Worksheet #5

1. Let $g(x) = \dfrac{x}{1+3x^2}$,

Quotient Rule

a. $g'(x) = \dfrac{(1+3x^2)(1) - x(6x)}{(1+3x^2)^2} = \dfrac{1+3x^2-6x^2}{(1+3x^2)^2} = \boxed{\dfrac{1-3x^2}{(1+3x^2)^2}}$

b. Horizontal Tangent Line means $g'(x)=0$.

Set $g'(x) = \dfrac{1-3x^2}{(1+3x^2)^2} \overset{\text{set}}{=} 0 \Rightarrow 1-3x^2=0$

$3x^2=1$

$x^2 = 1/3$

$x = \pm \dfrac{1}{\sqrt{3}}$

$f\left(\frac{1}{\sqrt{3}}\right) = \dfrac{\frac{1}{\sqrt{3}}}{1+3\left(\frac{1}{\sqrt{3}}\right)^2} = \dfrac{\frac{1}{\sqrt{3}}}{1+1} = \dfrac{1}{2\sqrt{3}}$

$f(-1/\sqrt{3}) = \dfrac{-1/\sqrt{3}}{1+3(-1/\sqrt{3})^2} = \dfrac{-1}{2\sqrt{3}}$

Points $\left(\frac{1}{\sqrt{3}}, f\left(\frac{1}{\sqrt{3}}\right)\right) = \boxed{\left(\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)}$

$\left(-\frac{1}{\sqrt{3}}, f(-1/\sqrt{3})\right) = \boxed{\left(-\frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right)}$

c. First simplify $g'(x) = \dfrac{1-3x^2}{(1+3x^2)^2} = \dfrac{1-3x^2}{1+6x^2+9x^4}$

$g''(x) = \dfrac{(1+6x^2+9x^4)(-6x) - (1-3x^2)(12x+36x^3)}{\underbrace{(1+6x^2+9x^4)^2}_{(1+3x^2)^4}}$ o.k

$= \dfrac{-6x-36x^3-54x^5 - (12x+\cancel{36x^3}-\cancel{36x^3}-108x^5)}{(1+6x^2+9x^4)^2}$ distribute minus

$= \dfrac{-6x-36x^3-54x^5 -12x+108x^5}{(1+6x^2+9x^4)^2}$

$= \boxed{\dfrac{54x^5-36x^3-18x}{(1+6x^2+9x^4)^2}}$

2. $\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

3. $\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x$

4. $f(x) = \sec x \tan x$ Product Rule

$f'(x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x)$

$= \boxed{\sec^3 x + \sec x \tan^2 x}$

5. $f(x) = \frac{\sin x + \cos x}{\sec x \tan x}$ Quotient + Product Rule(s)

$f'(x) = \frac{\sec x \tan x(\cos x - \sin x) - (\sin x + \cos x)\boxed{\sec^3 x + \sec x \tan^2 x}}{\sec^2 x \tan^2 x}$

$= \frac{\sec x \tan x \cos x - \sec x \tan x \sin x - (\sin x \sec^3 x + \sin x \sec x \tan^2 x + \cos x \sec^3 x + \cos x \sec x \tan^2}{\sec^2 x \tan^2 x}$

Simplify?

6. a. $f(x) = \sin x$

$f'(x) = \cos x$

Slope $f'(0) = \cos 0 = 1$

Point $(0, \sin 0) = (0,0)$

Point Slope Form $y - 0 = 1(x-0)$ $\boxed{y = x}$ Tangent Line.

b. $f(x) = \cos x$

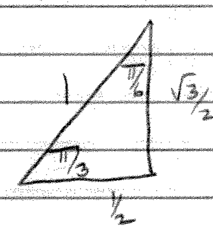
$f'(x) = -\sin x$

Slope $f'(\pi/6) = -\sin(\pi/6) = -1/2$

Point $(\pi/6, f(\pi/6)) = (\pi/6, \cos \pi/6) = (\pi/6, \sqrt{3}/2)$

Point-Slope Form $y - \sqrt{3}/2 = -1/2 (x - \pi/6)$

$$\boxed{y = -1/2\, x + \frac{\pi}{12} + \frac{\sqrt{3}}{2}}$$



c. $f(x) = \tan x$

$f'(x) = \sec^2 x$

Slope $f'(\pi/3) = \sec^2(\pi/3) = \dfrac{1}{\cos^2(\pi/3)} = \dfrac{1}{(1/2)^2} = \dfrac{1}{(1/4)} = 4.$

Point $(\pi/3, f(\pi/3)) = (\pi/3, \tan(\pi/3)) = (\pi/3, \sqrt{3})$

$$\tan \pi/3 = \frac{\sin \pi/3}{\cos \pi/3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Point Slope Form

$$y - \sqrt{3} = 4(x - \pi/3)$$

$$\boxed{y = 4x - 4\pi/3 + \sqrt{3}}$$

Algebra First then Power Rules

7. $f(x) = (x^4 - 5x^3 + 6)\sqrt{x} = (x^4 - 5x^3 + 6)(x^{1/2}) = x^{9/2} - 5x^{7/2} + 6x^{1/2}$

$f'(x) = \frac{9}{2}x^{7/2} - \frac{35}{2}x^{5/2} + 3x^{-1/2}$

(*) $f''(x) = \boxed{\frac{63}{4}x^{5/2} - \frac{175}{4}x^{3/2} - \frac{3}{2}x^{-3/2}}$

OR Product Rule then Algebra.

$f'(x) = (x^4 - 5x^3 + 6)\frac{1}{2\sqrt{x}} + \sqrt{x}(4x^3 - 15x^2)$

$= \underbrace{\tfrac{1}{2}x^{7/2}} - \tfrac{5}{2}x^{5/2} + 3x^{-1/2} + \underbrace{4x^{7/2}} - 15x^{5/2}$

$= \frac{9}{2}x^{7/2} - \frac{35}{2}x^{5/2} + 3x^{-1/2}$

$f''(x)$ = Same as above. (*)

8. $f(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3}$

a. $f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}\left(x^{1/3} + \frac{1}{x^{2/3}}\right) = \frac{4}{3}\left(\frac{x}{x^{2/3}} + \frac{1}{x^{2/3}}\right) = \boxed{\frac{4}{3}\left[\frac{x+1}{x^{2/3}}\right]}$

OR Product Rule First

$f'(x) = x^{1/3}(1) + (x+4)\frac{1}{3}(x^{-2/3})$

$= x^{1/3} + \frac{(x+4)}{3x^{2/3}}$

$= \frac{3x}{3x^{2/3}} + \frac{(x+4)}{3x^{2/3}}$

$= \frac{4x+4}{3x^{2/3}}$ Match ✓

8b. $f'(x) = \frac{4}{3}\left(\frac{x+1}{x^{2/3}}\right) \overset{set}{=} 0$.

$\Rightarrow x+1=0 \Rightarrow x=-1 \Rightarrow f(-1) = (-1)^{1/3}(-1+4) = (-1)(3) = -3$.

Point $\boxed{(-1,-3)}$

9. a. $\lim\limits_{x\to 0} \frac{x}{\sin x} = \lim\limits_{x\to 0} \frac{1}{\left(\frac{\sin x}{x}\right)} = \frac{\lim\limits_{x\to 0} 1}{\lim\limits_{x\to 0} \frac{\sin x}{x} \to 1} = \frac{1}{1} = \boxed{1}$

b. $\lim\limits_{x\to 0} \frac{\tan(3x)}{\sin x} = \lim\limits_{x\to 0} \frac{\left(\frac{\sin(3x)}{\cos(3x)}\right)}{\sin x} = \lim\limits_{x\to 0} \frac{\sin(3x)}{\sin x \cos(3x)}$

$= \lim\limits_{x\to 0} \frac{\sin(3x)}{\sin x}\frac{1/x}{1/x} \cdot \lim\limits_{x\to 0} \frac{1}{\cos(3x)} \to 1$ $\quad \frac{1}{\cos 0}$

$= \lim\limits_{x\to 0} \frac{\frac{\sin(3x)}{x}}{\frac{\sin x}{x}} \cdot (1)$

$= \lim\limits_{x\to 0} \frac{3\frac{\sin(3x)}{3x} \to 3}{\frac{\sin x}{x} \to 1} = \frac{3\lim\limits_{x\to 0} \frac{\sin 3x}{3x} \to 1}{\lim\limits_{x\to 0} \frac{\sin x}{x} \to 1} = \frac{3}{1} = \boxed{3}$

c. $\lim\limits_{x\to 0} \frac{\sin 5x}{x^3-4x} = \lim\limits_{x\to 0} \frac{\sin 5x}{x(x^2-4)} = \lim\limits_{x\to 0} \frac{5}{5}\frac{\sin 5x}{x} \cdot \lim\limits_{x\to 0} \frac{1}{x^2-4}$ $\quad$ DSP. $\frac{1}{-4}$

$= 5\lim\limits_{x\to 0} \frac{\sin 5x}{5x} \cdot \left(-\frac{1}{4}\right)$

$= \boxed{\frac{-5}{4}}$

10. Simplify $\quad 6(x+1)^2(1-2x)^4 + (x+1)^3\, 4(1-2x)^3(-2)$

$= 2(x+1)^2(1-2x)^3\left[3(1-2x) - 4(x+1)\right] = \boxed{2(x+1)^2(1-2x)^3(-10x-1)}$