# Math 121 Final Exam Fall 24 Answer Key

1(a)
$$\lim_{x\to 0}\frac{x^2+4x-\arctan(4x)}{1-3x-e^{-3x}} = \lim_{x\to 0}\frac{x^2+4x-\left((4x)-\frac{(4x)^3}{3}+\frac{(4x)^5}{5}-\frac{(4x)^7}{7}+\dots\right)}{1-3x-\left(1+(-3x)+\frac{(-3x)^2}{2!}+\frac{(-3x)^3}{3!}+\dots\right)}$$

$$1-3x+\frac{3^2x^2}{2!}-\frac{3^3x^3}{3!}+\dots$$

$$= \lim_{x\to 0}\frac{x^2+\cancel{4x}-\cancel{4x}+\frac{4^3x^3}{3}-\frac{4^5x^5}{5}+\frac{4^7x^7}{7}-\dots}{\cancel{1}-3\cancel{x}-\cancel{1}+3\cancel{x}-\frac{3^2x^2}{2!}+\frac{3^3x^3}{3!}-\dots}$$

$$= \lim_{x\to 0}\frac{x^2+\frac{4^3x^3}{3}-\frac{4^5x^5}{5}+\frac{4^7x^7}{7}-\dots}{-\frac{9x^2}{2!}+\frac{3^3x^3}{3!}-\frac{3^4x^4}{4!}+\dots}\quad \frac{1}{x^2}$$

$$= \lim_{x\to 0}\frac{1+\frac{4^3x}{3}-\frac{4^5x^3}{5}+\frac{4^7x^5}{7}-\dots}{-\frac{9}{2}+\frac{3^3x}{3!}-\frac{3^4x^2}{4!}+\dots}$$

$$\to 0$$

$$= \frac{1}{-\frac{9}{2}} = -\frac{2}{9}$$

1(b)
$$\lim_{x\to}\frac{x^2+4x-\arctan(4x)}{1-3x-e^{-3x}}\ \frac{0}{0} \underset{L'H}{=} \lim_{x\to 0}\frac{2x+4-\frac{4}{1+(4x)^2}}{-3+3e^{-3x}}\ \frac{0}{0}$$

chain rule

$$-4(1+16x^2)^{-1} \xrightarrow{\frac{d}{dx}} +4(1+16x^2)^{-2}(32x)$$

$$\underset{L'H}{=} \lim_{x\to 0}\frac{2+\frac{128x}{(1+16x^2)^2}}{-9e^{-3x}}$$

$$\to 0, \quad \to 1, \quad \to -9$$

$$= \frac{2}{-9} = -\frac{2}{9}$$

Match!

$$2\text{(a)} \quad \int \frac{x^3}{(x^2+4)^{7/2}}\,dx = \int \frac{x^3}{\left(\sqrt{x^2+4}\right)^7}\,dx = \int \frac{8\tan^3\theta}{\left(\sqrt{4\tan^2\theta+4}\right)^7} \cdot 2\sec^2\theta\,d\theta$$

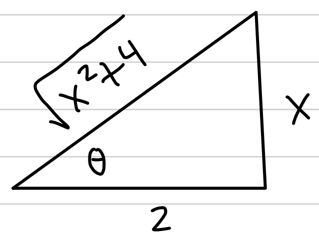
Don't Drop

$$\sqrt{4(\tan^2\theta+1)}$$

$$\sqrt{4\sec^2\theta}$$

$$\boxed{\begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta\,d\theta \end{array}}$$

$$\tan\theta = \frac{x}{2}$$




$$= \int \frac{\overset{2^3}{8}\tan^3\theta}{\left(\sqrt{4\sec^2\theta}\right)^7} \cdot 2\sec^2\theta\,d\theta$$

$$(2\sec\theta)^7$$

$$= \frac{\cancel{2^4}}{\underset{2^3}{\cancel{2^7}}} \int \frac{\tan^3\theta}{\underset{\sec^5\theta}{\cancel{\sec^7\theta}}} \cancel{\sec^2\theta}\,d\theta$$

$$= \frac{1}{8}\int \tan^3\theta \cdot \cos^5\theta\,d\theta$$

$$= \frac{1}{8}\int \frac{\sin^3\theta}{\cancel{\cos^3\theta}} \cdot \overset{\cos^2\theta}{\cancel{\cos^5\theta}}\,d\theta$$

$$\boxed{\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \\ -du = \sin\theta\,d\theta \end{array}}$$

$$= \frac{1}{8}\int \sin^3\theta \cdot \cos^2\theta\,d\theta$$

ODD Power

$$= \frac{1}{8}\int \sin^2\theta\ \cos^2\theta \cdot \sin\theta\,d\theta$$

$$= \frac{1}{8}\int (1-\cos^2\theta)\cdot\cos^2\theta \cdot \sin\theta\,d\theta$$

$$= -\frac{1}{8}\int (1-u^2)\,u^2\,du$$

$$= -\frac{1}{8}\int u^2 - u^4\,du$$

$$= -\frac{1}{8}\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$

$$= -\frac{1}{8}\left(\frac{\cos^3\theta}{3} - \frac{\cos^5\theta}{5}\right) + C$$

$$= -\frac{1}{8}\left(\frac{1}{3}\left(\frac{2}{\sqrt{x^2+4}}\right)^3 - \frac{1}{5}\left(\frac{2}{\sqrt{x^2+4}}\right)^5\right) + C$$

OR

$$= -\frac{1}{\cancel{8}} \cdot \frac{1}{3} \cdot \frac{\cancel{8}}{(x^2+4)^{3/2}} + \frac{1}{\cancel{8}} \cdot \frac{1}{5} \cdot \frac{\overset{4}{\cancel{32}}}{(x^2+4)^{5/2}} + c$$

$$= -\frac{1}{3(x^2+4)^{3/2}} + \frac{4}{5(x^2+4)^{5/2}} + C$$

2(b) $\int x \arcsin x \, dx = \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \, dx$

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = x \, dx \\ du = \frac{1}{\sqrt{1-x^2}} \, dx & v = \frac{x^2}{2} \end{array}}$$

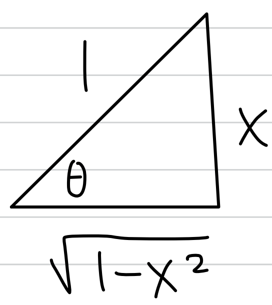
$$= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \cancel{\cos\theta} \, d\theta$$

$$\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} \to \cancel{\cos\theta}$$

$$\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta \, d\theta \end{array}} \to \theta = \arcsin x$$

$$= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \sin^2\theta \, d\theta$$

EVEN POWER




$$= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{1-\cos(2\theta)}{2} \, d\theta$$

Half-Angle

$$= \frac{x^2}{2} \arcsin x - \frac{1}{4}\left(\theta - \frac{\overbrace{\sin(2\theta)}^{\cancel{2}\sin\theta\cos\theta}}{\cancel{2}}\right) + C$$

Double Angle

$\sin\theta = x$ already

$\cos\theta = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$$\boxed{= \frac{x^2}{2} \arcsin x - \frac{1}{4} \arcsin x + \frac{1}{4} x\sqrt{1-x^2} + C}$$

note: careful not to accidentally write $\frac{d}{dx} \arcsin x$ is $\frac{1}{1+x^2}$

then you miss the Trig Sub

3(a) $\displaystyle\int_0^e \ln x \, dx = \lim_{t\to 0^+} \int_t^e \ln x \cdot 1 \, dx = \lim_{t\to 0^+} x\ln x \Big|_t^e - \int_t^e \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx$

Improper Type II

| | |
|---|---|
| $u = \ln x$ | $dv = 1\,dx$ |
| $du = \frac{1}{x}\,dx$ | $v = x$ |

$$= \lim_{t\to 0^+} x\ln x \Big|_t^e - x\Big|_t^e$$

$$= \lim_{t\to 0^+} e\,\overset{1}{\cancel{\ln e}} - \overset{\to 0}{\cancel{t\cdot\ln t}} - (e - \overset{\to 0}{\cancel{t}})$$

See (*)

$$= e - e = 0 \qquad \text{Converges}$$

(*) $\displaystyle\lim_{t\to 0^+} \overset{0\cdot(-\infty)}{t\cdot\ln t} = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} \overset{\frac{-\infty}{\infty}}{\underset{L'H}{=}} \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-1}{t^2}} \; {-t^2} = \lim_{t\to 0^+} -\overset{\to 0}{\cancel{t}} = 0$

**Key Note:** $\ln 0$ is undefined, so must "sneak attack" 0 using Limit $0^+$

3(b) $\displaystyle\int_0^{e^4} \frac{4}{x(16+(\ln x)^2)}\,dx = \lim_{t\to 0^+}\int_t^{e^4} \frac{4}{x(16+(\ln x)^2)}\,dx = \lim_{t\to 0^+}\int_{\ln t}^{4} \frac{4}{16+u^2}\,du$

Improper Type II

| | |
|---|---|
| $u = \ln x$ | $x = t \Rightarrow u = \ln t$ |
| $du = \frac{1}{x}\,dx$ | $x = e^4 \Rightarrow u = \cancel{\ln}(e^4) = 4$ |

$$= \lim_{t\to 0^+} \frac{\cancel{4}}{\cancel{4}} \arctan\left(\frac{u}{4}\right)\Big|_{\ln t}^{4}$$

$$= \lim_{t\to 0^+} \overset{\frac{\pi}{4}}{\cancel{\arctan\left(\tfrac{4}{4}\right)}} - \overset{-\frac{\pi}{2}}{\cancel{\arctan\left(\tfrac{\ln t}{4}\right)}}$$

$$= \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4} \qquad \text{Converges}$$

**Key Note:** $\ln 0$ is undefined, so must "sneak attack" 0 using Limit $0^+$

3(c) $\displaystyle\int_{-4}^{-3} \frac{8-x}{x^2+2x-8}\,dx = \int_{-4}^{-3} \frac{8-x}{(x-2)(x+4)}\,dx = \lim_{t\to -4^+}\int_t^{-3} \frac{8-x}{(x-2)(x+4)}\,dx$

Improper Type II

PFD

$$= \lim_{t\to -4^+}\int_t^{-3} \frac{1}{x-2} - \frac{2}{x+4}\,dx$$

Continued

3(c) Partial Fractions Decomposition

$$(x-2)(x+4)\left(\frac{8-x}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4}\right)(x-2)(x+4)$$

$$8-x = A(x+4)+B(x-2)$$
$$= Ax+4A+Bx-2B$$
$$= (A+B)x+(4A-2B)$$

Conditions

- $A+B=-1 \hookrightarrow B=-A-1$
- $4A-2B=8$

$$4A-2(-A-1)=8$$
$$4A+2A+2=8$$
$$6A=6$$
$$A=1 \hookrightarrow B=-2$$

$$= \lim_{t\to -4^+} \ln|x-2| - 2\ln|x+4| \Big|_t^{-3}$$

$$= \lim_{t\to-4^+} \underbrace{\ln|-5|}_{\substack{\ln 5 \\ \text{Finite}}} - \underbrace{2\ln|1|}_{\text{Finite}} - \left(\underbrace{\ln|t-2|}_{\substack{\ln 6 \\ \text{Finite}}} - 2\ln|t+4|\right)$$

$$= -(-(-\infty)) = -\infty \quad \text{Diverges}$$

Key Note: ln 0 is undefined, so must "sneak attack" 0 using Limit $0^+$

3(d) $$\int_5^{\infty} \frac{7}{x^2-4x+7}\,dx = \lim_{t\to\infty}\int_5^t \frac{7}{x^2-4x+7}\,dx = \lim_{t\to\infty}\int_5^t \frac{7}{(x-2)^2+3}\,dx$$

Improper Type I

$$\boxed{\begin{aligned} u&=x-2\\ du&=dx\end{aligned}}$$

$$\boxed{\begin{aligned} x=5 &\Rightarrow u=5-2=3\\ x=t &\Rightarrow u=t-2\end{aligned}}$$

$$= \lim_{t\to\infty}\int_3^{t-2} \frac{7}{u^2+3}\,du = \lim_{t\to\infty} \frac{7}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}}\Big|_3^{t-2}$$

$$= \lim_{t\to\infty}\frac{7}{\sqrt{3}}\left(\underbrace{\arctan\left(\tfrac{t-2}{\sqrt{3}}\right)}_{\frac{\pi}{2}} - \underbrace{\arctan\left(\tfrac{3}{\sqrt{3}}\right)}_{\frac{\pi}{3}}\right)$$

$$= \frac{7}{\sqrt{3}}\left(\underbrace{\frac{\pi}{2}-\frac{\pi}{3}}_{\frac{3\pi}{6}-\frac{2\pi}{6}}\right) = \frac{7\pi}{6\sqrt{3}} \quad \text{Converges}$$

Sums

4(a) $$\frac{2}{3}-\frac{2}{4}+\frac{2}{5}-\frac{2}{6}+\dots = 2\left(\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots\right) = 2\Big(\ln(1+1)\underbrace{-1+\tfrac{1}{2}}_{-\frac{1}{2}}\Big) = 2\left(\ln 2-\frac{1}{2}\right)$$

OR $-\left(1-\frac{1}{2}\right)$

$$\ln(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\dots$$

$$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}+\dots$$

missing

4(b) $\sum_{n=0}^{\infty} \frac{(-1)^n (\ln 9)^n}{3!\, 2^n\, n!} = \frac{1}{3!} \sum_{n=0}^{\infty} \frac{\left(-\frac{\ln 9}{2}\right)^n}{n!} = \frac{1}{6} e^{-\frac{\ln 9}{2}} = \frac{1}{6} \cancel{e}^{\cancel{\ln}(9^{-\frac{1}{2}})} = \frac{1}{6} \cdot \frac{1}{\cancel{\sqrt{9}}_{3}} = \frac{1}{18}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

4(c) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \pi^{2n}}{36^n (2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n+1)!}$ $\frac{\pi}{6}$ $\frac{\pi}{6}$

$2^{4n} = (2^2)^{2n} = 4^{2n}$

$= -\frac{6}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n+1}}{(2n+1)!} = -\frac{6}{\pi} \cancel{\sin}\left(\frac{\pi}{6}\right)^{\frac{1}{2}} = -\frac{3}{\pi}$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

4(d) $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \ldots = -\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) = -\cancel{\arctan(1)}^{\frac{\pi}{4}} = -\frac{\pi}{4}$

$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ OR $\arctan(-1)$ works too.

extra

4(e) $1 + 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \ldots = 1 + \cancel{\cos\pi}^{-1} = 1 - 1 = 0$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

4(g) $\sum_{n=0}^{\infty} \frac{(-4)^n - 2}{5^n} = \sum_{n=0}^{\infty} \frac{(-4)^n}{5^n} - \sum_{n=0}^{\infty} \frac{2}{5^n} = \sum_{n=0}^{\infty} \left(-\frac{4}{5}\right)^n - 2 \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$= \frac{1}{1-\left(-\frac{4}{5}\right)} - 2\left(\frac{1}{1-\frac{1}{5}}\right)^{\frac{5}{4}}$

$+\frac{9}{5}$ $\frac{4}{5}$

$= \frac{5}{9} - \frac{5}{2} = \frac{10}{18} - \frac{45}{18} = \frac{-35}{18}$ Match!

Note: Sum of 2 Convergent Series Converges

5(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (4x+1)^n}{(4n+1)^2\, 7^n}$ Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^n}(4x+1)^{n+1}}{(4n+5)^2\, 7^{n+1}}}{\frac{\cancel{(-1)^n}(4x+1)^n}{(4n+1)^2 \cdot 7^n}} \right| = \lim_{n\to\infty} \left|\frac{(4x+1)^{n+1}}{\cancel{(4x+1)^n}}\right| \cdot \frac{(4n+1)^2}{(4n+5)^2} \cdot \frac{\cancel{7^n}}{\cancel{7^{n+1}}_{7}}$$

$|\cancel{(4x+1)^n}(4x+1)|$ $\quad \left(\frac{4n+1}{4n+5}\right)^2 \to 1$

$$= \frac{|4x+1|}{7} < 1$$

Converges by Ratio Test when

$$\frac{|4x+1|}{7} < 1 \Rightarrow |4x+1| < 7 \Rightarrow -7 < 4x+1 < 7 \Rightarrow -8 < 4x < 6 \Rightarrow -2 < x < \frac{3}{2}$$

$-1 \quad -1 \quad -1$ $\qquad \frac{6}{4}$

Manually Test Convergence at End points

Take $x=-2$. Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overbrace{(4(-2)+1)^n}^{(-7)^n}}{(4n+1)^2\, 7^n} = \sum_{n=1}^{\infty} \frac{\overbrace{(-1)^n(-1)^n}^{\cancel{(-1)^{2n}}\to 1 \text{ even}}\cancel{7^n}}{(4n+1)^2\, \cancel{7^n}} = \sum_{n=1}^{\infty} \frac{1}{(4n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Converges p-Series $p = 2 > 1$

Bound Terms

Note: LCT also works

$$\frac{1}{(4n+1)^2} \le \frac{1}{(4n)^2} \le \frac{1}{n^2} \Rightarrow$$ Series also Converges by Comparison Test

Take $x = \frac{3}{2}$. Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n \overbrace{(\cancel{4}^{2}(\tfrac{3}{\cancel{2}})+1)^n}^{7^n}}{(4n+1)^2\, \cancel{7^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(4n+1)^2} \to \sum_{n=1}^{\infty} \frac{1}{(4n+1)^2}$$

Converges as shown above using CT.

Series Converges by Absolute Convergence Test

Note: or use AST

$[-2 \quad 0 \quad \frac{3}{2}]$; $2$, $\frac{3}{2}$, $\frac{7}{2}$ → Half $\frac{7}{4}$

Finally, $\boxed{I = \left[-2, \frac{3}{2}\right] \quad R = \frac{7}{4}}$

5(b) $\displaystyle\sum_{n=1}^{\infty} n^n (x-7)^n$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\overbrace{(n+1)^{n+1}}^{(n+1)^n(n+1)}(x-7)^{n+1}}{n^n\,(x-7)^n}\right| = \lim_{n\to\infty} \underbrace{\frac{(n+1)^n}{n^n}}_{\to e} \cdot \underbrace{(n+1)}_{\to\infty}|x-7| = \infty > 1$$

Diverges by Ratio Test for all $x$ unless $x-7=0$ or $x=7$

Finally, $\boxed{I = \{7\} \quad R = 0}$

5(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{(2n)!}$ BIG! Ratio Test

or $n!, n^n, (n!)^2 \ldots$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-5)^{n+1}}{(2(n+1))!}}{\frac{(x-5)^n}{(2n)!}}\right| = \lim_{n\to\infty} \left|\frac{\overbrace{(x-5)^{n+1}}^{|x-5|}}{\cancel{(x-5)^n}}\right| \cdot \frac{\cancel{(2n)!}}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)\cancel{(2n)!}}} = \lim_{n\to\infty} \frac{|x-5|}{\underbrace{(2n+2)(2n+1)}_{\infty}} \to 0 = 0 < 1$$

Always Converges by the Ratio Test for all Real Numbers $x$

Finally, $\boxed{I = (-\infty, \infty) \quad R = \infty}$

6(a) $\sum_{n=2}^{\infty}\left(1-\frac{7}{n^4}\right)^{n^4}$ Diverges by $n^{th}$ Term Divergence Test because

$$\lim_{n\to\infty}\left(1-\frac{7}{n^4}\right)^{n^4} = \lim_{x\to\infty}\left(1-\frac{7}{x^4}\right)^{x^4} = e^{\lim_{x\to\infty}\ln\left(\left(1-\frac{7}{x^4}\right)^{x^4}\right)}$$

$1^{\infty}$

$\infty \cdot 0$ $\quad \frac{0}{0}$

$$= e^{\lim_{x\to\infty} x^4\ln\left(1-\frac{7}{x^4}\right)} = e^{\lim_{x\to\infty}\frac{\ln\left(1-\frac{7}{x^4}\right)}{\frac{1}{x^4}}}$$

$$\underset{L'H}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{1-\frac{7}{x^4}}\cdot\left(\frac{28}{x^5}\right)}{-\frac{4}{x^5}}} = e^{-7} \neq 0$$

$\frac{1}{1-\cancel{\frac{7}{x^4}}} \to 1$, $\cancel{\frac{7}{x^4}} \to 0$, $\frac{28}{x^5}\cdot\left(-\frac{x^5}{4}\right) = -7$

6(b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4+7} \xrightarrow{A.S.} \sum_{n=1}^{\infty}\frac{1}{n^4+7} \approx \sum_{n=1}^{\infty}\frac{1}{n^4}$ Converges p-Series $p=4>1$

<u>Bound Terms</u>

$\frac{1}{n^7+4} \le \frac{1}{n^4}$ $\Rightarrow$ Absolute Series also Converges by the Comparison Test

The Original Series Converges by the Absolute Convergence Test $\xleftarrow{ACT}$

6(c) $\sum_{n=2}^{\infty}\frac{\ln n}{n^4}$ $\hookrightarrow$ Study Related Integral

$$\int_2^{\infty}\frac{\ln x}{x^4}\,dx = \lim_{t\to\infty}\int_2^t \ln x\cdot x^{-4}\,dx = \lim_{t\to\infty} -\frac{\ln x}{3x^3}\Big|_2^t + \frac{1}{3}\int_2^t x^{-4}\,dx$$

$$\boxed{\begin{matrix} u=\ln x & dv = x^{-4}\,dx \\ du=\frac{1}{x}\,dx & v=\frac{x^{-3}}{-3}\end{matrix}}$$

$$= \lim_{t\to\infty}\frac{-\ln x}{3x^3}\Big|_2^t + \frac{1}{3}\left(\frac{x^{-3}}{-3}\right)\Big|_2^t$$

$$= \lim_{t\to\infty} -\frac{\ln t}{3t^3} + \frac{\ln 2}{24} - \cancel{\frac{1}{9t^3}} + \frac{1}{72}$$

$$\underset{L'H}{=} \lim_{t\to\infty} -\frac{\frac{1}{t}}{9t^2} + \frac{\ln 2}{24} + \frac{1}{72}$$

$\cancel{\frac{-1}{9t^3}} \to 0$

$$= \frac{\ln 2}{24} + \frac{1}{72} \quad \text{Integral Converges}$$

$\Rightarrow$ The Series Converges by the Integral Test

(6d) $\sum_{n=2}^{\infty} \frac{n^4}{\ln n}$ Diverges by nTDT because

$$\lim_{n\to\infty} \frac{n^4}{\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{x^4}{\ln x} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{4x^3}{\frac{1}{x}} = \lim_{x\to\infty} 4x^4 = \infty \neq 0$$

6(e) $-3 - \frac{3}{2} - \underset{-\frac{3}{3}}{1} - \frac{3}{4} - \frac{3}{5} - \underset{-\frac{3}{6}}{\frac{1}{2}} \ldots = -3\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots\right)$

$$= -3 \sum_{n=1}^{\infty} \frac{1}{n}$$

Constant Multiple of a Divergent (Harmonic) p-Series $p=1$ is Divergent

6(f) $\sum_{n=1}^{\infty} \frac{1}{7^n} + \frac{4}{n^7} \overset{\text{split}}{=} \sum_{n=1}^{\infty} \frac{1}{7^n} + 4\sum_{n=1}^{\infty} \frac{1}{n^7}$

Converges by GST with $|r| = \left|\frac{1}{7}\right| = \frac{1}{7} < 1$

Constant Multiple of a Convergent p-Series $p = 7 > 1$ is Convergent

$$\boxed{\text{Original Series Converges because the Sum of two Convergent Series is Convergent}}$$

$$7(a)\quad \sum_{n=1}^{\infty} (-1)^n \frac{n^4+7}{n^7+4} \xrightarrow{AS} \sum_{n=1}^{\infty} \frac{n^4+7}{n^7+4} \approx \sum_{n=1}^{\infty} \frac{n^4}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$ Converges p-Series $p=3>1$

$$\lim_{n\to\infty} \frac{\frac{n^4+7}{n^7+4}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^7+7n^3}{n^7+4} = 1$$ Finite, Non-Zero

$\Rightarrow$ Absolute Series also Converges by Limit Comparison Test

$\Rightarrow$ Original Series is $\boxed{\text{Absolutely Convergent}}$ by Definition

$$7(b)\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n+7} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{4n+7} \approx \sum_{n=1}^{\infty} \frac{1}{n}$$ Diverges (Harmonic) p-Series $p=1$

$$\lim_{n\to\infty} \frac{\frac{1}{4n+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{4n+7} = \frac{1}{4}$$ Finite, Non-Zero

$\Rightarrow$ Absolute Series also Diverges by Limit Comparison Test

AST

1 Isolate $b_n = \frac{1}{4n+7} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{4n+7} = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{4(n+1)+7} = \frac{1}{4n+11} \le \frac{1}{4n+7} = b_n$$

Original Series Converges by the Alternating Series Test

Original Series is $\boxed{\text{Conditionally Convergent}}$ by Definition

8. Estimate

$$\int_0^1 x^3 \sin(x^2)\,dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(2n+1)!}\,dx$$

$x^{4n+2}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+6}}{(2n+1)!\,(4n+6)} \Bigg|_0^1 = \underbrace{\frac{x^6}{1\cdot 6}}_{n=0} - \underbrace{\frac{x^{10}}{(3!)\cdot(10)}}_{n=1} + \underbrace{\frac{x^{14}}{(5!)\cdot(14)}}_{n=2} - \underbrace{\frac{x^{18}}{(7!)\cdot(18)}}_{n=3} + \dots \Bigg|_0^1$$

$$= \frac{1}{6} - \frac{1}{60} + \frac{1}{1680} - \frac{1}{90{,}720} + \dots - \cancel{(0-0+0-\dots)}$$

Switch to Estimate

$$\approx \frac{280}{1680} - \frac{28}{1680} + \frac{1}{1680} = \frac{253}{1680}$$ Estimate

Using the ASET, we can Estimate the Full Sum using only the First three terms and the Error in Estimating will be at Most the Absolute Value of the first neglected term, here $\frac{1}{90{,}720} < \frac{1}{10{,}000}$ as desired

Key Note: Full Sum $\neq$ Estimate, check Notation

9(a) Two Methods $R=\infty$ for $\cos x$

1. Differentiation

$$\cos x = \frac{d}{dx}\left(\sin x\right) = \frac{d}{dx}\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n \cancel{(2n+1)} x^{2n}}{\underbrace{(2n+1)!}_{\cancel{(2n+1)}(2n)!}} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$ Match!

2. Chart Method / "By Definition"

| | |
|---|---|
| $f(x) = \cos x$ | $f(0) = \cos 0 = 1$ |
| $f'(x) = -\sin x$ | $f'(0) = -\sin 0 = 0$ |
| $f''(x) = -\cos x$ | $f''(0) = -\cos 0 = -1$ |
| $f'''(x) = \sin x$ | $f'''(0) = \sin 0 = 0$ |
| $f^{(4)}(x) = \cos x$ | $f^{(4)}(0) = \cos 0 = 1$ |
| $\vdots$ | $\vdots$ |

Maclaurin Series

$$\cancelto{1}{f(0)} + \cancelto{0}{f'(0)}x + \frac{\cancelto{-1}{f''(0)}}{2!}x^2 + \frac{\cancelto{0}{f'''(0)}}{3!}x^3 + \frac{\cancelto{1}{f^{(4)}(0)}}{4!}x^4 + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$ Match!

Other Option

OR 3. $\cos x = \int -\sin x \, dx$ OR short form $\cos x = \int -\sum_{n=0}^{\infty}$

$$= \int -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)dx$$

missing lead term 1

$$= \frac{-x^2}{2} + \frac{x^4}{\underbrace{3!\cdot 4}_{4!}} - \frac{x^6}{\underbrace{5!\cdot 6}_{6!}} + \frac{x^8}{\underbrace{7!\cdot 8}_{8!}} - \ldots + \cancelto{1}{C}$$

Plug in $x=0$ to both sides to solve for $+C$

$$\cancelto{1}{\cos 0} = \cancelto{0}{-0+0-0+0-\ldots} + C \Rightarrow C = 1$$

Finally, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ Match!

9(b) $\ln(9+x^2) = \int \frac{2x}{9+x^2}\,dx = \int 2x\left(\frac{1}{9+x^2}\right)dx = \int \frac{2x}{9}\left(\frac{1}{1+\frac{x^2}{9}}\right)dx$

$$= \int \frac{2x}{9}\left(\frac{1}{1-\left(-\frac{x^2}{9}\right)}\right)dx = \int \frac{2x}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{9}\right)^n dx$$

Need $\left|-\frac{x^2}{9}\right| < 1$

$|x^2| = |x|^2 < 9$

$\Rightarrow |x| < 3$

$\Rightarrow R = 3$

$$= \int \frac{2x}{9}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{9^n}\,dx = \int 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{9^{n+1}}\,dx$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{9^{n+1}(2n+2)} + C$$

$$\ln(9+x^2) = 2\left(\frac{x^2}{9\cdot 2} - \frac{x^4}{9^2 4} + \frac{x^6}{9^3\cdot 6} - \dots\right) + C$$

Test $x=0$

$$\ln(9+0) = 2(\cancel{0-0+0-\dots}) + C$$

0

$\Rightarrow \ln 9 = C$

Finally, $\ln(9+x^2) = 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{9^{n+1}(2n+2)} + \ln 9$

$R=\infty$

9(c) $\frac{1}{2}\left(e^x - e^{-x}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)$ Substitution OR Chart Method work(s)

$$= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots\right)\right)$$

$$= \frac{1}{2}\left(\cancel{1} + x + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}} + \frac{x^5}{5!} + \dots - \cancel{1} + x - \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} - \cancel{\frac{x^4}{4!}} + \frac{x^5}{5!} - \dots\right)$$

Note: All Even Powered Terms Cancel

$$= \cancel{\frac{1}{2}}\left(\cancel{2}x + \cancel{2}\cdot\frac{x^3}{3!} + \cancel{2}\cdot\frac{x^5}{5!} + \cancel{2}\cdot\frac{x^7}{7!} + \dots\right)$$

2 copies of odd terms left

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

All ODD Terms
Looks like sinx Series with no Alternating

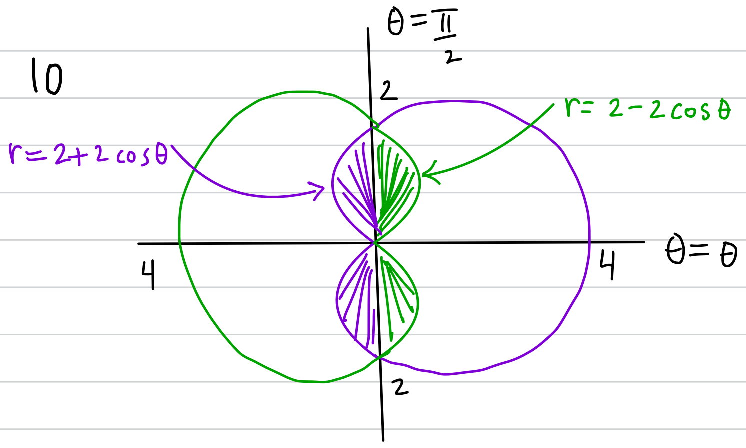
$$= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

Match

$R = \infty$

Note: OR use Chart Method / Maclaurin Series Definition to Match Formula

10




$$\text{Area} = \boxed{4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (\text{Radius})^2\, d\theta\right)}$$

Quadruple using Symmetry

$$= \boxed{4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2-2\cos\theta)^2\, d\theta\right)}$$

More ↓ ⋮

OR $= 4\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (2+2\cos\theta)^2\, d\theta\right)$ OR $= 4\left(\frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} (2+2\cos\theta)^2\, d\theta\right)$

OR $= 4\left(\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi} (2-2\cos\theta)^2\, d\theta\right)$

OR $= 2\left(\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2-2\cos\theta)^2\, d\theta\right)$ OR $= 2\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2+2\cos\theta)^2\, d\theta\right)$

Double using Symmetry

Not Needed just extra options to Compute

Compute

$$\overset{2}{\cancel{4}}\left(\frac{1+\cos(2\theta)}{\cancel{2}}\right)$$

$$\text{Area} = \overset{2}{\cancel{4}}\left(\cancel{\frac{1}{2}}\int_0^{\frac{\pi}{2}} (2-2\cos\theta)^2\, d\theta\right) = 2\int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + 4\cos^2\theta\, d\theta$$

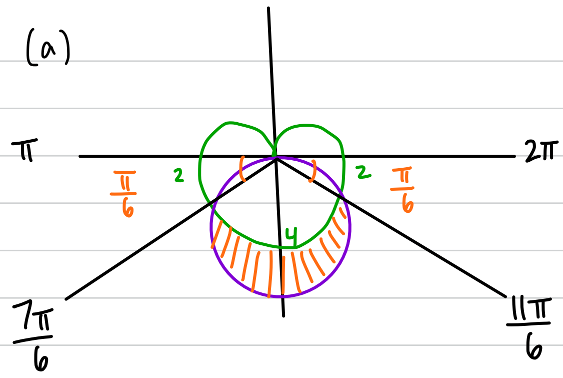
$$= 2\int_0^{\frac{\pi}{2}} \overbrace{4 - 8\cos\theta + 2}^{6} + 2\cos(2\theta)\, d\theta$$

$$= 2\left(6\theta - 8\sin\theta + \cancel{2}\left(\frac{\sin(2\theta)}{\cancel{2}}\right)\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= 2\left(\overset{3\pi}{6\left(\frac{\pi}{\cancel{2}}\right)} - 8\,\overset{1}{\cancel{\sin\left(\frac{\pi}{2}\right)}} + \overset{0}{\cancel{\sin\left(\cancel{2}\left(\frac{\pi}{\cancel{2}}\right)\right)}} - \left(0 - 8\,\overset{0}{\cancel{\sin 0}} + \overset{0}{\cancel{\sin 0}}\right)\right)$$

$$= 6\pi - 16$$ Match!

11 (a)




Intersect?

$$2-2\sin\theta=-6\sin\theta$$

$$-4\sin\theta=2$$

$$\sin\theta=-\tfrac{1}{2}$$

$$\hookrightarrow \theta=\tfrac{7\pi}{6},\ \tfrac{11\pi}{6}$$

$$\text{Area}=\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (\text{Outer Radius})^2-(\text{Inner Radius})^2\,d\theta$$

$$=\boxed{\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (-6\sin\theta)^2-(2-2\sin\theta)^2\,d\theta}$$

Do Not Evaluate

Symmetry:

$$=\boxed{\cancel{2}\left(\cancel{\frac{1}{2}}\int_{\frac{7\pi}{6}}^{\frac{\pi}{2}} (-6\sin\theta)^2-(2-2\sin\theta)^2\,d\theta\right)}$$

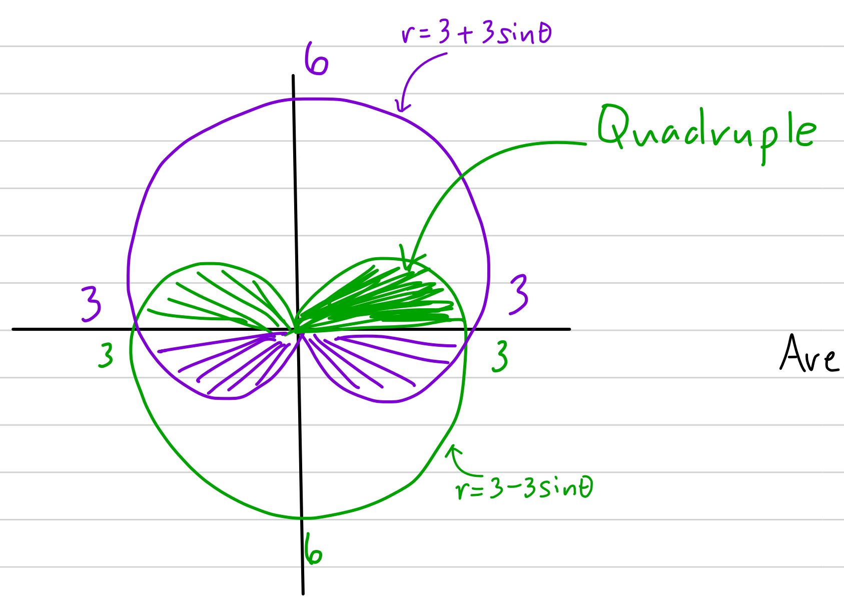
Double using Symmetry

OR//

$$=\boxed{\cancel{2}\left(\cancel{\frac{1}{2}}\int_{\frac{3\pi}{2}}^{\frac{11\pi}{6}} (-6\sin\theta)^2-(2-2\sin\theta)^2\,d\theta\right)}$$

Double using Symmetry

11 (b)



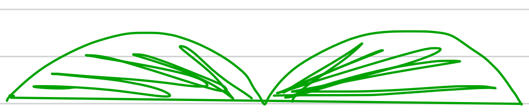

$$\text{Area}=\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$

Quadruple

$$=\boxed{4\left(\frac{1}{2}\int_{0}^{\frac{\pi}{2}} (3-3\sin\theta)^2\,d\theta\right)}$$

Symmetry

$$\text{Area}=\boxed{2\left(\frac{1}{2}\int_{0}^{\pi} (3-3\sin\theta)^2\,d\theta\right)}$$



Double

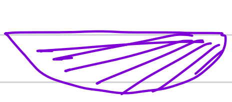
OR//

$$=\boxed{2\left(\frac{1}{2}\int_{\pi}^{2\pi} (3+3\sin\theta)^2\,d\theta\right)}$$

Double

OR//

$$=\boxed{4\left(\frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} (3+3\sin\theta)^2\,d\theta\right)}$$



Quadruple

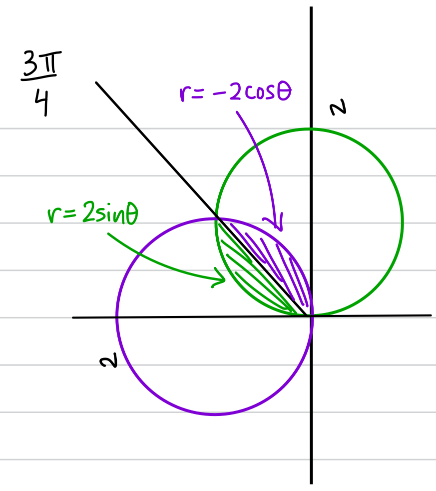
OR//

$$=\boxed{\frac{1}{2}\int_{0}^{\pi} (3-3\sin\theta)^2\,d\theta+\frac{1}{2}\int_{\pi}^{2\pi} (3+3\sin\theta)^2\,d\theta}$$


1 each

⋮

11 (c)




Intersect ?

$\cancel{2}\sin\theta = -\cancel{2}\cos\theta$

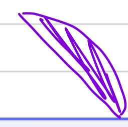
$\sin\theta = -\cos\theta$

$\tan\theta = -1$

$\hookrightarrow -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$

$$\text{Area} = 2\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (\text{Polar Radius})^2 \, d\theta\right)$$

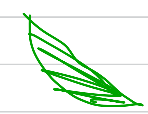
Double using Symmetry



Double

$$= \boxed{2\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (-2\cos\theta)^2 \, d\theta\right)}$$

OR//



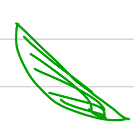
Double

$$= \boxed{2\left(\frac{1}{2}\int_{\frac{3\pi}{4}}^{\pi} (2\sin\theta)^2 \, d\theta\right)}$$

OR//

$$= \boxed{\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (-2\cos\theta)^2 \, d\theta + \frac{1}{2}\int_{\frac{3\pi}{4}}^{\pi} (2\sin\theta)^2 \, d\theta}$$

+



1 each