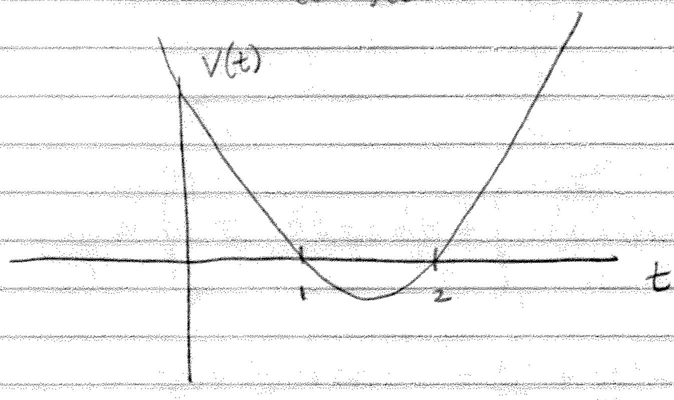
Worksheet #9

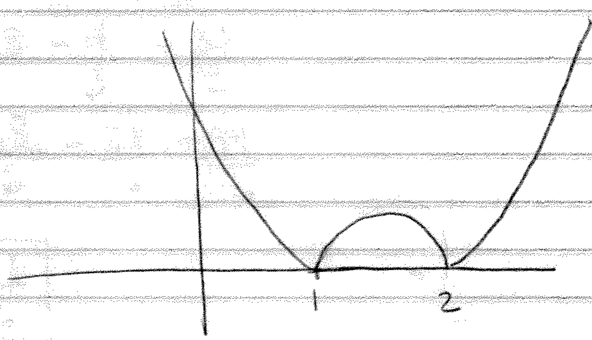
1. a. Graph $v(t) = t^2 - 3t + 2$
$= (t-2)(t-1)$



Object is moving backwards between time $t=1$ and $t=2$ seconds, when $v(t) < 0$.

b. $|v(t)| = |t^2 - 3t + 2| = \begin{cases} t^2 - 3t + 2 & \text{when } t \le 1 \text{ and } t \ge 2 \\ -(t^2 - 3t + 2) & \text{when } 1 < t < 2 \end{cases}$

c. Graph $|v(t)|$

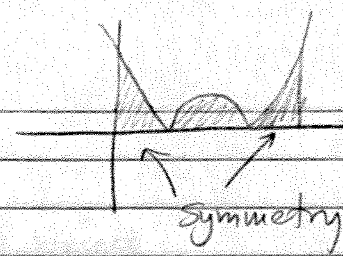


d. Displacement $= \displaystyle\int_0^3 v(t)\,dt = \int_0^3 t^2 - 3t + 2\,dt = \frac{t^3}{3} - \frac{3t^2}{2} + 2t \Big|_0^3$

$$= \left(9 - \frac{27}{2} + 6\right) - 0$$

$$= 15 - \frac{27}{2}$$

$$= \frac{30}{2} - \frac{27}{2} = \boxed{\frac{3}{2}} \text{ feet}$$



e. Total Distance $= \int_0^3 |v(t)|\,dt = \int_0^3 |t^2-3t+2|\,dt$

$$= \int_0^1 t^2-3t+2\,dt + \int_1^2 \underbrace{-(t^2-3t+2)}_{-t^2+3t-2}\,dt + \int_2^3 t^2-3t+2\,dt$$

$$= \frac{t^3}{3}-\frac{3t^2}{2}+2t\Big|_0^1 - \frac{t^3}{3}+\frac{3t^2}{2}-2t\Big|_1^2 + \frac{t^3}{3}-\frac{3t^2}{2}+2t\Big|_2^3$$

$$= \left(\frac{1}{3}-\frac{3}{2}+2\right)-(0-0+0)+\left(\frac{-8}{3}+\underbrace{6-4}_{2}\right)-\left(\frac{-1}{3}+\frac{3}{2}-2\right)+\left(\frac{\overset{9}{\cancel{27}}}{3}-\frac{27}{2}+6\right)-\left(\frac{8}{3}\underbrace{-6+4}_{-2}\right)$$

$$= \frac{1}{3}-\frac{3}{2}+\boxed{2}-\frac{8}{3}+\boxed{2}+\frac{1}{3}-\frac{3}{2}+\boxed{2}+\boxed{9}-\frac{27}{2}+\boxed{6}-\frac{8}{3}+\boxed{2}$$

$$= 23-\frac{14}{3}-\frac{33}{2}$$

$$= \frac{138}{6}-\frac{28}{6}-\frac{99}{6}$$

$$= \boxed{\frac{11}{6}}$$

28
99
127

138
−127
11

OR/// Double Left Piece

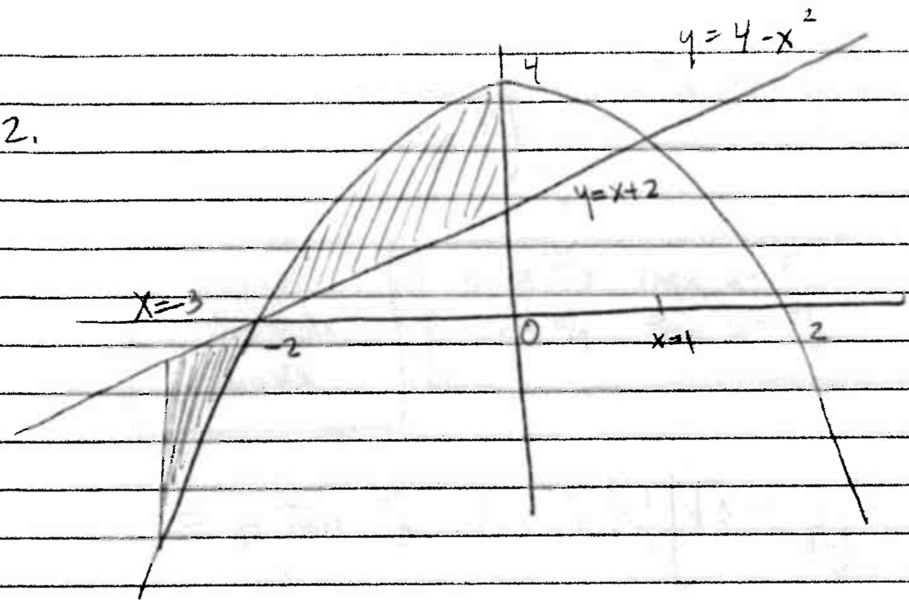
$$\int_0^3 |v(t)|\,dt = 2\int_0^1 t^2-3t+2\,dt + \int_1^2 -(t^2-3t+2)\,dt$$

$$= 2\left[t^3/3-\frac{3t^2}{2}+2t\Big|_0^1\right] - t^3/3+3t^2/2-2t\Big|_1^2$$

$$= 2\left[\frac{1}{3}-3/2+2-0\right]+\left(-8/3+6-4\right)-\left(-1/3+3/2-2\right)$$

$$= \underbrace{2/3}-3+4\underbrace{-8/3}+2\underbrace{+1/3}-3/2+2 = \frac{-5}{3}+5-\frac{3}{2} = \frac{-10}{6}+\frac{30}{6}-\frac{9}{6} = \boxed{11/6}$$

2.



Intersect

$4-x^2 = x+2$

$x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2, x=1$

$$\text{Area} = \int_{-3}^{-2} \text{top} - \text{bottom}\, dx + \int_{-2}^{0} \text{top} - \text{bottom}\, dx$$

$$= \int_{-3}^{-2} (x+2) - \overset{-4+x^2}{(4-x^2)}\, dx + \int_{-2}^{0} (4-x^2) - (x+2)\, dx$$

$$= \int_{-3}^{-2} x^2+x-2\, dx + \int_{-2}^{0} -x^2-x+2\, dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - 2x \Big|_{-3}^{-2} + \left(-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right)\Big|_{-2}^{0}$$

$$= \left(\frac{-8}{3} + 2 + 4\right) - \left(-9 + \underbrace{\frac{9}{2} + 6}_{-3}\right) + (0-0+0) - \left(\frac{+8}{3} \underbrace{-2-4}_{-6}\right)$$

$$= -8/3 + 6 + 3 - 9/2 - 8/3 + 6$$

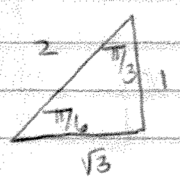
$$= -16/3 + 15 - 9/2$$

$$= -\frac{32}{6} + \frac{90}{6} - \frac{27}{6} = \boxed{\frac{31}{6}} \oplus$$

3. a. $\displaystyle\int_{\pi/18}^{\pi/9} \sec^2(3x)\,dx = \frac{1}{3}\int_{\pi/6}^{\pi/3} \sec^2 u\,du = \frac{1}{3}\tan u\Big|_{\pi/6}^{\pi/3}$

$\boxed{\begin{array}{c} u = 3x \\ du = 3\,dx \\ \frac{1}{3}du = dx \end{array}}$ $\boxed{\begin{array}{l} \text{if } x = \pi/18 \text{ then } u = \pi/6 \\ \quad x = \pi/9 \text{ then } u = \pi/3 \end{array}}$

$$= \frac{1}{3}\left[\tan\frac{\pi}{3} - \tan\frac{\pi}{6}\right] = \frac{1}{3}\left[\sqrt{3} - \frac{1}{\sqrt{3}}\right] = \frac{1}{3}\left[\frac{3}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right] = \frac{1}{3}\left(\frac{2}{\sqrt{3}}\right) = \boxed{\frac{2}{3\sqrt{3}}}$$



b. $\displaystyle\int \frac{1}{x^2}\sqrt{1 - 1/x}\,dx = \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C$

$$= \boxed{\frac{2}{3}\left(1 - 1/x\right)^{3/2} + C}$$

$\boxed{\begin{array}{c} u = 1 - 1/x \\ du = 1/x^2\,dx \end{array}}$ x^{-1}

c. $\displaystyle\int_2^4 \frac{x}{(3x^2-8)^2}\,dx = \frac{1}{6}\int_4^{40} \frac{1}{u^2}\,du = \frac{1}{6}\int_4^{40} u^{-2}\,du = \frac{1}{6}\left(\frac{u^{-1}}{-1}\right)\Big|_4^{40}$

$\boxed{\begin{array}{c} u = 3x^2 - 8 \\ du = 6x\,dx \\ \frac{1}{6}du = x\,dx \end{array}}$ $\boxed{\begin{array}{l} \text{if } x = 2 \text{ then } u = 4 \\ \quad x = 4 \text{ then } u = 40 \end{array}}$

$$= \frac{-1}{6u}\Big|_4^{40}$$

$$= \frac{-1}{240} - \left(-\frac{1}{24}\right)$$

$$= \frac{-1}{240} + \frac{10}{240} = \frac{9}{240} = \boxed{\frac{3}{80}}$$

d. $\displaystyle\int \frac{\tan\sqrt{x}\,\sec^2\sqrt{x}}{\sqrt{x}}\,dx = 2\int u\,du = u^2 + C = \boxed{\tan^2\sqrt{x} + C}$

$$\boxed{\begin{array}{l} u = \tan\sqrt{x} \\ du = \sec^2\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx \\ 2du = \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx \end{array}}$$

$2\tan\sqrt{x}\cdot\sec^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$ ✓

Algebra

e. $\displaystyle\int x(1+x)^{2/3}\,dx = \int (u-1)\,u^{2/3}\,du = \int u^{5/3} - u^{2/3}\,du = \frac{3}{8}u^{8/3} - \frac{3}{5}u^{5/3} + C$

$$= \boxed{\frac{3}{8}(1+x)^{8/3} - \frac{3}{5}(1+x)^{5/3} + C}$$

$$\boxed{\begin{array}{l} u = 1+x \Rightarrow x = u-1 \\ du = dx \end{array}}$$

Inverted Substitution

4. $f'(x) = \dfrac{\sec x\tan x}{\sqrt{\sec x + 8}}$ $\quad f(0) = 7.$

$$f(x) = \int f'(x)\,dx = \int \frac{\sec x\tan x}{\sqrt{\sec x+8}}\,dx = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = 2u^{1/2} + C$$

$$= 2\sqrt{\sec x + 8} + C$$

$$\boxed{\begin{array}{l} u = \sec x + 8 \\ du = \sec x\tan x\,dx \end{array}}$$

Use Initial Condition $\quad f(0) = 2\sqrt{\sec 0 + 8} + C \overset{\text{set}}{=} 7$

$\sec 0 = 1$

$2\sqrt{9} + C = 7$

$6 + C = 7 \Rightarrow C = 1$

Finally $f(x) = \boxed{2\sqrt{\sec x + 8} + 1}$

5. $g(x) = \int_x^9 \sqrt{1+\cos t}\; dt$

$g'(x) = \frac{d}{dx} \int_x^9 \sqrt{1+\cos t}\, dt$ FTC Part I

$= -\frac{d}{dx} \int_9^x \sqrt{1+\cos t}\; dt$ switch limits $\rightarrow$ need minus sign.

$= -\sqrt{1+\cos x}$

$g''(x) = -\frac{1}{2\sqrt{1+\cos x}}(-\sin x) = \boxed{\frac{\sin x}{2\sqrt{1+\cos x}}}$

CHALLENGE:

$\int \sqrt{1+\sqrt{x}}\; dx = \int \sqrt{u}\; 2(u-1)\, du = 2\int u^{3/2} - u^{1/2}\, du$

$= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C$

$= \boxed{\frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + C}$

$u = 1+\sqrt{x} \Rightarrow \sqrt{x} = u-1$

$du = \frac{1}{2\sqrt{x}}\, dx$ "invert"

$2\sqrt{x}\,du = dx$

$2(u-1)du = dx$

OR

$u = \sqrt{1+\sqrt{x}} \Rightarrow u^2 = 1+\sqrt{x} \Rightarrow \sqrt{x} = u^2-1$

$du = \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}}\, dx$

$4\sqrt{1+\sqrt{x}}\,\sqrt{x}\, du = dx$

$4u(u^2-1)du = dx$

$\boxed{\frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + C}$

$\int \sqrt{1+\sqrt{x}}\; dx = \int u \cdot 4u(u^2-1)du = 4\int u^4 - u^2\, du = 4\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C = \frac{4}{5}\left(\sqrt{1+\sqrt{x}}\right)^5 - \frac{4}{3}\left(\sqrt{1+\sqrt{x}}\right)^3 + C$