Worksheet #6 Answer Key

1.a $f(x) = (x-3)^4 (x+2)^6$ Horizontal Tangent $\longleftrightarrow f'(x) = 0$.

$$f'(x) = (x-3)^4 \cdot 6(x+2)^5 + (x+2)^6 \cdot 4(x-3)^3$$

$$= 2(x-3)^3(x+2)^5 \left[3(x-3) + 2(x+2)\right]$$

$3x - 9 + 2x + 4$

$$= 2(x-3)^3(x+2)^5 [5x-5]$$

$$= 10(x-3)^3(x+2)^5(x-1) \overset{\text{set}}{=} 0 \Rightarrow \boxed{x = 3,\ x = -2,\ x = 1}$$

b. $f(x) = \dfrac{(x+1)^4}{(x+2)^5}$

$$f'(x) = \frac{(x+2)^5 \cdot 4(x+1)^3 - (x+1)^4 \cdot 5(x+2)^4}{(x+2)^{10}}$$

$4x + 8 - 5x - 5$

$$= \frac{(x+2)^4(x+1)^3\left[4(x+2) - 5(x+1)\right]}{(x+2)^{10}}$$

$$= \frac{(x+2)^4(x+1)^3(3-x)}{(x+2)^{10}}$$ cancel $(x+2)$ terms

$$= \frac{(x+1)^3(3-x)}{(x+2)^6} \overset{\text{set}}{=} 0 \Rightarrow \boxed{x = -1,\ x = 3}$$

2a. $f(x) = \cos^4(x^3-5) = \left[\cos(x^3-5)\right]^4$

$$f'(x) = 4\left[\cos(x^3-5)\right]^3 \cdot (-\sin(x^3-5))(3x^2) = \boxed{-12x^2\cos^3(x^3-5)\cdot\sin(x^3-5)}$$

b. $g(x) = \sqrt{x^{7/9} + (x^2+1)^{4/3}}$

$$g'(x) = \frac{1}{2\sqrt{x^{7/9} + (x^2+1)^{4/3}}} \cdot \left[\frac{7}{9}x^{-2/9} + \frac{4}{3}(x^2+1)^{1/3}(2x)\right] = \boxed{\frac{\frac{7}{9}x^{-2/9} + \frac{8}{3}x(x^2+1)^{1/3}}{2\sqrt{x^{7/9} + (x^2+1)^{4/3}}}}$$

multiply to get $\left(\frac{2\sqrt{x+2}}{2\sqrt{x+2}}\right)$ common denomi[nator]

2c. $$\frac{d}{dx}\left[(x+1)^7\sqrt{x+2}\right] = (x+1)^7\frac{1}{2\sqrt{x+2}} + \sqrt{x+2}\cdot 7(x+1)^6$$

$$= \frac{(x+1)^7}{2\sqrt{x+2}} + \frac{14(x+2)(x+1)^6}{2\sqrt{x+2}}$$

$$= \frac{(x+1)^7 + 14(x+2)(x+1)^6}{2\sqrt{x+2}}$$

$$= \frac{(x+1)^6\left[(x+1)+14(x+2)\right]}{2\sqrt{x+2}}$$

$x+1+14x+28$

$$= \boxed{\frac{(x+1)^6\left[15x+29\right]}{2\sqrt{x+2}}}$$

3a. $$y^2 + \sec(x^2y) = 1$$

$$\frac{d}{dx}\left[y^2+\sec(x^2y)\right] = \frac{d}{dx}\left[1\right]$$

$$2y\frac{dy}{dx} + \sec(x^2y)\tan(x^2y)\left[x^2\frac{dy}{dx} + y\cdot 2x\right] = 0$$

$$2y\frac{dy}{dx} + x^2\sec(x^2y)\tan(x^2y)\frac{dy}{dx} + 2xy\sec(x^2y)\tan(x^2y) = 0.$$

Isolate + solve for $\frac{dy}{dx}$

$$\left[2y + x^2\sec(x^2y)\tan(x^2y)\right]\frac{dy}{dx} = -2xy\sec(x^2y)\tan(x^2y)$$

Solve $$\frac{dy}{dx} = \boxed{\frac{-2xy\sec(x^2y)\tan(x^2y)}{2y+x^2\sec(x^2y)\tan(x^2y)}}$$

3b. $x^2y^4 + y^5 = x^3 + \frac{1}{y^2}$

y^{-2}

$$\frac{d}{dx}\left[x^2y^4+y^5\right] = \frac{d}{dx}\left[x^3+\frac{1}{y^2}\right]$$

$$x^2\,4y^3\frac{dy}{dx} + y^4\cdot 2x + 5y^4\frac{dy}{dx} = 3x^2 - 2y^{-3}\cdot\frac{dy}{dx}$$

Isolate and Solve $\frac{dy}{dx}$

$$4x^2y^3\frac{dy}{dx} + 5y^4\frac{dy}{dx} + \frac{2}{y^3}\frac{dy}{dx} = 3x^2 - 2xy^4$$

$$\left[4x^2y^3 + 5y^4 + \frac{2}{y^3}\right]\frac{dy}{dx} = 3x^2 - 2xy^4$$

Solve $\frac{dy}{dx} = \boxed{\dfrac{3x^2-2xy^4}{4x^2y^3+5y^4+2/y^3}}$

c. $\tan\left(\frac{x}{y}\right) + \sqrt{x} = \sqrt{y}$

$$\frac{d}{dx}\left[\tan\left(\frac{x}{y}\right)+\sqrt{x}\right] = \frac{d}{dx}\left[\sqrt{y}\right]$$

$$\sec^2\left(\frac{x}{y}\right)\cdot\left[\frac{y(1) - x\frac{dy}{dx}}{y^2}\right] + \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{y}}\frac{dy}{dx}$$

split distribute

$$\sec^2\left(\frac{x}{y}\right)\left[\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}\right] + \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{y}}\frac{dy}{dx}$$

$$\frac{1}{y}\sec^2\left(\frac{x}{y}\right) - \frac{x}{y^2}\sec^2\left(\frac{x}{y}\right)\frac{dy}{dx} + \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{y}}\frac{dy}{dx}$$

3c (continued)

$$\cancel{-}\frac{x}{y^2}\sec^2\left(\frac{x}{y}\right)\frac{dy}{dx}\cancel{-}\frac{1}{2\sqrt{y}}\frac{dy}{dx}=\cancel{-}\frac{1}{y}\sec^2\left(\frac{x}{y}\right)\cancel{-}\frac{1}{2\sqrt{x}}$$

clear minus signs and factor

$$\left[\frac{x}{y^2}\sec^2\left(\frac{x}{y}\right)+\frac{1}{2\sqrt{y}}\right]\frac{dy}{dx}=\frac{1}{y}\sec^2\left(\frac{x}{y}\right)+\frac{1}{2\sqrt{x}}$$

Solve $$\frac{dy}{dx}=\boxed{\frac{\frac{1}{y}\sec^2\left(x/y\right)+\frac{1}{2\sqrt{x}}}{x/y^2\,\sec^2\left(\frac{x}{y}\right)+\frac{1}{2\sqrt{y}}}}$$

4a. $f(x)=\dfrac{\sin x\cos^3 x}{\sqrt{3/x-4/x^3}}$ Quotient Rule, then product + chain rules

$3x^{-1}-4x^{-3}$

$$f'(x)=\frac{\sqrt{3/x-4/x^3}\left[\sin x\cdot 3\cos^2 x\cdot(-\sin x)+\cos^3 x\cdot\cos x\right]-\sin x\cos^3 x\cdot\frac{1}{2\sqrt{3/x-4/x^3}}\left(-3x^{-2}+12x^{-4}\right)}{\left(\sqrt{3/x-4/x^3}\right)^2}$$

squeeze

$\left(3/x-4/x^3\right)$

$4x^{-1}$ $\quad$ $6x^{-6}$

4b. $y = \tan^4\left(\frac{4}{x} + \cos x\right)\sqrt{6/x^6 + \sec(3x)}$ $\qquad$ Product Rule, then Chain Rule.

$$\frac{dy}{dx} = \tan^4\left(\frac{4}{x} + \cos x\right) \cdot \frac{1}{2\sqrt{6/x^6 + \sec(3x)}} \cdot \left[-36x^{-7} + \sec(3x)\tan(3x)\cdot(3)\right] +$$

$$\sqrt{6/x^6 + \sec(3x)} \cdot 4\tan^3\left(4/x + \cos x\right)\cdot \sec^2\left(4/x + \cos x\right)\cdot\left(-4x^{-2} - si\right.$$

$6x^{-6}$

4c. $\frac{d}{dx} \sec\left[\frac{6/x^6 + \tan(3x)}{4/x + \cos x}\right]$ $\qquad$ Chain Rule, then Quotient Rule + Chain Ru

$4x^{-1}$

$$= \sec\left[\frac{6/x^6 + \tan(3x)}{4/x + \cos x}\right]\tan\left[\frac{6/x^6 + \tan(3x)}{4/x + \cos x}\right]\cdot$$

$$\boxed{\frac{\left(4/x + \cos x\right)\left(-36x^{-7} + \sec^2(3x)(3)\right) - \left(\frac{6}{x^6} + \tan(3x)\right)\cdot\left(-4x^{-2} - \sin x\right)}{\left(4/x + \cos x\right)^2}}$$

5. $h(x) = f(x) \cdot g(x)$

$h'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$ Product Rule.

- $h'(1) = f(1) \cdot g'(1) + g(1) \cdot f'(1)$

$= 4 \cdot 7 + 2 \cdot (-3) = 28 - 6 = \boxed{22}$

$K(x) = \dfrac{g(x)}{f(x)}$ Quotient Rule → Watch Order!

$K'(x) = \dfrac{f(x) \cdot g'(x) - g(x) \cdot f'(x)}{[f(x)]^2}$

- $K'(3) = \dfrac{f(3) \cdot g'(3) - g(3) f'(3)}{[f(3)]^2}$

$= \dfrac{\cancel{3 \cdot 0} - (-1)(-2)}{[3]^2} = \boxed{\dfrac{-2}{9}}$

$P(x) = f(x) \cdot f(x)$

$P'(x) = f(x) \cdot f'(x) + f(x) \cdot f'(x)$

$= 2 f(x) \cdot f'(x)$

OR

$P(x) = [f(x)]^2$

$P'(x) = 2 \cdot f(x) \cdot f'(x)$

- $P'(1) = 2 f(1) f'(1)$

$= 2(4)(-3) = \boxed{-24}$

$Q(x) = f \circ g(x) = f(g(x))$

$Q'(x) = f'(g(x)) \cdot g'(x)$

- $Q'(2) = f'(g(2)) \cdot g'(2)$

  $= f'(1) \cdot 5$

  $= (-3) \cdot 5 = \boxed{-15}$

$W(x) = g \circ g(x) = g(g(x))$

$W'(x) = g'(g(x)) \cdot g'(x)$

- $W'(1) = g'(g(1)) \cdot g'(1)$

  $= g'(2) \cdot 7$

  $= 5 \cdot 7 = \boxed{35}$