Worksheet #7

1. (a) $\lim\limits_{x\to 0} \dfrac{3x^2-x^4}{\sin^2(7x)} = \lim\limits_{x\to 0} \dfrac{x^2(3-x^2)}{\sin^2(7x)} = \lim\limits_{x\to 0} \dfrac{x}{\sin(7x)} \cdot \dfrac{x}{\sin(7x)} \cdot (3-x^2)$

$= \lim\limits_{x\to 0} \dfrac{7x}{7\sin(7x)} \cdot \dfrac{7x}{7\sin(7x)} \cdot (3-x^2) = \dfrac{1}{49} \underbrace{\lim\limits_{x\to 0} \dfrac{7x}{\sin(7x)}}_{1} \cdot \underbrace{\lim\limits_{x\to 0} \dfrac{7x}{\sin(7x)}}_{1} \cdot \underbrace{\lim\limits_{x\to 0} 3-x^2}_{3}$

$= \boxed{\dfrac{3}{49}}$

(b) $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{\sin(8x)} = \lim\limits_{x\to 0} \dfrac{\frac{\sin(3x)}{x}}{\frac{\sin(8x)}{x}} = \lim\limits_{x\to 0} \dfrac{\frac{3\sin(3x)}{3x}}{\frac{8\sin(8x)}{8x}} = \dfrac{3}{8} \dfrac{\underbrace{\lim\limits_{x\to 0} \frac{\sin(3x)}{3x}}_{1}}{\underbrace{\lim\limits_{x\to 0} \frac{\sin(8x)}{8x}}_{1}} = \boxed{\dfrac{3}{8}}$

(c) $\lim\limits_{x\to \infty} \dfrac{8x^2-17}{3x^4+2012x+6} \dfrac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} = \lim\limits_{x\to 0} \dfrac{\underbrace{8/x^2}_{0} - \underbrace{17/x^4}_{0}}{3 + \underbrace{2012/x^3}_{0} + \underbrace{6/x^4}_{0}} = \dfrac{0}{3} = \boxed{0}$

(d) $\lim\limits_{x\to \infty} \dfrac{x^7-4x+7}{x^2+9} \dfrac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim\limits_{x\to \infty} \dfrac{x^5 - \underbrace{4/x}_{0} + \underbrace{7/x^2}_{0}}{1 + \underbrace{9/x^2}_{0}} = \boxed{\infty}$

2 (a) $f(x) = (9-x^2)^8 \cdot (x^3-6x)^9$

$f'(x) = (9-x^2)^8 \cdot 9(x^3-6x)^8 \cdot (3x^2-6) + (x^3-6x)^9 \cdot 8(9-x^2)^7(-2x)$

(b) $f(t) = \sin^3\left(\cos\left(\dfrac{1}{t^{7/8}}\right)\right)$ $\quad t^{-7/8}$

$f'(t) = 3\sin^2\left(\cos\left(\frac{1}{t^{7/8}}\right)\right) \cdot \cos\left(\cos\left(\frac{1}{t^{7/8}}\right)\right) \cdot \left(-\sin\left(\frac{1}{t^{7/8}}\right)\right) \cdot \left(-\frac{7}{8} t^{-15/8}\right)$

2(c) $f(x) = \sqrt{\dfrac{\sin x}{x - \cos^2 x}}$

$$f'(x) = \frac{1}{2\sqrt{\dfrac{\sin x}{x-\cos^2 x}}}\left[\frac{(x-\cos^2 x)(-\cos x) - \sin x\,(1 - 2\cos x(-\sin x))}{(x-\cos^2 x)^2}\right]$$

(2d) $f(x) = \dfrac{1}{(\tan(7x) + 1/x)^{5/7}} = \left(\tan(7x) + 1/x\right)^{-5/7}$

$$f'(x) = -5/7\left[\tan(7x) + \frac{1}{x}\right]^{-12/7} \cdot \left(\sec^2(7x)\cdot 7 - 1/x^2\right)$$

3. $G(x) = \dfrac{5x}{x^2+1}$ Closed Interval Method $[0,2]$

Step1: Find Critical Numbers $G'(x) = \dfrac{\overbrace{(x^2+1)5 - (5x)(2x)}^{5x^2+5\ -10x^2}}{(x^2+1)^2} = \dfrac{5-5x^2}{(x^2+1)^2} \overset{set}{=} 0$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

Step2: Evaluate G at Critical Number(s)

$G(1) = \dfrac{5}{1+1} = \boxed{5/2}$ ← Absolute Max Value

$\cancel{G(-1) = \dfrac{-5}{1+1} = -5/2}$ outside closed interval $[0,2]$

Step 3: Evaluate G at Endpoints.

$G(0) = \boxed{0}$ ← Absolute Min Value

$G(2) = \dfrac{10}{4+1} = \dfrac{10}{5} = 2$

Step 4: Pick off Values (above).

4. $f(x) = \dfrac{x^2-9}{x^2-4}$

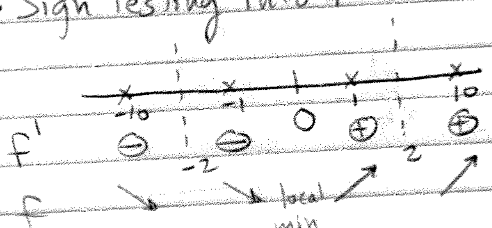
• Domain: $\{x \mid x \neq \pm 2\}$

• V.A. $x = \pm 2$

• H.A. $\displaystyle\lim_{x\to\pm\infty} \frac{x^2-9}{x^2-4} \cdot \frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\pm\infty} \frac{1 - \overset{\to 0}{9/x^2}}{1 - \underset{\to 0}{4/x^2}} = 1 \Rightarrow$ H.A. @ $y=1$

• First Derivative $f'(x) = \dfrac{10x}{\underset{\oplus}{(x^2-4)^2}} \overset{\text{set}}{=} 0 \Rightarrow x = 0.$

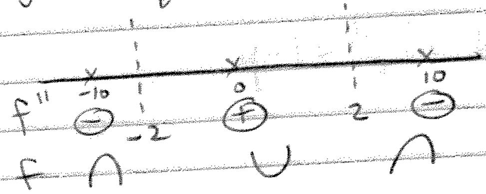
• Sign Testing into f'



f Increasing $(0,2) \cup (2,\infty)$
Decreasing $(-\infty,-2) \cup (-2,0)$
Local Min $(0, f(0)) = (0, 9/4)$
Local Max None.

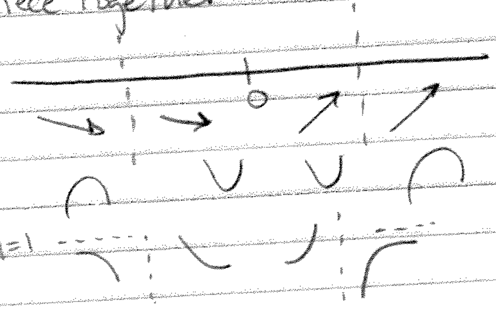
• Second Derivative $f''(x) = \dfrac{-10(3x^2+4)}{(x^2-4)^3} \overset{\text{set}}{=} 0$ No Possible Inflection Pts.

• Sign Testing into f''

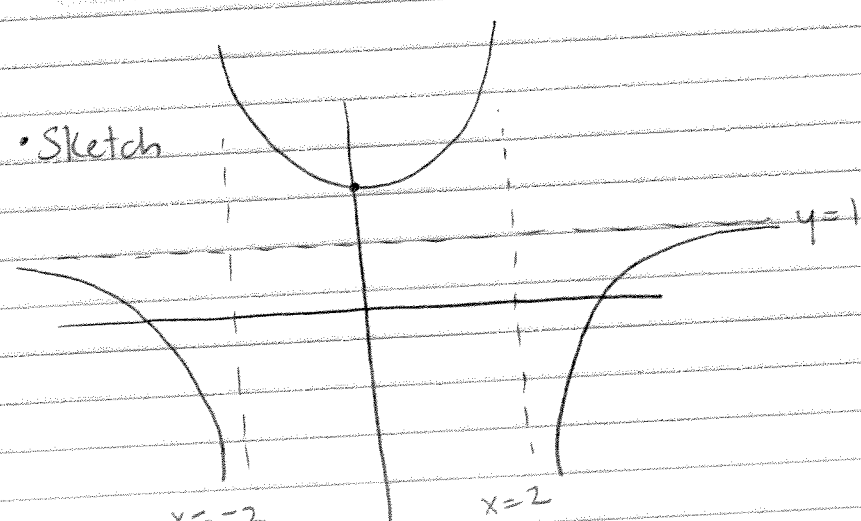


CU $(-2,2)$
CD $(-\infty,-2) \cup (2,\infty)$

• Piece Together



• Sketch



5. $y^3+\cos(xy) = 2+xy^2$

(a) $\frac{d}{dx}\left[y^3+\cos(xy)\right] = \frac{d}{dx}\left[2+xy^2\right]$

$$3y^2\frac{dy}{dx} - \sin(xy)\left[x\frac{dy}{dx}+y\right] = x\,2y\frac{dy}{dx}+y^2(1)$$

$$\underbrace{3y^2\frac{dy}{dx}} - \underbrace{x\sin(xy)\frac{dy}{dx}} - y\sin(xy) = \underbrace{2xy\frac{dy}{dx}} + y^2$$

$$\left[3y^2 - x\sin(xy) - 2xy\right]\frac{dy}{dx} = y^2 + y\sin(xy)$$

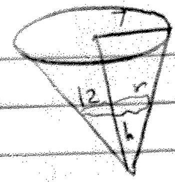
Solve $\frac{dy}{dx} = \frac{y^2+y\sin(xy)}{3y^2-x\sin(xy)-2xy}$

(b) $\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1+\cancel{1\cdot\sin(0)}^{\,0}}{3-0-0} = \frac{1}{3}$

slope $= 1/3$ Point $= (0,1)$

$y-1 = \frac{1}{3}(x-0) \Rightarrow \boxed{y=\frac{1}{3}x+1}$

6(a) · Diagram



· Variables Let r = radius of water level at time t.

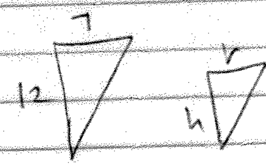
h = height of water level at time t.

V = Volume of water level at time t

Given $\frac{dr}{dt} = -2$ ft/min

$\frac{dV}{dt} = ?$ when $r = 2$ft.

· Equation $V = \frac{1}{3}\pi r^2 h$



Similar Triangles

$\frac{r}{7} = \frac{h}{12} \Rightarrow h = \frac{12}{7} r$

$V = \frac{1}{3}\pi r^2 \left(\frac{12}{7} r\right)$

$= \frac{4}{7}\pi r^3$

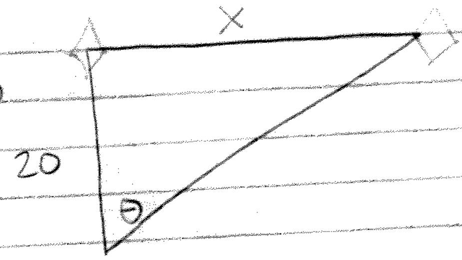
· Differentiate. $\frac{d}{dt}\left[V\right] = \frac{d}{dt}\left[\frac{4}{7}\pi r^3\right]$

$\frac{dV}{dt} = \frac{12\pi}{7} r^2 \frac{dr}{dt}$

· Substitute
· Solve $\frac{dV}{dt} = \frac{12\pi}{7}(2)^2 \cdot (-2) = \frac{-96\pi}{7}$ ft^3/min

· Answer The volume of the water is decreasing $\frac{96\pi}{7}$ cubic feet every minute.

6(b) • Diagram



• Variable Let x = distance the kite travelled horizontally

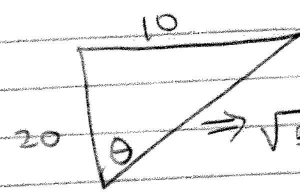
θ = angle formed by string and vertical.

Given $\frac{dx}{dt} = 5$ ft/sec. $\Rightarrow$ after 2 seconds, travelled 10 ft.

$\frac{d\theta}{dt} = ?$ when $x = 10$ ft.

• Equation $\tan\theta = \frac{x}{20}$

• Differentiate $\frac{d}{dt}\left[\tan\theta\right] = \frac{d}{dt}\left[\frac{x}{20}\right]$



$\Rightarrow \sqrt{500} = 10\sqrt{5}$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

$$\sec\theta = \frac{10\sqrt{5}}{20} = \frac{\sqrt{5}}{2}$$

• Substitute $\left(\frac{\sqrt{5}}{2}\right)^2 \frac{d\theta}{dt} = \frac{1}{20} \cdot 5$

• Solve $\frac{5}{4}\frac{d\theta}{dt} = \frac{1}{4}$

$$\frac{d\theta}{dt} = \frac{1}{5} \text{ rad./sec.}$$

• Answer: The angle between string and vertical is increasing at a rate of $\frac{1}{5}$ radians per second.

7. $W(x) = \cos^2(2x) + \tan(2x) + 3\sec x$

$$W'(x) = 2\cos(2x)\cdot(-\sin(2x)) + \sec^2(2x)\cdot 2 + 3\sec x \tan x$$

$2\cdot\pi/6$

$$W'(\pi/6) = -2\cos(\pi/3)\sin(\pi/3) + 2\sec^2(\pi/3) + 3\sec(\pi/6)\tan(\pi/6)$$

$$= -\cancel{2}\left(\frac{1}{\cancel{2}}\right)\cdot\left(\frac{\sqrt{3}}{2}\right) + 2\,(2)^2 + \cancel{3}\cdot\left(\frac{2}{\cancel{\sqrt{3}}}\right)\cdot\frac{1}{\cancel{\sqrt{3}}}$$

$$= -\frac{\sqrt{3}}{2} + 8 + 2$$

$$= -\frac{\sqrt{3}}{2} + 10 = \boxed{\frac{20-\sqrt{3}}{2}}$$

$\frac{20}{2}$