Worksheet #11

1. $f(x) = \dfrac{e^x}{e^x+1}$

(a) Quotient Rule $\quad f'(x) = \dfrac{(e^x+1)e^x - e^x \cdot e^x}{(e^x+1)^2} = \dfrac{\cancel{e^{2x}} + e^x - \cancel{e^{2x}}}{(e^x+1)^2} = \boxed{\dfrac{e^x}{(e^x+1)^2}}$

(b) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{e^{x+h}}{e^{x+h}+1} - \dfrac{e^x}{e^x+1}}{h} = \lim_{h\to 0} \frac{e^{x+h}(e^x+1) - e^x(e^{x+h}+1)}{\underbrace{h\,(e^{x+h}+1)(e^x+1)}_{\text{common denom}}}$$

$$= \lim_{h\to 0} \frac{\overbrace{\cancel{e^{x+h}e^x}}^{e^{2x+h}} + e^{x+h} \overset{\text{cancel}}{-} \overbrace{\cancel{e^x e^{x+h}}}^{e^{2x+h}} - e^x}{h\,(e^{x+h}+1)(e^x+1)}$$

$$= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h(e^{x+h}+1)(e^x+1)} = \lim_{h\to 0} \frac{e^x(e^h-1)}{h(e^{x+h}+1)(e^x+1)}$$

Exponential Algebra

partner

$$= \lim_{h\to 0} \underbrace{\left(\frac{e^h-1}{h}\right)}_{1 \text{ by choice of } e} \left(\frac{e^x}{(e^{x+\overset{0}{h}}+1)(e^x+1)}\right) = \boxed{\frac{e^x}{(e^x+1)^2}}$$

Match!

Recall: $\dfrac{d}{dx} a^x = a^x \cdot \left[\underbrace{\lim_{h\to 0} \dfrac{a^h-1}{h}}_{\text{want } 1}\right]$ $\quad$ General Exponential

and e was chosen as the number that made that messy limit 1.

$$\boxed{\lim_{h\to 0} \frac{e^h-1}{h} = 1}$$

2. (a) $y = \ln x$

(b) $e^y = e^{\ln x} = x$

(c) $\frac{d}{dx} e^y = \frac{d}{dx} x$

$e^y \frac{dy}{dx} = 1$

(d) Solve $\frac{dy}{dx} = \frac{1}{e^y}$

(e) Substitute $\frac{dy}{dx} = \frac{1}{e^{\ln x}} = \frac{1}{x}$ ✓

3. $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du = -\ln|u| + C = \boxed{-\ln|\cos x| + C}$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \\ -du &= \sin x \, dx \end{aligned}}$$

4. $\int_0^{\ln 2} \frac{e^{3x}}{\sqrt{8+e^{3x}}} dx = \frac{1}{3}\int_9^{16} \frac{1}{\sqrt{u}} du = \frac{2}{3}\sqrt{u}\Big|_9^{16} = \frac{2}{3}\sqrt{16} - \frac{2}{3}\sqrt{9} = \frac{8}{3} - 2 = \boxed{\frac{2}{3}}$

$u^{-1/2}$ $\quad$ 4 $\quad$ 3 $\quad$ $4/3$

$$\boxed{\begin{aligned} u &= 8 + e^{3x} \\ du &= 3e^{3x} dx \\ \tfrac{1}{3} du &= e^{3x} dx \end{aligned}}$$

$$\boxed{\begin{aligned} &x = 0 \Rightarrow u = 8 + e^0 = 8 + 1 = 9 \\ &x = \ln 2 \Rightarrow u = 8 + e^{3\ln 2} = 8 + e^{\ln(2^3)} = 8 + 8 = 16 \end{aligned}}$$

can drop |·|

5. $\displaystyle\int_0^1 \frac{e^x}{2+e^x}\,dx = \int_3^{2+e} \frac{1}{u}\,du = \ln|u|\Big|_3^{2+e} = \boxed{\ln|2+e| - \ln 3} = \ln\left(\frac{2+e}{3}\right)$

$\boxed{\begin{array}{l} u = 2+e^x \\ du = e^x\,dx \end{array}}$ $\boxed{\begin{array}{l} x=0 \Rightarrow u = 2+\cancel{e^0}^{\,1} = 2+1 = 3 \\ x=1 \Rightarrow u = 2+e \end{array}}$

6. $\displaystyle\int_{e^3}^{e^9} \frac{1}{5x}\,dx = \frac{1}{5}\int_{e^3}^{e^9} \frac{1}{x}\,dx = \frac{1}{5}\ln|x|\Big|_{e^3}^{e^9} = \frac{1}{5}\cancel{\ln e^9}^{\,9} - \frac{1}{5}\cancel{\ln e^3}^{\,3} = \frac{9}{5} - \frac{3}{5} = \boxed{\frac{6}{5}}$

OR do u-sub, but easier to pull constant out

7. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|e^x + e^{-x}| + C}$

Nice!

Do Not simplify further

$\boxed{\begin{array}{l} u = e^x + e^{-x} \\ du = e^x - e^{-x}\,dx \end{array}}$

chain rule

8. $\displaystyle\int_e^{e^4} \frac{3}{x\sqrt{\ln x}}\,dx = 3\int_1^4 \frac{1}{\sqrt{u}}\,du = 6\sqrt{u}\,\Big|_1^4 = 6\cancel{\sqrt{4}}^{\,2} - 6\cancel{\sqrt{1}}^{\,1} = 12 - 6 = \boxed{6}$

$u^{-1/2} \to \frac{u^{1/2}}{1/2}$

$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}}$ $\boxed{\begin{array}{l} x=e \Rightarrow u = \ln e = 1 \\ x=e^4 \Rightarrow u = \ln e^4 = 4 \end{array}}$

Warning: $\displaystyle\int \frac{1}{\sqrt{u}}\,du \neq \ln\sqrt{u} + C$

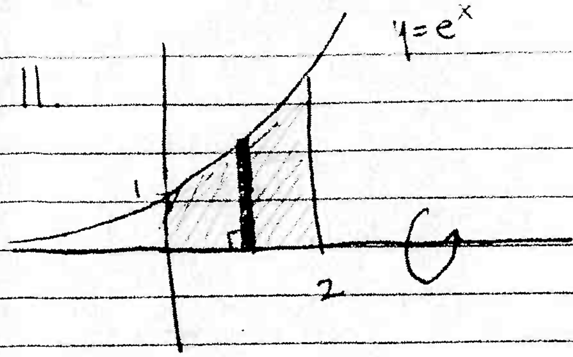
9. $\int \frac{\sqrt{1+e^{-x}}}{e^x}\,dx = -\int \sqrt{u}\,du = -\frac{2}{3}u^{3/2} + C = \boxed{\frac{-2}{3}(1+e^{-x})^{3/2} + C}$

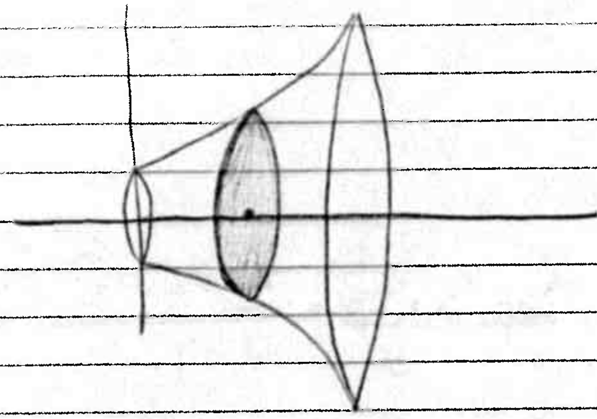
$$\boxed{\begin{aligned} u &= 1+e^{-x} \\ du &= -e^{-x}dx \\ -du &= \frac{1}{e^x}dx \end{aligned}}$$

10. $\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{u^3}{3} + C = \boxed{\frac{(\ln x)^3}{3} + C}$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}dx \end{aligned}}$$

11.

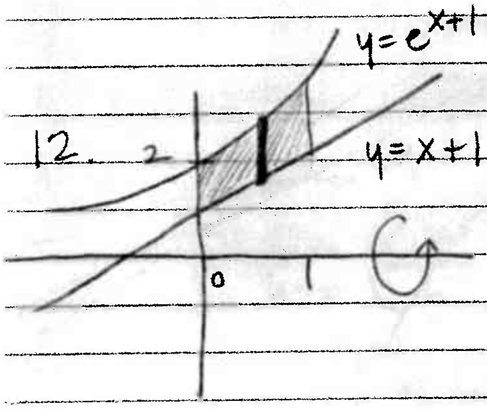


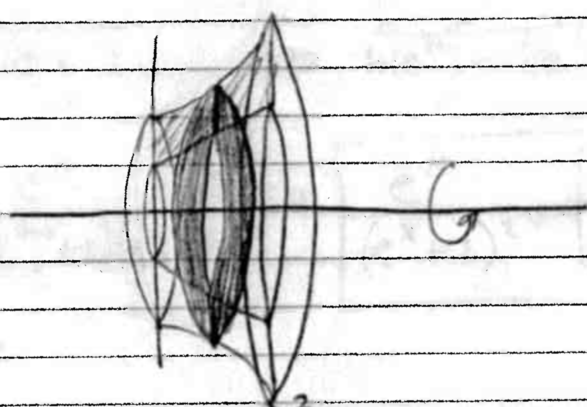


$$V=\pi\int_0^2 (e^x)^2\,dx=\pi\int_0^2 e^{2x}\,dx=\frac{\pi}{2}e^{2x}\Big|_0^2=\frac{\pi}{2}e^4-\frac{\pi}{2}\cancel{e^0}^{\,1}$$

$$=\boxed{\frac{\pi}{2}\left(e^4-1\right)}\ \oplus$$

12.





$$V=\pi\int_0^1 (\text{outer radius})^2-(\text{inner radius})^2\,dx$$

$$=\pi\int_0^1 (e^x+1)^2-\underbrace{(x+1)^2}_{x^2+2x+1}\,dx=\pi\int_0^1 e^{2x}+2e^x+\cancel{1}-x^2-2x-\cancel{1}\,dx$$

$$=\pi\left[\frac{e^{2x}}{2}+2e^x-\frac{x^3}{3}-x^2\right]\Bigg|_0^1=\pi\left[\left(\frac{e^2}{2}+2e-\tfrac{1}{3}-1\right)-\left(\underbrace{\frac{\cancel{e^0}^{\,1}}{2}}_{-1/2}+\underbrace{2\cancel{e^0}^{\,1}}_{-2}-0-0\right)\right]$$

$$-\frac{2}{6}-\frac{3}{6}-\frac{18}{6}=-\frac{23}{6}$$

$$=\boxed{\pi\left[\frac{e^2}{2}+2e-\frac{23}{6}\right]}$$