Inverse Trigonometric Functions Overview

In this handout, we will review two main Inverse Trigonometric Functions, Inverse Sine and Inverse Tangent by studying their related

1. Function properties
2. Limits
3. Derivatives
4. Integrals

Inverse Sine:

Start by considering the graph for the sine function, $y = \sin x$.

Recall, that a *one-to-one* or 1-1 function $f(x)$ is one where different inputs map to unique, distinct outputs. That is, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

Notice that this function $y = \sin x$ is **not** a one-to-one function on its entire domain $(-\infty, \infty)$. We can justify that statement by showing that two distinct input values map to the same output value. In particular, if we consider $x = 0$ and $x = \pi$, then they both map to the same output value $\sin 0 = \sin \pi = 0$.

We can also think about how the graph of $y = \sin x$, drawn below in purple, does not pass the Horizontal Line Test. Recall that the Horizontal Line Test would require that every horizontal line cross the graph of the function in at most one point. We represent a sample Horizontal Line in the following graph with a random Horizontal Line $y = \frac{1}{2}$, drawn in green.

Graph:

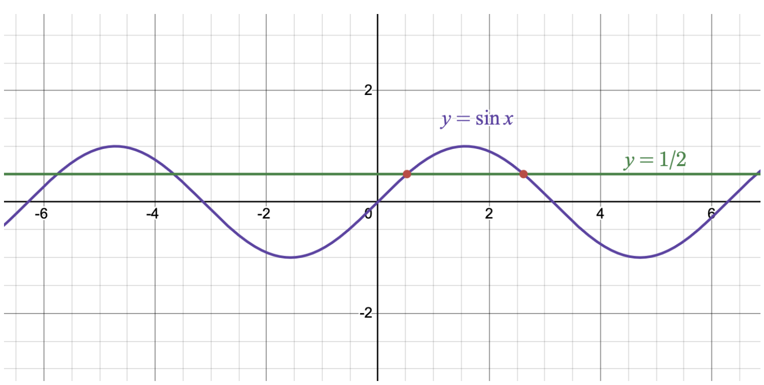

Figure 1: Sine Function $y = \sin x$

Goal: To find an Inverse function for the sine function $y = \sin x$, that is, find a **reversing** function to solve an equation like $\sin x = -\frac{1}{2}$ ...and eventually to lead to some important, new integrals.

IMPORTANT: To study the inverse sine function, we will start by *Restricting the Domain* of $y = \sin x$ to the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ where $y = \sin x$ will now be one-to-one.

We pick a largest, yet simplest restricted interval, including 0 (for nice symmetry and values), so that the restricted function now passes the Horizontal Line Test.

Graph:

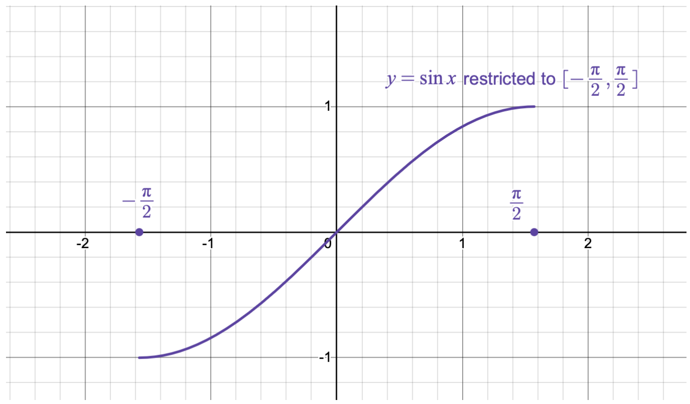

Figure 2: Sine Function $y = \sin x$ restricted to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Definition: The Inverse Sine Function $y = \sin^{-1} x$ is defined as the unique **Angle** (or Arc) y in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ such that $\sin y = x$ for x in the interval $[-1, 1]$.

$$\sin^{-1} x = y \quad \text{means} \quad \sin y = x$$

Think: Trigonometric functions will take in **Angles** for input and spit out **Values**. And Inverse Trigonometric Functions will take in **Values** and spit out **Angles**.

$$x \underset{\sin}{\overset{\arcsin}{\rightleftarrows}} y$$

$$\text{Values} \underset{\sin}{\overset{\arcsin}{\rightleftarrows}} \text{Angles}$$

Notation: Inverse Sine can also be called Arcsine meaning

$y = \sin^{-1} x$ can be interchangably written as $y = \arcsin x$.

Warning: be careful not to confuse the inverse notation with a reciprocal flip. That is, $y = \sin^{-1} x \neq \dfrac{1}{\sin x}$

Recall that the graph of Inverse Functions are mirror symmetrical flips across the line $y = x$.

Graph:

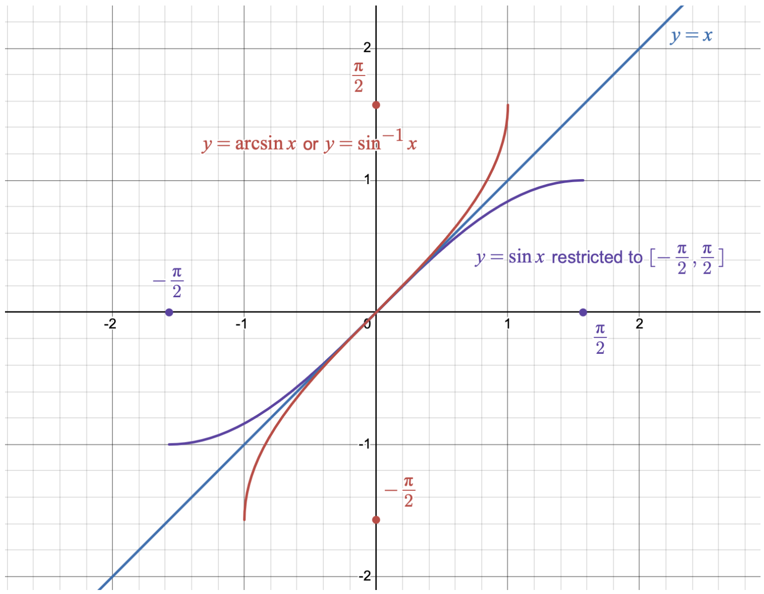

Figure 3: Symmetric Flips of $y = \sin x$ and $y = \arcsin x$

Here is a clear graph showing only $y = \arcsin x$.

Graph:

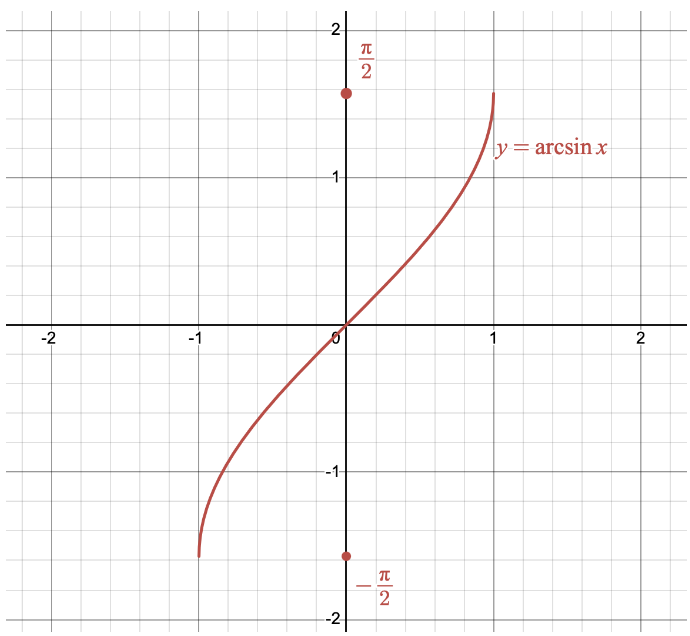

Figure 4: Inverse Sine Function $y = \arcsin x$

Let's study some properties of this Inverse Sine Function $y = \arcsin x$.

Domain= $[-1, 1]$

Recall, the *Domain* of a function is the collection of all possible input values which yield a finite output or the values for which the function is defined. Here, $\arcsin x$ is only defined for values in $[-1, 1]$

Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Recall, the *Range* of a function is the collection of all possible output values for a given function. Here $\arcsin x$ yields all output values in a restricted range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note: As Inverse Functions it makes sense, from their graphs/definitions, that

Domain $\sin x$ (restricted)	=	Range $\arcsin x$
Domain $\arcsin x$	=	Range $\sin x$ (restricted)

Value(s):

$\arcsin 0 = 0$	$\arcsin 1 = \frac{\pi}{2}$	$\arcsin(-1) = -\frac{\pi}{2}$	$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$	$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

Question: Can you use symmetry to solve other values, like $\arcsin\left(-\frac{1}{2}\right)$?

Tip: When we are computing an inverse sine value, try to think in reverse. For example, think of $\arcsin \frac{1}{2}$ as *What Angle do you know . . . such that . . .* sine of *THAT Angle* equals $\frac{1}{2}$

$$\sin \text{ (what angle?)} = \frac{1}{2} \quad \Rightarrow \quad \text{angle} = \frac{\pi}{6} \quad \text{because } \sin \frac{\pi}{6} = \frac{1}{2} \quad \Rightarrow \quad \arcsin \frac{1}{2} = \frac{\pi}{6}$$

Note: it is helpful to review Trig values in the forward direction, as it builds reverse fluency

Inverses:

$\sin(\arcsin x) = x$	for x in $[-1, 1]$
$\arcsin(\sin x) = x$	for x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Note: these inverse properties state that the sine and inverse sine invert each other, they literally reverse or unwind the other function value. They don't just "cancel".

Limits: Study the Graph

Limit	Tip
$\lim_{x\to 1^-} \arcsin x = \frac{\pi}{2}$	as input values approach 1 from the left the $\arcsin x$ output approaches $\frac{\pi}{2}$
$\lim_{x\to -1^+} \arcsin x = -\frac{\pi}{2}$	as input values approach -1 from the right the $\arcsin x$ output approaches $-\frac{\pi}{2}$

Derivatives:

Derivative	Tip
$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$	check the flip and the square root . . . and the minus sign
$\frac{d}{dx} \arcsin\left(u(x)\right) = \frac{1}{\sqrt{1-(u(x))^2}} \cdot u'(x)$	**CHAIN RULE** flips with the root, leaving *inside* function as is. . . times the derivative of the *inside nested* function

Recall: the Derivative Chain Rule can be written as

$$\boxed{\frac{d}{dx}\left(f(g(x))\right) = \underbrace{f'(g(x))}_{\substack{\text{deriv of outside}\\ \text{leave inside}}} \cdot \underbrace{g'(x)}_{\text{deriv of inside}}}$$

Let us justify why the derivative of $\arcsin x$ equals $\frac{1}{\sqrt{1-x^2}}$.

That is, Prove: $\boxed{\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}}$

Let $y = \arcsin x$

Invert $\sin y = x$ because $\sin y = \cancel{\sin}(\cancel{\arcsin} x) = x$

Differentiate $\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$ yielding $\cos y \frac{dy}{dx} = 1$

Finally, solve $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-(\sin y)^2}} = \frac{1}{\sqrt{1-x^2}}$

Note: the identity $\cos^2 y + \sin^2 y = 1$ yields $\cos y = \pm\sqrt{1-\sin^2 y}$ but we use the positive root in our restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

For the Chain Rule, note that the order of composed functions matters. The following are not the same.

Example: $\frac{d}{dx} \arcsin\left(e^x\right) = \frac{1}{\sqrt{1-\left(e^x\right)^2}} \cdot e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$

Example: $\frac{d}{dx} e^{\arcsin x} = e^{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}$

Here again, the following are not the same.

Example: $\frac{d}{dx} \left(\arcsin x\right)^3 = 3\left(\arcsin x\right)^2 \cdot \frac{1}{\sqrt{1-x^2}}$

Example: $\frac{d}{dx} \arcsin\left(x^3\right) = \frac{1}{\sqrt{1-\left(x^3\right)^2}} \cdot \left(3x^2\right) = \frac{3x^2}{\sqrt{1-x^6}}$

Integrals: We now have the function which is the Antiderivative of $\frac{1}{\sqrt{1-x^2}}$

Integral	Tip
$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C$	antiderivative of $\frac{1}{\sqrt{1-x^2}}$ is the Inverse Sine

Compare: $\int \frac{x}{\sqrt{1-x^2}}\, dx$ uses u-sub, whereas the new integral $\int \frac{1}{\sqrt{1-x^2}}\, dx$ does not.

Helpful WARNING/INCORRECT: Be careful not to *split* the square root in the denominator or *split* the denominator

$$\int \frac{1}{\sqrt{1-x^2}}\, dx \neq \int \frac{1}{1-x}\, dx \neq \int \frac{1}{1} - \frac{1}{x}\, dx \ldots$$

Note: be careful not to punch every integral with a denominator with the logarithm. That is, be careful to **only** use the log antiderivative rule for 1 over x **to the exact power 1**.

Helpful WARNING/INCORRECT: $\int \frac{1}{\sqrt{1-x^2}}\, dx \neq \ln\sqrt{1-x^2} + C$

Instead, using the correct (now) Snap Fact $\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C$ Memorize!

INDEFINITE Integrals with u-substitution: Always remember to add +C right away, as soon as you compute the Most General Antiderivative. The original variable always reappears when we re-substitute back for u.

Recall: u-substitution is a temporary convenience that hides a nested, meaty chunk of your integrand to first simplify the integral, and second, to match the derivative chunk, all with the overall goal of reversing the Chain Rule.

IMPORTANT: Keep a look out for opportunities by creating *Hidden Squares*

Example:

$$\int \frac{x^2}{\sqrt{1-x^6}}\,dx = \int \frac{x^2}{\sqrt{1-\left(x^3\right)^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{3}\arcsin u + C = \boxed{\frac{1}{3}\arcsin\left(x^3\right) + C}$$

$$\begin{array}{|rl|} u & = x^3 \\ du & = 3x^2\,dx \\ \frac{1}{3}du & = x^2 dx \end{array}$$

Recall Algebra Rule: $\left(x^a\right)^b = x^{ab}$

Think: why does the u-sub $u = 1 - x^6$ not work from the start?

Example:

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx = \int \frac{e^{2x}}{\sqrt{1-\left(e^{2x}\right)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\arcsin u + C = \boxed{\frac{1}{2}\arcsin\left(e^{2x}\right) + C}$$

$$\begin{array}{|rl|} u & = e^{2x} \\ du & = 2e^{2x}\,dx \\ \frac{1}{2}du & = e^{2x} dx \end{array}$$

Think: why does the u-sub $u = 1 - e^{4x}$ not work from the start?

Note: Many different integrals lead to the same u-sub integral leading to arcsin.

Inverse Tangent:

Start by considering the graph for the tangent function, $y = \tan x$.

Notice that this function $y = \tan x$ is **not** a one-to-one function on its entire domain. We can justify that statement by showing that two distinct input values map to the same output value. In particular, if we consider $x = 0$ and $x = \pi$, then they both map to the same output value $\tan 0 = \tan \pi = 0$.

We can also think about how the graph of $y = \tan x$, drawn in purple, does not pass the Horizontal Line Test. We represent a sample Horizontal Line in the following graph with a random Horizontal Line $y = 3$, drawn in green.

Graph:

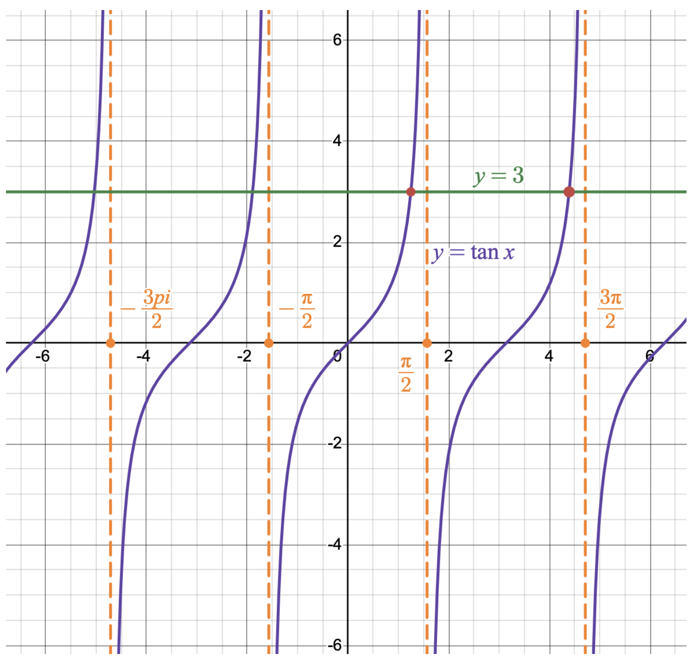

Figure 5: Tangent Function $y = \tan x$

Goal: To find an Inverse function for the tangent function $y = \tan x$, that is, find a **reversing** function to solve an equation like $\tan x = -1$ ...and eventually to lead to some important, new integrals.

IMPORTANT: To study the inverse tangent function, we will start by *Restricting the Domain* of $y = \tan x$ to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ where $y = \tan x$ will now be one-to-one.

We pick a largest, yet simplest restricted interval, including 0 (for nice symmetry and values), so that the restricted function passes the Horizontal Line Test.

Graph:

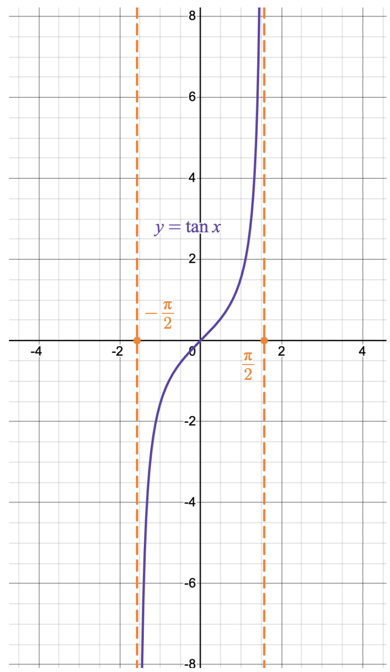

Figure 6: Tangent Function $y = \tan x$ restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Definition: The Inverse Tangent Function $y = \tan^{-1} x$ is defined as the unique **Angle** (or Arc) y in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan y = x$ for x in the interval $(-\infty, \infty)$.

$$\tan^{-1} x = y \quad \text{means} \quad \tan y = x$$

Similar to before

$$x \underset{\tan}{\overset{\arctan}{\rightleftarrows}} y$$

$$\text{Values} \underset{\tan}{\overset{\arctan}{\rightleftarrows}} \text{Angles}$$

Notation: Inverse Tangent can also be called Arctangent.

$y = \tan^{-1} x$ can be interchangably written as $y = \arctan x$.

Warning: be careful not to confuse the inverse notation with a reciprocal flip. That is, $y = \tan^{-1} x \neq \dfrac{1}{\tan x}$

Recall that the graph of Inverse Functions are mirror symmetrical flips across the line $y = x$.

Graph:

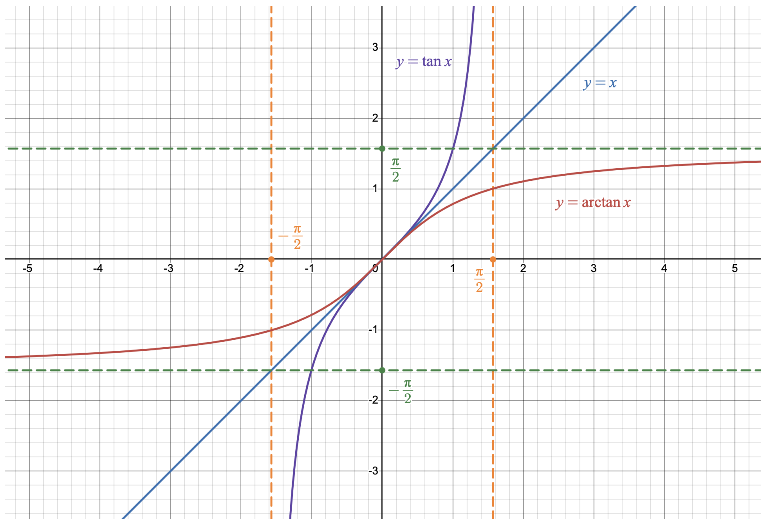

Figure 7: Symmetric Flips of $y = \tan x$ and $y = \arctan x$

Here is a clear graph showing only $y = \arctan x$.

Graph:

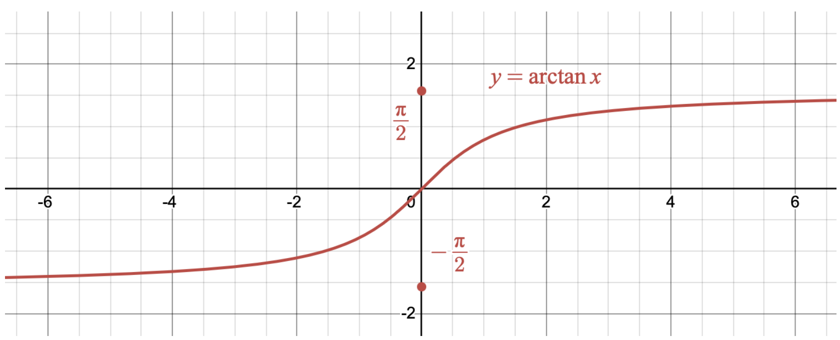

Figure 8: Inverse Tangent Function $y = \arctan x$

Let's study some properties of this Inverse Tangent Function $y = \arctan x$.

Domain$= (-\infty, \infty)$

Here, $\arctan x$ is defined for all Real numbers, which is different than $\arcsin x$.

Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Here $\arctan x$ yields all output values in a restricted range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, again different than $\arcsin x$.

Note: As Inverse Functions it makes sense, from their graphs/definitions, that

Domain $\tan x$ (restricted)	$=$	Range $\arctan x$
Domain $\arctan x$	$=$	Range $\tan x$ (restricted)

Value(s):

$\arctan 0 = 0$	$\arctan 1 = \dfrac{\pi}{4}$	$\arctan(\sqrt{3}) = \dfrac{\pi}{3}$	$\arctan\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

Question: Can you use symmetry to solve other values, like $\arctan(-1)$?

Tip: When computing an inverse tangent value, try to think in reverse. Think of $\arctan \dfrac{1}{\sqrt{3}}$ as *What Angle do you know . . . such that . . .* tangent of *THAT Angle* equals $\dfrac{1}{\sqrt{3}}$

$$\tan\text{(what angle?)} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \text{angle} = \frac{\pi}{6} \quad \text{because } \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

Inverses:

$\tan(\arctan x) = x$	for x in $(-\infty, \infty)$
$\arctan(\tan x) = x$	for x in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Note: these inverse properties state that the tangent and inverse tangent invert each other, they literally reverse or unwind the other function value. They don't just "cancel".

Limits: Study the Graph $\longrightarrow$ there are two nice Horizontal Asymptotes $y = \pm\dfrac{\pi}{2}$

Limit	Tip
$\displaystyle\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$	as input values grow uncontrollably large, the output grows to $\dfrac{\pi}{2}$
$\displaystyle\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$	as input values grow uncontrollably large negative the output approaches $-\dfrac{\pi}{2}$

These Limits will be very useful in future integrals.

Derivatives:

Derivative	Tip
$\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$	check the flip and plus sign
$\dfrac{d}{dx}\arctan(u(x)) = \dfrac{1}{1+(u(x))^2}\cdot u'(x)$	**CHAIN RULE** flips, leaving *inside* function as is... times the derivative of the *inside nested* function

Let us justify why the derivative of $\arctan x$ equals $\dfrac{1}{1+x^2}$.

That is, Prove: $\boxed{\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}}$

Let $y = \arctan x$

Invert $\tan y = x$ because $\tan y = \tan(\arctan x) = x$

Differentiate $\dfrac{d}{dx}(\tan y) = \dfrac{d}{dx}(x)$ yielding $\sec^2 y\,\dfrac{dy}{dx} = 1$

Finally, Solve $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+(\tan y)^2} = \dfrac{1}{1+x^2}$

Note: the identity $1+\tan^2 y = \sec^2 y$.

For the Chain Rule, note that the order of composed functions matters. The following are not the same.

Example: $\dfrac{d}{dx}\arctan\left(\sqrt{x}\right) = \dfrac{1}{1+\left(\sqrt{x}\right)^2}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}(1+x)}$

Example: $\dfrac{d}{dx}\sqrt{\arctan x} = \dfrac{1}{2\sqrt{\arctan x}}\cdot\dfrac{1}{1+x^2}$

Integrals: We now have the function which is the Antiderivative of $\dfrac{1}{1+x^2}$

Integral	Tip
$\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C$	antiderivative of $\dfrac{1}{1+x^2}$ is the Inverse Tangent

Compare: $\displaystyle\int \frac{x}{1+x^2}\,dx$ uses u-sub, whereas the new integral $\displaystyle\int \frac{1}{1+x^2}\,dx$ does not.

Helpful **WARNING/INCORRECT :** Be careful not to *split* the denominator

$$\int \frac{1}{1+x^2}\,dx \neq \int \frac{1}{1} + \frac{1}{x^2}\,dx \ldots$$

Note: be careful not to punch every integral with a denominator with the logarithm. That is, be careful to **only** use the log antiderivative rule for 1 over x **to the exact power 1**.

Helpful **WARNING/INCORRECT :** $\displaystyle\int \frac{1}{1+x^2}\,dx \neq \ln\left(1+x^2\right) + C$

Instead, using the correct (now) Snap Fact $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C$ Memorize!

INDEFINITE Integrals with u-substitution:

IMPORTANT: Keep a look for opportunities by creating *Hidden Squares*

Example:

$$\begin{aligned}\int \frac{x^3}{1+x^8}\,dx &= \int \frac{x^3}{1+\left(x^4\right)^2}\,dx = \frac{1}{4}\int \frac{1}{1+u^2}\,du \\ &= \frac{1}{4}\arctan u + C = \boxed{\frac{1}{4}\arctan\left(x^4\right) + C}\end{aligned}$$

$$\boxed{\begin{aligned} u &= x^4 \\ du &= 4x^3\,dx \\ \tfrac{1}{4}du &= x^3 dx \end{aligned}}$$

Think: why does the u-sub $u = 1 + x^8$ not work from the start?

Example:

$$\begin{aligned}\int \frac{e^{3x}}{1+e^{6x}}\,dx &= \int \frac{e^{3x}}{1+\left(e^{3x}\right)^2}\,dx = \frac{1}{3}\int \frac{1}{1+u^2}\,du \\ &= \frac{1}{3}\arctan u + C = \boxed{\frac{1}{3}\arctan\left(e^{3x}\right) + C}\end{aligned}$$

$$\boxed{\begin{aligned} u &= e^{3x} \\ du &= 3e^{3x}\,dx \\ \tfrac{1}{3}du &= e^{3x} dx \end{aligned}}$$

Think: why does the u-sub $u = 1 + e^{6x}$ not work from the start?

Note: Many different integrals lead to the same u-sub integral leading to arctan.

"a-Rules" for Inverse Trig: Note, some special integrals can be generalized to a Snap-Fact, which we will refer to as the "a-Rules", so we can avoid justifying the u-substitution and algebra each time

Integral	Tip
$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$	coefficient included for arctan, constant $a>0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C$	**no** coefficient included for arcsin, constant $a>0$

Here are some examples using the a-Rules as Snap-Facts.

Example: $\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right)+C$

Example: $\int \frac{1}{\sqrt{25-x^2}}\,dx = \arcsin\left(\frac{x}{5}\right)+C$

See Class notes/handouts for sample proofs of the a-Rules

DEFINITE Integral using u-substitution and a-Rule:

Example:

$$\begin{aligned}\int_e^{e^3} \frac{1}{x\left(3+(\ln x)^2\right)}\,dx &= \int_1^3 \frac{1}{3+u^2}\,du \overset{a\text{-rule}}{=} \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_1^3 \\ &= \frac{1}{\sqrt{3}}\left(\arctan\left(\frac{3}{\sqrt{3}}\right)-\arctan\left(\frac{1}{\sqrt{3}}\right)\right) \\ &= \frac{1}{\sqrt{3}}\left(\arctan\sqrt{3}-\arctan\left(\frac{1}{\sqrt{3}}\right)\right) \\ &= \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{6\sqrt{3}}\end{aligned}$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx\end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} &x=e \Rightarrow u=\ln e = 1 \\ &x=e^3 \Rightarrow u = \ln e^3 = 3\end{aligned}}$$