frac{x^2}{3^2\cdot 2} + \frac{x^3}{3^3\cdot 3} - \cdots + C$

Test $x=0$: $\ln 3 = 0 - 0 + 0 - \cdots + C \Rightarrow C = \ln 3$ here

Finally, $\ln(x+3) = \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)}$

8(b). $\ln(x+3) = \ln(3+x) = \ln\left(3\left(1+\frac{x}{3}\right)\right) = \ln 3 + \ln\left(1+\frac{x}{3}\right)$

Substitution here

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{3}\right)^{n+1}}{n+1} = \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)}$$ Match!

Just for Fun: Chart Method / Definition of Maclaurin Series

| | |
|---|---|
| $f(x) = \ln(x+3)$ | $f(0) = \ln 3$ |
| $f'(x) = \frac{1}{x+3} = (x+3)^{-1}$ | $f'(0) = \frac{1}{3}$ |
| $f''(x) = -(x+3)^{-2} = \frac{-1}{(x+3)^2}$ | $f''(0) = \frac{-1}{3^2} = \frac{-1}{9}$ |
| $f'''(x) = 2(x+3)^{-3} = \frac{2}{(x+3)^3}$ | $f'''(0) = \frac{2}{3^3} = \frac{2}{27}$ |
| $f^{(4)}(x) = -6(x+3)^{-4} = \frac{-6}{(x+3)^4}$ | $f^{(4)}(0) = \frac{-6}{3^4} = \frac{-6}{81}$ |
| $\vdots$ | $\vdots$ |

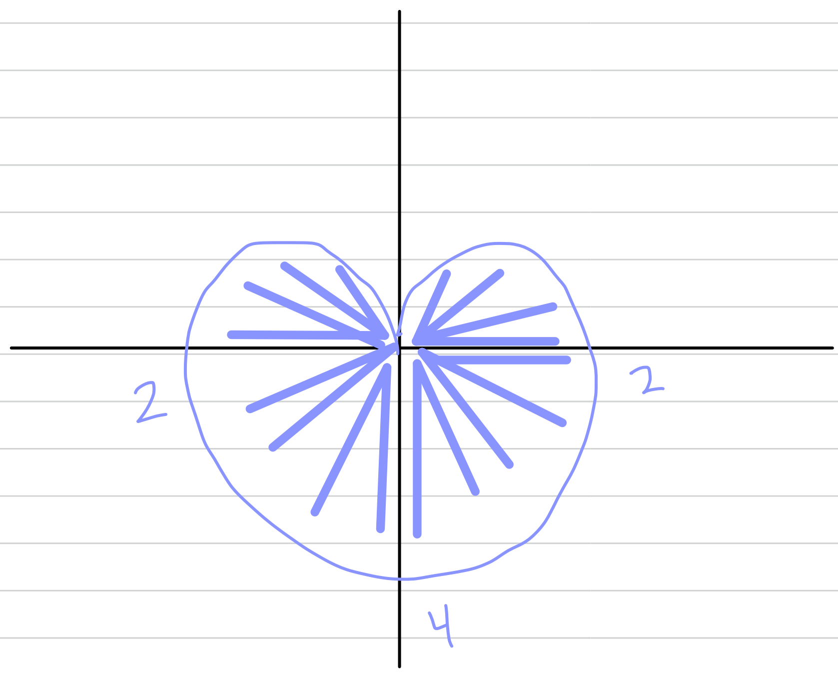
Maclaurin Series

$$\overset{\ln 3}{\cancel{f(0)}} + \overset{\frac{1}{3}}{\cancel{f'(0)}}x + \frac{\overset{-1/9}{\cancel{f''(0)}}}{2!}x^2 + \frac{\overset{\cancel{2}/27}{\cancel{f'''(0)}}}{3!_{\,3\cdot\cancel{2}}}x^3 + \frac{\overset{\cancel{-6}/81}{\cancel{f^{(4)}(0)}}}{4!_{\,\underbrace{4\cdot 3\cdot 2}_{\cancel{6}}}}x^4 + \cdots$$

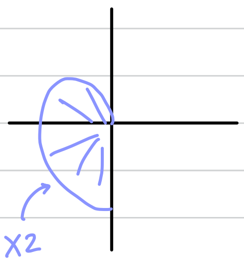
$$= \ln 3 + \frac{x}{3} - \frac{x^2}{3^2\cdot 2} + \frac{x^3}{3^3\cdot 3} - \frac{x^4}{3^4\cdot 4} + \cdots$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}\cdot(n+1)}$$ Match!

9. $r = 2 - 2\sin\theta$




OR// use Double by Symmetry




$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (\text{Polar Radius})^2 \, d\theta$$

OR// Distribute ½ works too.

$$= \frac{1}{2}\int_0^{2\pi} (2 - 2\sin\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} 4 - 8\sin\theta + 4\sin^2\theta \, d\theta$$

$$\cancel{4}^{\,2}\left(\frac{1-\cos(2\theta)}{\cancel{2}}\right) \quad \text{Half-Angle}$$

$$= \frac{1}{2}\int_0^{2\pi} \overbrace{4 - 8\sin\theta + 2}^{6} - 2\cos(2\theta) \, d\theta$$

$$= \frac{1}{2}\left[6\theta + 8\cos\theta - \sin(2\theta)\right]\Big|_0^{2\pi}$$

$$= \frac{1}{2}\left[6(2\pi) + 8\underbrace{\cos(2\pi)}_{1} - \underbrace{\sin(4\pi)}_{0} - \left(0 + 8\underbrace{\cos 0}_{1} - \underbrace{\sin 0}_{0}\right)\right]$$

$$= \frac{1}{2}\left[12\pi + \cancel{8} - \cancel{8}\right] = 6\pi$$

Extra Side Note: We know from class that the standard Cardioids $r = 1 \pm \cos\theta$ and $r = 1 - \sin\theta$ all have bounded are equaling $3\pi/2$. Here $r = 2 - 2\cos\theta = 2(1 - \cos\theta)$ and the integral can be factored algebraically with a scaling effect of $2^2 = 4$

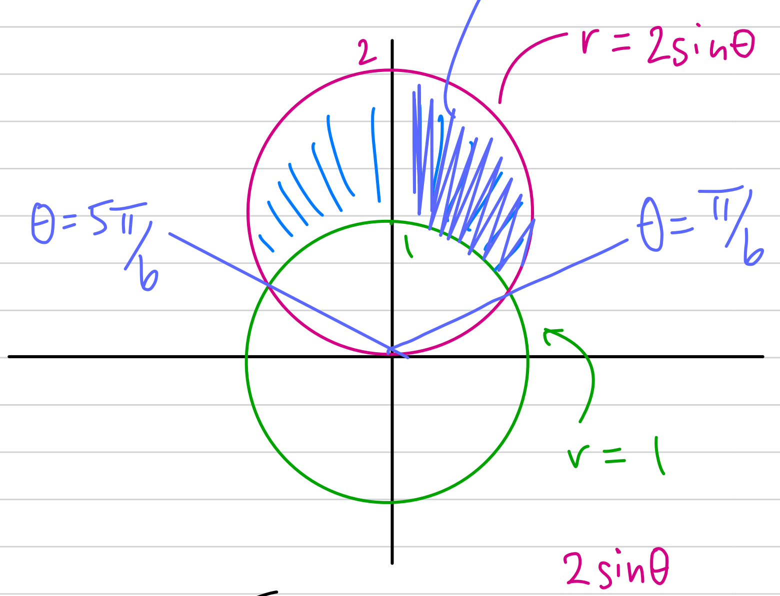
Here $\text{Area} = \frac{1}{2}\int_0^{2\pi} (2-2\sin\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \left[2(1-\sin\theta)\right]^2 d\theta$

$3\pi/2$

$$= \frac{1}{2}\int_0^{2\pi} 4\cdot(1-\cos\theta)^2\, d\theta = 4\left[\frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2\, d\theta\right] = 6\pi$$ Match!

Makes Sense.

10(a) $r=1$ $\quad r = 2\sin\theta$ $\quad$ Double by Symmetry $\quad$ Intersect?




$2\sin\theta = 1$

$\sin\theta = 1/2$

$\theta = \pi/6,\ \frac{5\pi}{6}$ (by symmetry)

$$\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (\overset{2\sin\theta}{\text{Outer Polar Radius}})^2 - (\overset{1}{\text{Inner Polar Radius}})^2\, d\theta$$

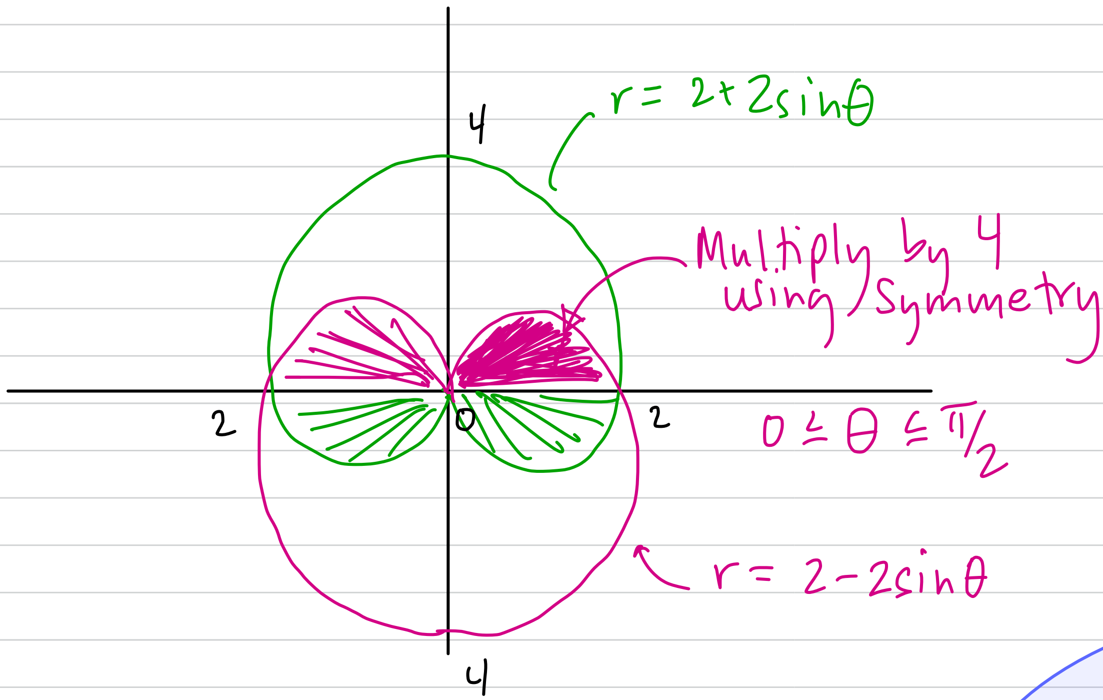
$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (2\sin\theta)^2 - 1^2\, d\theta$$

Alternate Integral

$$= 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (2\sin\theta)^2 - 1^2\, d\theta\right]$$

Double using Symmetry

10(b) $r = 2 + 2\sin\theta \qquad r = 2 - 2\sin\theta$

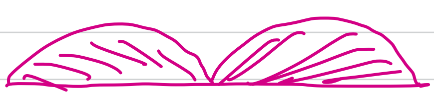



$$\text{Area} = 4\left[\frac{1}{2}\int_0^{\pi/2} \left(\overset{2-2\sin\theta}{\text{Polar Radius}}\right)^2 d\theta\right] = \boxed{4\left[\frac{1}{2}\int_0^{\pi/2} (2-2\sin\theta)^2\, d\theta\right]}$$

using Symmetry

Alternate Integrals

$$2\left[\frac{1}{2}\int_0^{\pi} (2-2\sin\theta)^2\, d\theta\right]$$



OR

$$2\left[\frac{1}{2}\int_{\pi}^{2\pi} (2+2\sin\theta)^2\, d\theta\right]$$

OR

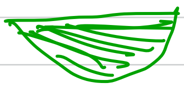
$$4\left[\frac{1}{2}\int_{\pi/2}^{\pi} (2-2\sin\theta)^2\, d\theta\right]$$

OR

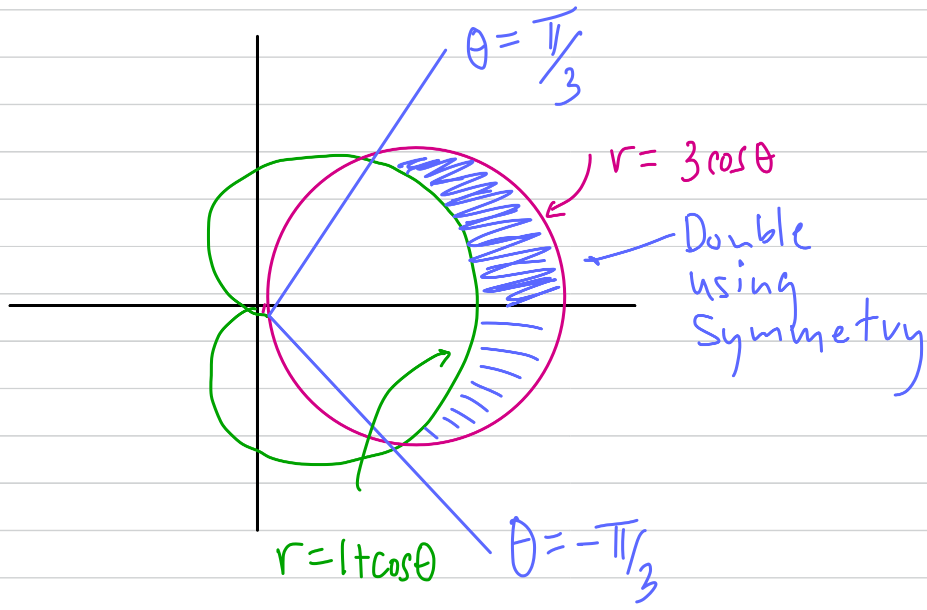
$$4\left[\frac{1}{2}\int_{\pi}^{3\pi/2} (2+2\sin\theta)^2\, d\theta\right]$$

OR

$$4\left[\frac{1}{2}\int_{3\pi/2}^{2\pi} (2+2\sin\theta)^2\, d\theta\right]$$



10(c) $r = 1+\cos\theta$ $\quad r = 3\cos\theta$




Intersect?

$1+\cos\theta = 3\cos\theta$

$1 = 2\cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{3}$

$$\text{Area} = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (\underbrace{\text{Outer Polar Radius}}_{3\cos\theta})^2 - (\underbrace{\text{Inner Polar Radius}}_{1+\cos\theta})^2 \, d\theta$$

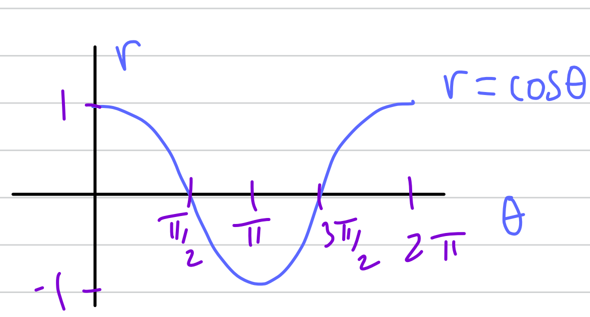
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (3\cos\theta)^2 - (1+\cos\theta)^2 \, d\theta$$

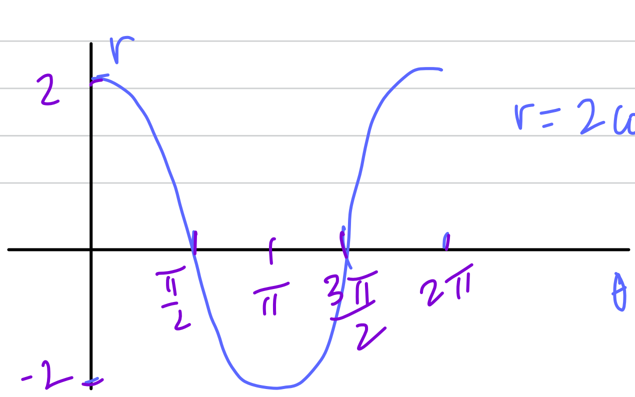
Alternate Integral

$$= 2\left[\frac{1}{2}\int_{0}^{\pi/3} (3\cos\theta)^2 - (1+\cos\theta)^2 \, d\theta\right]$$

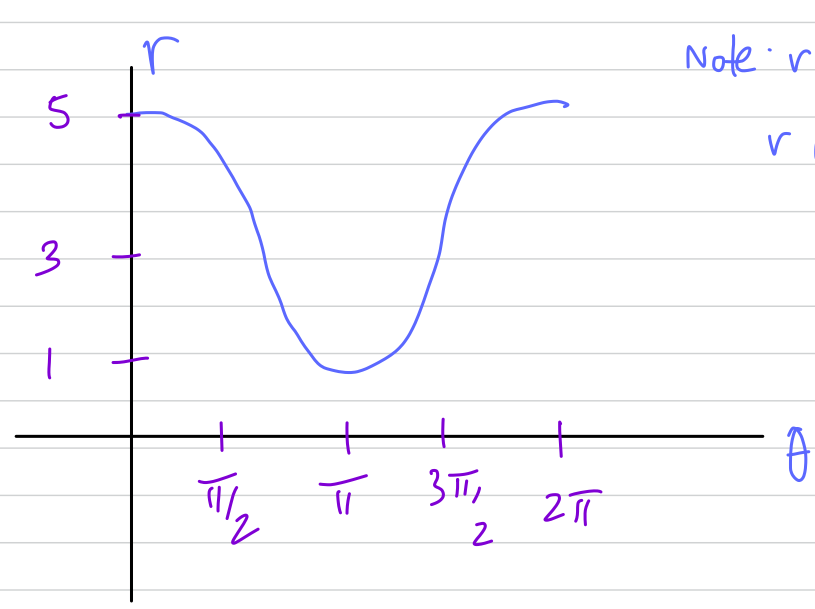
Double using Symmetry

10(d) $r = 3+2\cos\theta$ $\quad$ Cartesian Plots First




Shift up 3 units.



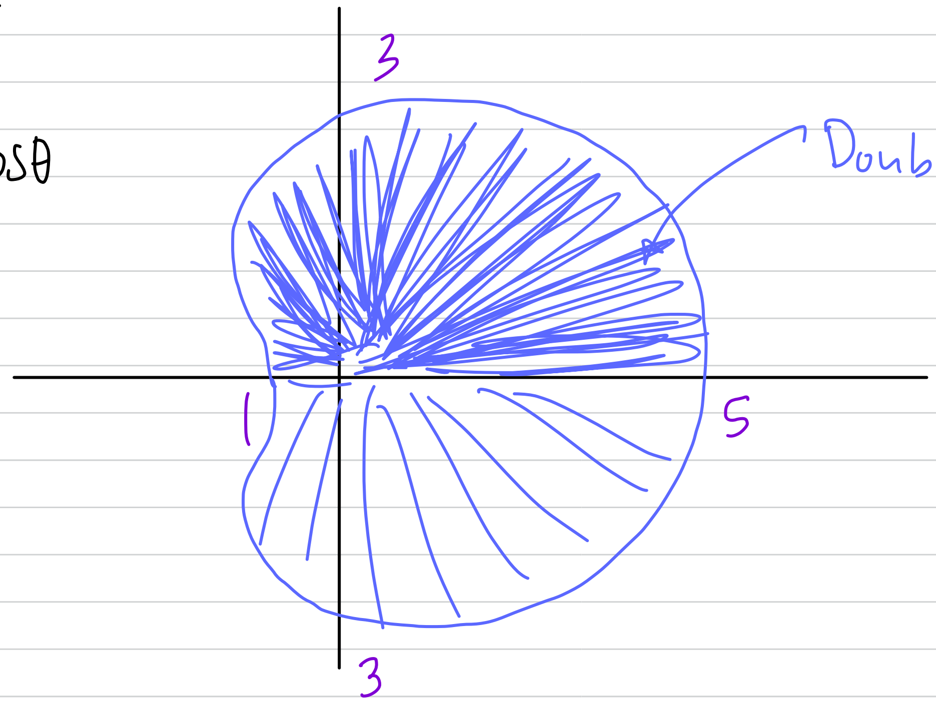

Note: r all (+)

r never 0

r: $5 \to 3 \to 1 \to 3 \to 5$

Polar Plot

$r = 3 + 2\cos\theta$




Double using Symmetry

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (\text{Polar Radius})^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (3 + 2\cos\theta)^2 \, d\theta$$

Alternate Integral

$$= 2\left[\frac{1}{2}\int_0^{\pi} (3 + 2\cos\theta)^2 \, d\theta\right]$$

Double by Symmetry