Math 121 Final December 16, 2025

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, or $\arctan(\sqrt{3})$, should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] (a) Use Series to show that $\lim\limits_{x\to 0} \dfrac{\cos(3x) - 1 - \arctan(2x) + 2x}{e^{-4x} - 1 + 4x} = \boxed{-\dfrac{9}{16}}$

(b) Compute $\lim\limits_{x\to 0} \dfrac{\cos(3x) - 1 - \arctan(2x) + 2x}{e^{-4x} - 1 + 4x}$ again using L'Hôpital's Rule.

2. [18 Points] Compute the following integral. Simplify.

(a) $\displaystyle\int \frac{1}{(x^2+4)^2}\, dx$ (b) $\displaystyle\int x^4 \arcsin x \, dx$

3. [24 Points] For each of the following **Improper** integrals, determine whether it Converges or Diverges. If it converges, find its value. Simplify.

(a) $\displaystyle\int_0^e x^3 \ln x \, dx$ (b) $\displaystyle\int_0^{e^3} \frac{1}{x\left(3 + (\ln x)^2\right)}\, dx$ (c) $\displaystyle\int_{-7}^0 \frac{x+15}{x^2+6x-7}\, dx$

4. [24 Points] Find the **Sum** of each of the following series (which do converge). Simplify.

(a) $-\dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dfrac{1}{9} - \ldots$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, (\ln 9)^n}{n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, \pi^{2n}}{9^n\, (2n+1)!}$

(d) $-1 + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{5} + \ldots$ (e) $1 + 1 + 1 - \dfrac{\pi^2}{2!} + \dfrac{\pi^4}{4!} - \dfrac{\pi^6}{6!} + \ldots$ (f) Show $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n - 2}{5^n} = \boxed{-\frac{35}{18}}$

5. [22 Points] (a) Find the Interval/Radius of Convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, (5x+7)^n}{(5n+7)^2\, 8^n}$

(b) Create a Power Series centered at $a = 5$ which has a Radius of Convergence, $R = 0$. Once you create your series, then proceed to justify that the Interval of Convergence is indeed $I = \{5\}$.

6. [25 Points] Determine whether each series **Converges** or **Diverges**. Name any Convergence Test(s) you use; justify all of your work.

(a) $\displaystyle\sum_{n=2}^{\infty} \left(1 - \frac{5}{n^3}\right)^{n^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^7} + \frac{(-5)^n}{7^{2n}}$

(c) Create an Infinite Series which **Converges by the Comparison Test**. Continue on to Prove the Series Converges by the Comparison Test.

(d) Use the **Absolute Convergence Test** to show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - 6n + 13}$ **Converges**. You are required to use the **Integral Test** on the Absolute Series.

7. [24 Points] Determine whether the series is **Absolutely Convergent** or **Conditionally Convergent** or **Divergent**. Name any Convergence Test(s) you use, justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^5 + 7}{n^7 + 5}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{n!\,(\ln n)}{n^n \cdot e^n}$

(c) Create an Alternating series which is **Conditionally Convergent**. Continue on to justify that this series is Conditionally Convergent.

8. [8 Points] Use Series to Estimate $\displaystyle\int_0^{\frac{1}{2}} x \ln\left(1 + x^2\right)\, dx$ with error less than $\dfrac{1}{5000}$. Simplify.

Tips: $(64) \cdot (12) = 768$ and $(256) \cdot (24) = 6144$ and $2^6 = 64$ and $2^8 = 256$

9. [14 Points] Answer in Sigma $\displaystyle\sum_{n=0}^{\infty}$ notation. **State** the Radius of Convergence for each.

(a) Find the MacLaurin Series for $\ln(4 + x^2)$. Hint: $\ln(4 + x^2) = \displaystyle\int \frac{2x}{4 + x^2}\, dx = \ldots$. Solve C

(b) Show that the MacLaurin Series for $\frac{1}{2}\left(e^x + e^{-x}\right)$ equals $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$. Justify.

10. [8 Points] **COMPUTE** the Area bounded inside the Cardioid $r = 1 - \cos\theta$. Sketch and shade the bounded region.

11. [18 Points] For **each** of the following problems, do the following **THREE** things:

1. **Sketch** the Polar curve(s) and **Shade** the described bounded region.

2. Set-Up but **DO NOT EVALUATE** an Integral representing the area of the described bounded region.

3. Set-Up but **DO NOT EVALUATE** another **slightly different** Integral representing the same area of the described bounded region.

(a) The **Area** bounded Outside the polar curve $r = 2 + 2\cos\theta$ and Inside $r = 6\cos\theta$.

(b) The **Area** bounded Inside **Both** of the curves $r = 2 + 2\sin\theta$ and $r = 2 - 2\sin\theta$.

(c) The **Area** bounded Outside the polar curve $r = 1 - \sin\theta$ and Inside $r = -3\sin\theta$.