Math 121 Final Exam Fall 25 Answer Key

1 (a)

$$\lim_{x\to0}\frac{\overset{1}{\cos(3x)}\overset{-1}{-1}\overset{+0}{-\arctan(2x)}\overset{-0}{+2x}}{\underset{1}{e^{-4x}}\underset{-1}{-1}\underset{+0}{+4x}}\qquad\frac{0}{0}$$

$$=\lim_{x\to0}\frac{\cancel{1}-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\dots-\cancel{1}-\left(\cancel{(2x)}-\frac{(2x)^3}{3}+\frac{(2x)^5}{5}-\dots\right)+\cancel{2x}}{\cancel{1}+\cancel{(-4x)}+\frac{(-4x)^2}{2!}+\frac{(-4x)^3}{3!}+\dots-\cancel{1}+\cancel{4x}}$$

$$=\lim_{x\to0}\frac{-\frac{9x^2}{2!}+\frac{3^4x^4}{4!}-\dots\overset{--}{+}\frac{2^3x^3}{3}-\frac{2^5x^5}{5}+\dots}{\frac{16x^2}{2!}-\frac{4^3x^3}{3!}+\frac{4^4x^4}{4!}-\dots}\qquad\begin{matrix}\frac{1}{x^2}\\[1em]\frac{1}{x^2}\end{matrix}$$

$$=\lim_{x\to0}\frac{-\frac{9}{2}+\cancelto{0}{\frac{3^4x^2}{4!}-\dots+\frac{2^3x}{3}-\frac{2^5x^3}{5}+\dots}}{\frac{16}{2}\cancelto{0}{-\frac{4^3x}{3!}+\frac{4^4x^2}{4!}-\dots}}$$

$$=\frac{-\frac{9}{2}}{\frac{16}{2}}=-\frac{9}{\cancel{2}}\cdot\frac{\cancel{2}}{16}=\boxed{-\frac{9}{16}}$$

1 (b)

$$\lim_{x\to0}\frac{\overset{1}{\cos(3x)}-\overset{0}{\arctan(2x)}+\overset{0}{2x}\overset{-1}{-1}}{\underset{1}{e^{-4x}}\underset{-1}{-1}\underset{0}{+4x}}\ \overset{0/0}{\underset{L'H}{=}}\ \lim_{x\to0}\frac{-3\overset{0}{\sin(3x)}-\overset{-2}{\frac{2}{1+(2x)^2}}\overset{+2}{+2}}{\underset{-4}{-4e^{-4x}}\underset{+4}{+4}}\qquad\frac{0}{0}$$

$$\underset{\text{prep}}{=}\lim_{x\to0}\frac{-3\sin(3x)-2(1+4x^2)^{-1}+2}{-4e^{-4x}+4}\qquad\frac{0}{0}$$

$$\underset{L'H}{=}\lim_{x\to0}\frac{-9\cos(3x)\overset{--}{+}2(1+4x^2)^{-2}\cdot(8x)}{16e^{-4x}}$$

$$\underset{\text{rewrite}}{=}\lim_{x\to0}\frac{-9\cancelto{1}{\cos(3x)}+\cancelto{0}{\frac{16x}{(1+4x^2)^2}}}{16\cancelto{1}{e^{-4x}}}=\boxed{-\frac{9}{16}}$$

Match

2(a)
$$\int \frac{1}{(x^2+4)^2}\,dx = \int \frac{1}{(4\tan^2\theta+4)^2}\cdot 2\sec^2\theta\,d\theta = \int \frac{1}{(4(\tan^2\theta+1))^2}\cdot 2\sec^2\theta\,d\theta$$

$$(4\sec^2\theta)^2$$

Trig Sub

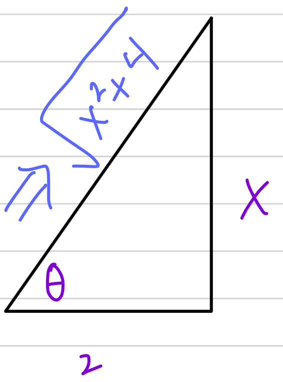
$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\,d\theta \end{aligned}}$$

$\tan\theta = \frac{x}{2} \Rightarrow \theta = \arctan\left(\frac{x}{2}\right)$

$$= \int \frac{1}{\cancel{16}\sec^4\theta}\cdot \cancel{2}\sec^2\theta\,d\theta = \frac{1}{8}\int \frac{1}{\sec^2\theta}\,d\theta$$

Flip to Cosine

2 copies left in denominator



$$= \frac{1}{8}\int \cos^2\theta\,d\theta = \frac{1}{8}\int \frac{1+\cos(2\theta)}{2}\,d\theta$$

Half-Angle Identity

$$= \frac{1}{16}\int 1+\cos(2\theta)\,d\theta = \frac{1}{16}\left(\theta + \frac{\overset{\cancel{2}\sin\theta\cos\theta}{\sin(2\theta)}}{\cancel{2}}\right) + C$$

Double Angle Identity

$$= \frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \left(\frac{x}{\sqrt{x^2+4}}\right)\left(\frac{2}{\sqrt{x^2+4}}\right)\right] + C$$

$$= \boxed{\frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right] + C}$$

2(b)
$$\int x^4\cdot\arcsin x\,dx = \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{x^5}{\sqrt{1-x^2}}\,dx$$

IBP

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = x^4\,dx \\ du = \dfrac{1}{\sqrt{1-x^2}} & v = \dfrac{x^5}{5} \end{array}}$$

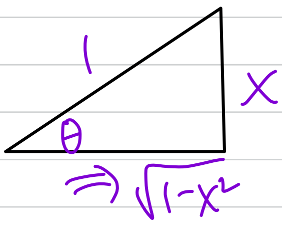
↓

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{\sin^5\theta}{\sqrt{1-\sin^2\theta}}\,\cancel{\cos\theta}\,d\theta$$

$$\sqrt{\cos^2\theta}$$
$$\cancel{\cos\theta}$$

Trig. Sub

$$\boxed{\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\,d\theta \end{aligned}}$$



$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \sin^5\theta\,d\theta$$

ODD Power

$\sin^4\theta\ \sin\theta$

$(\sin^2\theta)^2$

$$\boxed{\begin{aligned} w &= \cos\theta \\ dw &= -\sin\theta\, d\theta \\ -dw &= \sin\theta\, d\theta \end{aligned}}$$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int (1-\cos^2\theta)^2 \sin\theta\, d\theta$$

$$= \frac{x^5}{5}\arcsin x \overset{--}{+} \frac{1}{5}\int (1-w^2)^2\, du$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\int 1-2w^2+w^4\, du$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\overset{\cos\theta}{w} - \frac{2}{3}w^3 + \frac{w^5}{5}\right] + C$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\overset{\sqrt{1-x^2}}{\cos\theta} - \frac{2}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right] + C$$

$$\boxed{= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\sqrt{1-x^2} - \frac{2}{3}\left(\sqrt{1-x^2}\right)^3 + \frac{1}{5}\left(\sqrt{1-x^2}\right)^5\right] + C}$$

Important: All Improper Integrals must have Limit Definition Set-up in First Step

3(a) $\displaystyle\int_{0}^{e} x^3 \cdot \ln x\, dx = \lim_{t\to 0^+}\int_t^e x^3\cdot \ln x\, dx = \lim_{t\to 0^+} \frac{x^4}{4}\cdot \ln x\Big|_t^e - \frac{1}{4}\int_t^e \overset{x^3}{\frac{x^4}{x}}\, dx$

IBP

$$\boxed{\begin{aligned} u &= \ln x & dv &= x^3\, dx \\ du &= \frac{1}{x}\, dx & v &= \frac{x^4}{4} \end{aligned}}$$

$$= \lim_{t\to 0^+} \frac{x^4}{4}\ln x\Big|_t^e - \frac{x^4}{16}\Big|_t^e$$

Indeterminate Product Need L'H

$$= \lim_{t\to 0^+} \frac{e^4}{4}\cdot \overset{1}{\cancel{\ln e}} - \overset{0\cdot(-\infty)\ \to\ 0}{\cancel{\frac{t^4}{4}\cdot \ln t}} - \left(\frac{e^4}{16} - \overset{0}{\cancel{\frac{t^4}{9}}}\right) = \underset{\frac{4e^4}{16}}{\frac{e^4}{4}} - 0 - \frac{e^4}{16} = \boxed{\frac{3e^4}{16}}$$

see ☆

Converges

$$☆\quad \lim_{t\to 0^+} \overset{0\cdot(-\infty)}{t^4\cdot \ln t} = \lim_{t\to 0^+} \overset{\frac{-\infty}{\infty}}{\frac{\ln t}{\frac{1}{t^4}}} \overset{\text{L'H}}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-4}{t^5}} \overset{\frac{-t^5}{4}}{} = \lim_{t\to 0^+} \frac{-t^{\cancel{5}}}{4\cancel{t}} = \lim_{t\to 0^+} \overset{0}{\cancel{\frac{-t^4}{4}}} = 0$$

flip down

$t^{-4} \hookrightarrow -4t^{-5}$

Key Note: ln 0 is undefined, so must "sneak attack" 0 using Limit 0^+

3(b) $\int_{0}^{e^3} \frac{1}{x(3+(\ln x)^2)}\,dx = \lim_{t\to 0^+}\int_{t}^{e^3}\frac{1}{x(3+(\ln x)^2)}\,dx = \lim_{t\to 0^+}\int_{\ln t}^{3}\frac{1}{3+u^2}\,du$

$u=\ln x$
$du=\frac{1}{x}\,dx$

$x=t \Rightarrow u=\ln t$
$x=e^3 \Rightarrow u=\cancel{\ln e^3}=3$

$$= \lim_{t\to 0^+}\frac{1}{\sqrt3}\arctan\left(\frac{u}{\sqrt3}\right)\Bigg|_{\ln t}^{3}$$

$$= \lim_{t\to 0^+}\frac{1}{\sqrt3}\left(\overset{\frac{\pi}{3}}{\cancel{\arctan\left(\overset{\sqrt3}{\cancel{\frac{3}{\sqrt3}}}\right)}} - \overset{-\frac{\pi}{2}}{\cancel{\arctan\left(\frac{\overset{-\infty}{\cancel{\ln \overset{0^+}{\cancel{t}}}}}{\sqrt3}\right)}}\right)$$

Must Justify Size Argument

$$= \frac{1}{\sqrt3}\left(\frac{\pi}{3}\overset{--}{+}\frac{\pi}{2}\right) = \frac{1}{\sqrt3}\left(\frac{2\pi}{6}+\frac{3\pi}{6}\right) = \frac{5\pi}{6\sqrt3}$$

Converges

3(c) $\int_{-7}^{0}\frac{x+15}{x^2+6x-7}\,dx = \lim_{t\to -7^+}\int_{t}^{0}\frac{x+15}{x^2+6x-7}\,dx = \lim_{t\to -7^+}\int_{t}^{0}\frac{x+15}{(x-1)(x+7)}\,dx$

PFD

$$\cancel{(x-1)(x+7)}\left[\frac{x+15}{\cancel{(x-1)(x+7)}} = \frac{A}{x-1}+\frac{B}{x+7}\right](x-1)(x+7)$$

$1x+15 = A(x+7)+B(x-1)$
$= Ax+7A+Bx-B$
$= (A+B)x+(7A-B)$

Conditions

- $A+B=1 \Rightarrow B=1-A$
- $7A-B=15$

$7A-(1-A)=15$
$7A-1+A=15$
$8A=16 \quad B=1-2=-1$
$A=2$

PFD

$$= \lim_{t\to -7^+}\int_{t}^{0}\frac{2}{x-1}-\frac{1}{x+7}\,dx$$

$$= \lim_{t\to -7^+} 2\ln|x-1|-\ln|x+7|\Bigg|_{t}^{0}$$

Finish All Finite Values

$$= \lim_{t\to -7^+} \underset{\text{Finite}}{2\overset{0}{\cancel{\ln}}|-1|} - \underset{\text{Finite}}{\ln 7} - \left(\underset{\text{Finite}}{\overset{2\ln 8}{\cancel{2\ln|t-1|}}} - \overset{-\infty}{\cancel{\ln\left|\overset{-7^+}{\cancel{t}}+7\right|}}^{\,0^+}\right)$$

Show Size Argument

$$= -(-(-\infty)) = -\infty$$

Diverges

Key Note: ln 0 is undefined, so must "sneak attack" 0 using Limit 0^+

4(a) $$-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots = (\overset{\frac{\pi}{4}}{\cancel{\arctan 1}})-1 = \boxed{\frac{\pi}{4}-1} \overset{\text{OR}}{=} \boxed{\frac{\pi-4}{4}}$$

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\dots$$

$$\arctan 1 = \underset{\substack{\uparrow \\ \text{missing}}}{\boxed{1}}-\frac{1}{3}+\frac{1}{5}-\dots$$

4(b) $$\sum_{n=0}^{\infty}\frac{(-1)^n 2^n (\ln 9)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-2\ln 9)^n}{n!} = e^{-2\ln 9} = \cancel{e}^{\cancel{\ln}(9^{-2})} = 9^{-2} = \boxed{\frac{1}{81}}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

4(c) $$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n}}{\underset{3^{2n}}{9^n}(2n+1)!} = \overset{\substack{\text{extra} \\ \downarrow}}{-}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{3}\right)^{2n}\cdot\frac{\pi}{3}}{(2n+1)!\cdot\frac{\pi}{3}} = -\frac{3}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}$$

extra

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= -\frac{3}{\pi}\overset{\frac{\sqrt{3}}{2}}{\cancel{\sin\left(\frac{\pi}{3}\right)}} = \boxed{\frac{-3\sqrt{3}}{2\pi}}$$

4(d) $$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\dots = -\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots\right)$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\dots \qquad = -\ln(1+\boxed{1}) = \boxed{-\ln 2}$$

4(e) $$1+1+\underbrace{1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\dots}_{\longrightarrow} = 1+\cancel{1}+\overset{-1}{\cancel{\cos\pi}} = 1+\cancel{1}\cancel{-1} = \boxed{1}$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots$$

show split

4(f) $$\sum_{n=0}^{\infty}\frac{(-4)^n-2}{5^n} = \sum_{n=0}^{\infty}\frac{(-4)^n}{5^n}-\sum_{n=0}^{\infty}\frac{2}{5^n} = \sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)^n-2\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n.$$ show split

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$$

$$= \overset{\frac{5}{9}}{\frac{1}{1-\left(-\frac{4}{5}\right)}}-2\overset{\frac{5}{4}}{\left(\frac{1}{1-\frac{1}{5}}\right)}$$

$+\frac{9}{5}$ $\qquad$ $\frac{4}{5}$

$$= \frac{5}{9}-\frac{5}{2} = \frac{10}{18}-\frac{45}{18} = \boxed{\frac{-35}{18}}$$ Match!

Note: Sum of 2 Convergent Series Converges

5(a) $\sum_{n=1}^{\infty} \frac{(-1)^n (5x+7)^n}{(5n+7)^2 \cdot 8^n}$ Ratio Test

Converges by Ratio Test when

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}} (5x+7)^{n+1}}{(5(n+1)+7)^2 \cdot 8^{n+1}}}{\frac{\cancel{(-1)^n} (5x+7)^n}{(5n+7)^2 \, 8^n}} \right| \xrightarrow{\text{flip}} = \lim_{n\to\infty} \overset{|5x+7|}{\left| \frac{(5x+7)^{n+1}}{(5x+7)^n} \right|} \cdot \left(\frac{\cancel{5n+7}}{\cancel{5n+12}} \right)^2 \overset{1}{} \cdot \frac{\cancel{8^n}}{\underset{\cancel{8^n} \cdot 8}{8^{n+1}}} = \frac{|5x+7|}{8} < 1$$

$$\frac{|5x+7|}{8} < 1 \Rightarrow |5x+7| < 8 \Rightarrow -8 < 5x+\cancel{7} < 8 \Rightarrow -15 < 5x < 1 \Rightarrow -3 < x < \frac{1}{5}$$

$$\underset{-7}{} \quad \underset{-\cancel{7}}{} \quad \underset{-7}{} \qquad \underset{\overline{5}}{} \quad \underset{\overline{5}}{} \quad \underset{\overline{5}}{}$$

Manually Check Convergence at Endpoints

Take $x = -3$ Series becomes

Convergent p-Series $p = 2 > 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n [5(-3)+7]^n}{(5n+7)^2 \cdot \; n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-8)^n}{(5n+7)^2 \cdot 8^n} = \sum_{n=1}^{\infty} \frac{\overset{\text{Even } \to 1}{\cancel{(-1)^{2n}}} \; (-1)^n (-1)^n \cancel{8^n}}{(5n+7)^2 \cdot \cancel{8^n}} = \sum_{n=1}^{\infty} \frac{1}{(5n+7)^2} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Bound Terms

$$\frac{1}{(5n+7)^2} \le \frac{1}{n^2}$$

LCT also works

OR LCT Limit

$$\lim_{n\to\infty} \frac{\frac{1}{(5n+7)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(5n+7)^2} = \lim_{n\to\infty} \left(\frac{\cancel{n}}{\cancel{5n+7}} \right)^2 \to \left(\frac{1}{5}\right)^2 \to \frac{1}{25} = \frac{1}{25} \text{ Finite Non-Zero}$$

$\Rightarrow$ Series Converges by LCT

$\Rightarrow$ Series Converges by CT

Take $x = \frac{1}{5}$ Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\cancel{5}(\tfrac{1}{\cancel{5}})+7]^n}{(5n+7)^2 \, 8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{8^n}}{(5n+7)^2 \, \cancel{8^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(5n+7)^2} \xrightarrow{AS} \sum_{n=1}^{n} \frac{1}{(5n+7)^2}$$

Already Shown above to be Convergent using CT or LCT

1. Isolate $b_n = \frac{1}{(5n+7)^2} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\underset{\infty}{\cancel{(5n+7)^2}}} \to 0 = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{(5(n+1)+7)^2} = \frac{1}{(5n+12)^2} \le \frac{1}{(5n+7)^2} = b_n$$

AST OR ACT

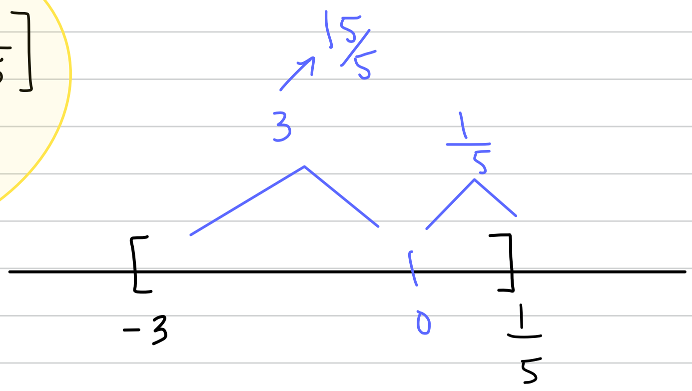
Series Converges by AST

ACT ⇙

Original Series Converges by ACT

Finally, Interval of Convergence $I = \left[-3, \frac{1}{5}\right]$

Radius of Convergence $R = \frac{8}{5}$



Total Length $\frac{16}{5}$ ↳ Half $\frac{8}{5}$

5 (b) Create a Series

Sample

BIG numerator

$$\sum_{n=0}^{\infty} n^n (x-5)^n \quad \text{Ratio Test} \quad \text{OR} \quad \sum_{n=0}^{\infty} n!\,(x-5)^n \quad \text{OR} \quad \sum_{n=0}^{\infty} (2n)!\,(x-5)^n \ldots \text{etc.}$$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\overset{(n+1)^n(n+1)}{(n+1)^{n+1}}\,(x-5)^{n+1}}{n^n\,(x-5)^n}\right| = \lim_{n\to\infty} \overset{e}{\cancel{\frac{(n+1)^n}{n^n}}}\cdot \overset{\infty}{\cancel{(n+1)}}\,|x-5| = \infty > 1$$

Diverges by Ratio Test for all x unless $x-5=0$ or $x=5$

Finally, $I=\{5\}$ ←
$R=0$

6(a) $\displaystyle\sum_{n=2}^{\infty}\left(1-\frac{5}{n^3}\right)^{n^3}$ Diverges by nth Term Divergence Test because

$$\lim_{n\to\infty}\overset{1^\infty}{\left(1-\frac{5}{n^3}\right)^{n^3}} = \lim_{x\to\infty}\overset{1^\infty}{\left(1-\frac{5}{x^3}\right)^{x^3}} = e^{\lim_{x\to\infty}\ln\left(\left(1-\frac{5}{x^3}\right)^{x^3}\right)} = e^{\lim_{x\to\infty} \overset{\infty\cdot 0}{x^3\ln\left(1-\frac{5}{x^3}\right)}} \quad \text{flip}$$

$-5x^{-3} \hookrightarrow 15x^{-4}$

$$= e^{\lim_{x\to\infty}\frac{\ln\left(1-\frac{5}{x^3}\right)}{\frac{1}{x^3}}} \;\frac{0}{0} \;\underset{L'H}{=}\; e^{\lim_{x\to\infty}\frac{\frac{1}{1-\cancel{\frac{5}{x^3}}}\left(\frac{\overset{-5}{\cancel{+15}}}{\cancel{x^4}}\right)}{\frac{-3}{x^4}}} = e^{-5} \neq 0$$

$x^{-3} \hookrightarrow -3x^{-4}$

6(b) $$\sum_{n=1}^{\infty}\frac{5}{n^7}+\frac{(-5)^n}{7^{2n}} \;\text{split}\; = \sum_{n=1}^{\infty}\frac{5}{n^7} + \sum_{n=1}^{\infty}\frac{(-5)^n}{7^{2n}}$$

Show Split using Arithmetic of Series

$$= 5\sum_{n=1}^{\infty}\frac{1}{n^7} + \sum_{n=1}^{\infty}\frac{(-5)^n}{49^n}$$

$r=\frac{-5}{49}$

$$\frac{-5}{49}+\frac{5^2}{(49)^2}-\frac{5^3}{(49)^3}+\ldots$$

Constant Multiple of a Convergent p-Series

$p=7>1$ is Convergent

Convergent by GST with $|r|=\left|\frac{-5}{49}\right|=\frac{5}{49}<1$

Careful with Absolute Values

Sum of Two Convergent Series is Convergent

Create

6(c) $\sum_{n=1}^{\infty} \frac{1}{n^4+7} \approx \sum_{n=1}^{\infty} \frac{1}{n^4}$ Converges p-Series $p=4>1$

<u>Bound Terms</u>

$\frac{1}{n^7+4} \le \frac{1}{n^4}$ ⇒ Original Series also Converges by the Comparison Test

OR

6(c) $\sum_{n=1}^{\infty} \frac{1}{n^5+n^4+n^3+n^2+n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n^5}$ Converges p-Series $p=5>1$

<u>Bound Terms</u>

$\frac{1}{n^5+n^4+n^3+n^2+n+1} \le \frac{1}{n^5}$ ⇒ Original Series also Converges by the Comparison Test

6(d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-6n+13} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{n^2-6n+13}$ Required to use Integral Test ☆

Study $\int_1^{\infty} \frac{1}{x^2-6x+13}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{(x-3)^2+4}\,dx$

$(x-3)^2 = x^2 \overset{--}{-6x} + 9$

$+4$

$\boxed{\begin{array}{l} u = x-3 \\ du = dx \end{array}}$

$\boxed{\begin{array}{l} x=1 \Rightarrow u = 1-3 = -2 \\ x=t \Rightarrow u = t-3 \end{array}}$

$$= \lim_{t\to\infty} \int_{-2}^{t-3} \frac{1}{u^2+4}\,du$$

$$= \lim_{t\to\infty} \frac{1}{2}\arctan\left(\frac{u}{2}\right)\Bigg|_{-2}^{t-3}$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\cancelto{\frac{\pi}{2}}{\arctan}\left(\frac{\cancelto{\infty}{t-3}}{\cancelto{\infty}{2}}\right) - \cancelto{-\frac{\pi}{4}}{\arctan}\left(\cancelto{-1}{\frac{-2}{2}}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} \overset{--}{+} \frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{3\pi}{4}\right) = \frac{3\pi}{8}$$ Integral Converges

⇒ Absolute Series Converges by Integral Test

Finally, the Original Series Converges by the Absolute Convergence Test

Important Final Conclusion

Note: Yes, the Absolute Series Converges by LCT and / or CT, but this problem required Integral Test.

7(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^5+7}{n^7+5} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{n^5+7}{n^7+5} \approx \sum_{n=1}^{\infty} \frac{n^5}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges p-Series $p=2>1$

$$\lim_{n\to\infty} \frac{\frac{n^5+7}{n^7+5}}{\frac{1}{n^2}} \overset{n^2}{} = \lim_{n\to\infty} \overset{1}{\cancel{\frac{n^7+7n^2}{n^7+5}}} = 1$$ Finite, Non-Zero

⇒ Absolute Series also Converges by Limit Comparison Test

⇒ Original Series is Absolutely Convergent by Definition

7(b) $\sum_{n=1}^{\infty} \frac{n!\ln n}{n^n \cdot e^n}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!\ln(n+1)}{(n+1)^{n+1}\cdot e^{n+1}}}{\frac{n'\ln n}{n^n\cdot e^n}} \right|$$

$$= \lim_{n\to\infty} \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n\cancel{(n+1)}}} \cdot \frac{\overbrace{(n+1)!}^{\cancel{(n+1)}\cancel{n!}}}{\cancel{n!}} \cdot \overset{1}{\cancel{\frac{\ln n}{\ln(n+1)}}} \cdot \underset{\frac{1}{e}}{\cancel{\frac{e^n}{e^{n+1}}}}$$

See ☆

$$= \lim_{n\to\infty} \overset{\frac{1}{e}}{\cancel{\frac{n^n}{(n+1)^n}}} \cdot \frac{1}{e} = \frac{1}{e^2} < 1$$

⇒ Series is Absolutely Convergent by the Ratio Test

☆ $$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{x+1}{x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{1}{1} = 1$$

Sample

7(c) $\sum_{n=1}^{\infty}\frac{(-1)^n}{7n+5} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty}\frac{1}{7n+5} \approx \sum_{n=1}^{\infty}\frac{1}{n}$ Diverges (Harmonic) p-Series $p=1$

$$\lim_{n\to\infty}\frac{\frac{1}{7n+5}}{\frac{1}{n}} = \lim_{n\to\infty}\cancelto{\frac{1}{7}}{\frac{n}{7n+5}} = \frac{1}{7}$$ Finite, Non-Zero

⇒ Absolute Series also Diverges by Limit Comparison Test

AST

1 Isolate $b_n = \frac{1}{7n+5} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\cancelto{0}{\frac{1}{\underset{\infty}{7n+5}}} = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{7(n+1)+5} = \frac{1}{7n+12} \le \frac{1}{7n+5} = b_n$$

OR $f(x) = \frac{1}{7x+5}$ has $f'(x) = \frac{-7}{(7x+5)^2} < 0$

Original Series Converges by the Alternating Series Test

Original Series is Conditionally Convergent by Definition

Note: (Direct) Comparison Test CT will not be helpful on the Absolute Series here because although $\frac{1}{7n+5} < \frac{1}{n}$ is a true bound, it is not a "helpful" bound since "smaller than Diverge is Inconclusive" for CT

Other C.C. Examples

$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ OR $\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+2}$ OR $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+7}}$ OR $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$

8. $$\int_0^{\frac12} x\ln(1+x^2)\,dx = \int_0^{\frac12} x\sum_{n=0}^{\infty}\frac{(-1)^n(x^2)^{n+1}}{n+1}\,dx = \int_0^{\frac12} x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}\,dx = \int_0^{\frac12}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{n+1}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+4}}{(n+1)(2n+4)}\Bigg|_0^{\frac12} = \overset{n=0}{\frac{x^4}{1\cdot4}} - \overset{n=1}{\frac{x^6}{2\cdot6}} + \overset{n=2}{\frac{x^8}{3\cdot8}} - \dots\Bigg|_0^{\frac12}$$

$$= \frac{\overset{\frac{1}{16}}{\left(\frac12\right)^4}}{\underset{1}{4}} - \frac{\overset{\frac{1}{64}}{\left(\frac12\right)^6}}{\underset{1}{12}} + \frac{\overset{\frac{1}{256}}{\left(\frac12\right)^8}}{\underset{1}{24}} - \dots - \cancelto{0}{\left(0-0+0-\dots\right)}$$

$$= \frac{1}{64} - \frac{1}{768} + \frac{1}{6{,}144} - \dots \approx \frac{1}{64} - \frac{1}{728} = \frac{12}{768} - \frac{1}{768} = \boxed{\frac{11}{768}}$$ ← ESTIMATE

Keep

MAX ERROR

Key Note: Full Sum ≠ Estimate, check Notation

Using the Alternating Series Estimation Theorem (ASET), we can Estimate the Full Series Sum using only the first two terms with Error at Most $\frac{1}{6144} < \frac{1}{5000}$ as desired.

9(a) $$\ln(4+x^2)=\int \frac{2x}{4+x^2}\,dx = \int 2x\left(\frac{1}{4+x^2}\right)dx = \int \frac{2x}{4}\left(\frac{1}{1+\frac{x^2}{4}}\right)dx = \int \frac{2x}{4}\left(\frac{1}{1-\left(\frac{-x^2}{4}\right)}\right)dx$$

$$= \int \frac{2x}{4}\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n dx = \int \frac{2x}{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^n}\,dx = \int 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{4^{n+1}}\,dx$$

Need $\left|-\frac{x^2}{4}\right|<1$

$\Rightarrow |x|^2<4$

$\Rightarrow |x|<2$

$R=2$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{4^{n+1}(2n+2)}+C = 2\left(\underset{n=0}{\frac{x^2}{4\cdot 2}} - \underset{n=1}{\frac{x^4}{4^2\cdot 4}} + \underset{n=2}{\frac{x^6}{4^3\cdot 6}} - \ldots\right)+C$$

$R=2$ STILL After Integration

Expand in Long Form to solve for $+C$

Test the center $x=0$ to solve for $+C$

$$\ln(4+0) = 2\left(\cancelto{0}{0-0+0-\ldots}\right)+C \Rightarrow C=\ln 4$$

Finally, $$\ln(4+x^2) = 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{4^{n+1}\underset{2(n+1)}{(2n+2)}}+\ln 4 \quad \text{OR} \quad = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{4^{n+1}(n+1)}+\ln 4$$

Can use Log Algebra to check Series with an optional method $\ln(A\cdot B)=\ln A+\ln B$

$$\ln(4+x^2)=\ln\left[4\cdot\left(1+\frac{x^2}{4}\right)\right]=\ln 4+\ln\left(1+\frac{x^2}{4}\right)=\ln 4+\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{x^2}{4}\right)^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{4^{n+1}(n+1)}+\ln 4$$ Match!

9(b) $$\frac{1}{2}\left(e^x+e^{-x}\right)=\frac{1}{2}\left(1+\cancel{x}+\frac{x^2}{2!}+\cancel{\frac{x^3}{3!}}+\frac{x^4}{4!}+\ldots+1-\cancel{x}+\frac{x^2}{2!}-\cancel{\frac{x^3}{3!}}+\frac{x^4}{4!}-\ldots\right)$$

All odd powered terms cancel
Two copies of each even powered term remain

$$=\frac{1}{\cancel{2}}\left(\cancel{2}+\cancel{2}\frac{x^2}{2!}+\cancel{2}\frac{x^4}{4!}+\cancel{2}\frac{x^6}{6!}+\ldots\right)$$ 2's Cancel

$$= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

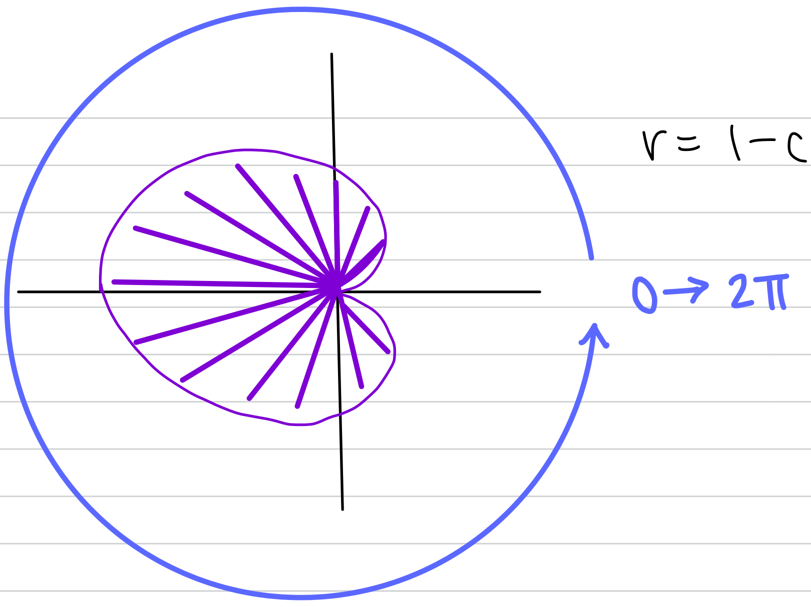
Same terms as cosine but not alternating.

Alternate Method:

Work up a "chart Method" using the Definition of Maclaurin Series $f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$

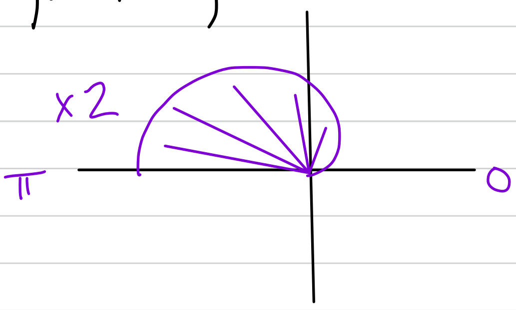
Recall: the "Chart Method" is always available if have nice derivatives

10.



$r = 1 - \cos\theta$

OR use Double by Symmetry



$$\text{Area} = \frac{1}{2}\int_0^{2\pi}(\text{Polar Radius})^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$ FOIL

$$\frac{1}{2} + \frac{\cos(2\theta)}{2} \quad \text{split}$$

$$= \frac{1}{2}\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(1 - 2\cos\theta + \cancel{\frac{1+\cos(2\theta)}{2}}\right)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\frac{3}{2} - 2\cos\theta + \frac{\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\left(\frac{3}{2}\theta - 2\sin\theta + \frac{\sin(2\theta)}{4}\right)\Bigg|_0^{2\pi}$$

$$= \frac{1}{2}\left(\left(\frac{3}{\cancel{2}}(\cancel{2}\pi) - \cancelto{0}{2\sin(2\pi)} + \cancelto{0}{\frac{\sin(4\pi)}{4}}\right) - \left(0 - \cancelto{0}{2\sin 0} + \cancelto{0}{\frac{\sin 0}{4}}\right)\right)$$

$$= \boxed{\frac{3\pi}{2}}$$

11(a) $r = 2 + 2\cos\theta \quad r = 6\cos\theta$

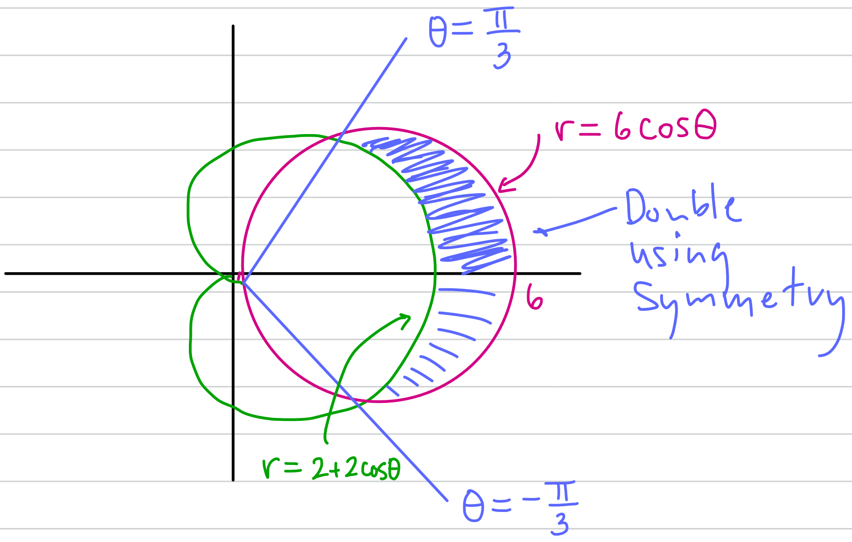
Intersect?

$2 + 2\cos\theta = 6\cos\theta$

$2 = 4\cos\theta$

$\cos\theta = \frac{1}{2}$

↳ $\theta = \pm\frac{\pi}{3}$



$$\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(\cancelto{6\cos\theta}{\text{Outer Polar Radius}})^2 - (\cancelto{2+2\cos\theta}{\text{Inner Polar Radius}})^2\,d\theta$$

$$= \boxed{\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(6\cos\theta)^2 - (2+2\cos\theta)^2\,d\theta}$$

STOP Do Not Evaluate

11(a) Continued

Alternate Integrals

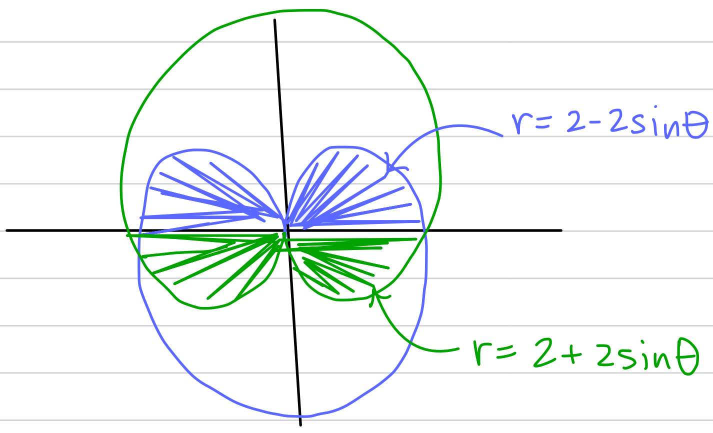
$$2\Bigg(\frac{1}{2}\int_0^{\frac{\pi}{3}}(6\cos\theta)^2-(2+2\cos\theta)^2\,d\theta$$

Double using Symmetry

OR

$$2\Bigg(\frac{1}{2}\int_{-\frac{\pi}{3}}^{0}(6\cos\theta)-(2+2\cos\theta)^2\,d\theta\Bigg)$$

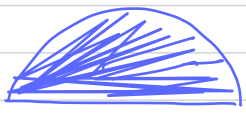
11(b) $r=2+2\sin\theta \quad r=2-2\sin\theta$



2−2sinθ

$$\text{Area}=\frac{1}{2}\int_\alpha^\beta(\text{Polar Radius})^2\,d\theta = 4\Bigg(\frac{1}{2}\int_0^{\frac{\pi}{2}}(2-2\sin\theta)^2\,d\theta\Bigg)$$

using Symmetry



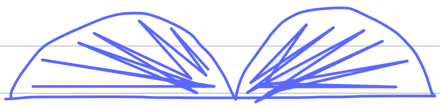
STOP

Do Not Evaluate

Alternate Integrals

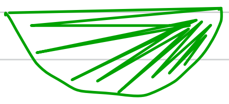
OR

$$=2\Bigg(\frac{1}{2}\int_0^{\pi}(2-2\sin\theta)^2\,d\theta$$

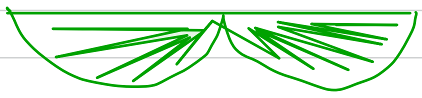


⋮

$$=4\Bigg(\frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}}(2+2\sin\theta)^2\,d\theta\Bigg)$$

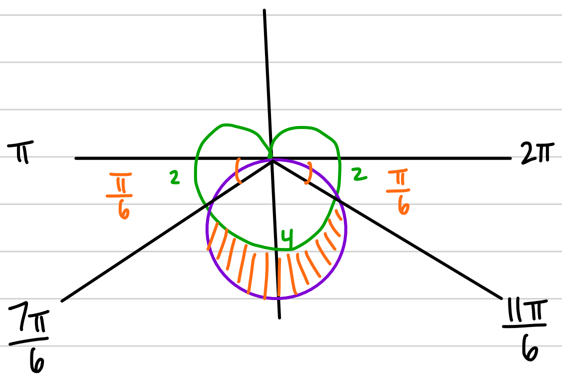


$$=2\Bigg(\frac{1}{2}\int_{\pi}^{2\pi}(2+2\sin\theta)^2\,d\theta$$



$$=4\Bigg(\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi}(2+2\sin\theta)^2\,d\theta$$

11(c)



Intersect?

$$1-\sin\theta=-3\sin\theta$$
$$1=-2\sin\theta$$
$$\sin\theta=-\frac{1}{2}$$
$$\theta=\frac{7\pi}{6},\ \frac{11\pi}{6}$$

$$\text{Area}=\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}(\text{Outer Radius})^2-(\text{Inner Radius})^2\,d\theta$$

$$=\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}(-3\sin\theta)^2-(1-\sin\theta)^2\,d\theta$$

STOP

Do Not Evaluate

Alternate Integrals

Symmetry:

$$=\cancel{2}\left(\cancel{\frac{1}{2}}\int_{\frac{7\pi}{6}}^{\frac{\pi}{2}}(-3\sin\theta)^2-(1-\sin\theta)^2\,d\theta\right)$$

Double using Symmetry

OR

$$=\cancel{2}\left(\cancel{\frac{1}{2}}\int_{\frac{3\pi}{2}}^{\frac{11\pi}{6}}(-3\sin\theta)^2-(1-\sin\theta)^2\,d\theta\right)$$

Double using Symmetry