Extra Examples of Absolute and Conditional Convergence

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Start by reviewing some recent definitions:

Definition: Given a series $\sum_{n=1}^{\infty} a_n$, then the Absolute Series (A.S.) is given by $\sum_{n=1}^{\infty} |a_n|$.

The Absolute Series is the same as the Original Series (O.S.), but with all positive terms.

Definition: A series $\sum_{n=1}^{\infty} a_n$ is called **Absolutely Convergent** if the Absolute Series Converges.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ Absolutely Converges because its A.S. $\sum_{n=1}^{\infty} \frac{1}{n^5}$ Converges (p-series, $p = 5 > 1$).

Helpful: it is sometimes easier to analyze the Absolute Series, because it has all positive terms, but also because there are several tests that can be applied to strictly positive-termed series (Integral Test, p-Series, Comparison and Limit Comparison Test). Note that if the original, given, series already had all positive terms, then it is equal to its Absolute Series, and Absolute Convergence is the same as Convergence.

Definition: A series $\sum_{n=1}^{\infty} a_n$ is called **Conditionally Convergent** if the Original Series Converges, **BUT** the Absolute Series Diverges.

The classic Conditionally Convergent example is the Alternating Harmonic series:

We showed that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$ Converges by the Alternating Series Test, but the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \ldots$ Diverges as a p-Series with $p = 1$ (how else?).

Essentially the Harmonic Series has all positive terms that are adding up just too quickly to be controlled in a large sum. Even though the terms are shrinking to zero, they are not shrinking fast enough as n explodes, whereas the Alternating Series has a better chance to converge because every other term is subtracted.

Plan of attack: **Absolutely Convergent, Conditionally Convergent, or Divergent?**

- If the question is asking if the Original Series is Absolutely Convergent, then we must analyze the Absolute Series. I suggest analyzing the Absolute Series first.
- If the Absolute Series is Convergent, then we are done, and we label the *original series* as Absolutely Convergent (by definition).
- If the Absolute Series is Divergent, then we know the Original Series is *NOT* Absolutely Convergent. So we must step back to analyze the Original Series. If the Original Series is Convergent, usually using the Alternating Series Test, then we can declare the Original Series to be Conditionally Convergent (by definition).
- If the Original Series was Divergent, then we likely may have already spotted that using the n^{th} Term Divergence Test.

Two organizational Charts:

O.S. $\displaystyle\sum_{n=1}^{\infty} a_n \xhookrightarrow{1^{st}}$ A.S. $\displaystyle\sum_{n=1}^{\infty} |a_n|$

A.C. Chart

fill $\downarrow$ work

no need to analyze O.S.

A.S Converges by (fill here) ______

O.S.
Absolutely
Convergent
by Definition

O.S. $\displaystyle\sum_{n=1}^{\infty} a_n \xhookrightarrow{1^{st}}$ A.S. $\displaystyle\sum_{n=1}^{\infty} |a_n|$

$\swarrow 2^{nd}$

C.C. Chart

fill $\downarrow$ work (O.S.)

fill $\downarrow$ work (A.S.)

O.S. Converges by AST

A.S Diverges by (fill here) ______

$\Rightarrow$ O.S. is **NOT** A.C.

$\Rightarrow$ must analyze O.S.

O.S.
Conditionally
Convergent
by Definition

Determine whether the given series is **Absolutely Convergent, Conditionally Convergent, or Diverges**. Name any Convergence test(s) that you use, and justify all of your work.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\,\frac{n^2+6}{n^6+2}$

$$\sum_{n=1}^{\infty}(-1)^n\,\frac{n^2+6}{n^6+2} \quad \xrightarrow[\text{A.S.}]{1^{st}} \quad \sum_{n=1}^{\infty}\frac{n^2+6}{n^6+2} \approx \sum_{n=1}^{\infty}\frac{n^2}{n^6} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

Conv. p-Series
$p = 4 > 1$

$$\lim_{n\to\infty}\frac{\frac{n^2+6}{n^6+2}}{\frac{1}{n^4}} = \lim_{n\to\infty}\frac{n^6+6n^4}{n^6+2}\cdot\frac{\left(\frac{1}{n^6}\right)}{\left(\frac{1}{n^6}\right)} = \lim_{n\to\infty}\frac{1+\overset{0}{\cancel{\frac{6}{n^2}}}}{1+\overset{0}{\cancel{\frac{2}{n^6}}}} = 1$$

Finite
Non-Zero

no need to analyze O.S.

$\boxed{\text{A.S. Converges}}$ by LCT

$\boxed{\begin{array}{c}\text{O.S.}\\ \text{Absolutely}\\ \text{Convergent}\\ \textit{by Definition}\end{array}}$

Notes:

- Since the question is A.C., C.C., or diverge, then go straight to analyzing the Absolute Series, because IF the Absolute Series converges, then we're done and the Original Series is A.C. by definition.
- IF the question is instead just *regular converge or diverge?* then we can make one more conclusion here... since the Absolute Series Converges, we know that the Original Series Converges by the Absolute Convergence Test.
- Remember Absolute Convergence is a declaration *by Definition*, not by a Convergence test.

Determine whether the given series is **Absolutely Convergent, Conditionally Convergent, or Diverges**. Name any Convergence test(s) that you use, and justify all of your work.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+3}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+3} \quad \overset{1^{st}}{\underset{\text{A.S.}}{\hookrightarrow}} \quad \sum_{n=1}^{\infty} \frac{1}{5n+3} \approx \sum_{n=1}^{\infty} \frac{1}{n}$$

Diverges. p-Series
Harmonic $p = 1$

$\swarrow 2^{nd}$

1. $b_n = \dfrac{1}{5n+3} > 0$

2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\cancel{5n+3}} = 0$

3. Terms Decreasing

$b_{n+1} = \dfrac{1}{5n+8} \leq \dfrac{1}{5n+3} = b_n$

$$\lim_{n\to\infty} \frac{\frac{1}{5n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+3} \cdot \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{5+\cancelto{0}{\frac{3}{n}}} = \frac{1}{5}$$

Finite

Non-Zero

$\boxed{\text{O.S Converges}}$ by AST

$\boxed{\text{A.S Diverges}}$ by LCT

$$\boxed{\begin{array}{c}\text{O.S.}\\ \text{Conditionally}\\ \text{Convergent}\\ \textit{by Definition}\end{array}}$$

- Since the question is A.C., C.C., or Diverge, then we still need to analyze the Absolute Series, even if we have good instincts that it will diverge. We need to *prove* that the A.S. Diverges. Then we need to go determine if the Original Series Converges.
- Be careful: There is NO ~~Absolute Divergence Test~~. If the Absolute Series Diverges, then it is **not true** that the Original Series diverges. (Think about the Alternating Harmonic Series example above.) Then we must study the Original Series to check if it converges.
- Remember Conditional Convergence is a declaration *by Definition*, not by a Convergence test.