Worksheet 4 Fall 2012 Math 111 Answer Key

1a. $\lim_{x\to 7} \frac{x^2-4x-21}{x^2-3x} \overset{DSP}{=} \frac{0}{28} = \boxed{0}$

$49-21$

b. $\lim_{x\to 4} \frac{x^2-3x-4}{|x-4|}$ $\frac{0}{0}$ $\boxed{DNE}$ since RHL ≠ LHL

$$|x-4| = \begin{cases} x-4 & \text{if } x-4 \geq 0 \quad x\geq 4 \\ -(x-4) & \text{if } x-4<0 \quad x<4 \end{cases}$$

RHL: $\lim_{x\to 4^+} \frac{x^2-3x-4}{|x-4|} = \lim_{x\to 4^+} \frac{\cancel{(x-4)}(x+1)}{\cancel{x-4}} = \lim_{x\to 4^+} x+1 = 5$

LHL: $\lim_{x\to 4^-} \frac{x^2-3x-4}{|x-4|} = \lim_{x\to 4^-} \frac{\cancel{(x-4)}(x+1)}{-\cancel{(x-4)}} = \lim_{x\to 4^-} -(x+1) = -5$

c. $\lim_{x\to 4} \frac{x^2-2x-8}{x^2-5x+4} \overset{0/0}{=} \lim_{x\to 4} \frac{\cancel{(x-4)}(x+2)}{\cancel{(x-4)}(x-1)} = \lim_{x\to 4} \frac{x+2}{x-1} = \frac{6}{3} = \boxed{2}$

$16-8-8$ $16-20+4$

d. $\lim_{x\to -5} \frac{\frac{1}{1-x} - \frac{1}{6}}{x^2+3x-10} \overset{0/0}{=} \lim_{x\to -5} \frac{\left[\frac{6-(1-x)}{(1-x)6}\right]}{x^2+3x-10} = \lim_{x\to -5} \frac{\cancel{5+x}}{(1-x)(6)\cancel{(x+5)}(x-2)}$

$(x+5)(x-2)$

$= \lim_{x\to -5} \frac{1}{6(1-x)(x-2)} = \frac{1}{6(6)(-7)} = \boxed{\frac{-1}{252}}$

$36 \times 7 = 252$

e. $\lim_{x\to 3} \frac{x^2-12x+27}{x^2-6x+9} \overset{0/0}{=} \lim_{x\to 3} \frac{\cancel{(x-3)}(x-9)}{\cancel{(x-3)}(x-3)} = \lim_{x\to 3} \frac{x-9}{x-3}$ DNE since RHL ≠ LHL

RHL: $\lim_{x\to 3^+} \frac{x-9}{x-3} = \frac{-6}{0^+} = -\infty$

LHL: $\lim_{x\to 3^-} \frac{x-9}{x-3} = \frac{-6}{0^-} = +\infty$

1 f. $\lim_{x\to 3} \frac{x^2-12x+27}{x^2-9} = \lim_{x\to 3} \frac{(x-9)\cancel{(x-3)}}{(x+3)\cancel{(x-3)}} = \frac{-6}{6} = \boxed{-1}$

g. $\lim_{x\to 4} \frac{x+2}{4-x}$ $\quad 6/0$ DNE since RHL ≠ LHL

RHL: $\lim_{x\to 4^+} \frac{x+2}{4-x} = \frac{6}{0^-} = -\infty$

LHL: $\lim_{x\to 4^-} \frac{x+2}{4-x} = \frac{6}{0^+} = +\infty$.

h. $\lim_{x\to -4} \frac{x+2}{x+4}$ $\quad -2/0$ DNE since RHL ≠ LHL

RHL: $\lim_{x\to -4^+} \frac{x+2}{x+4} = \frac{-2}{0^+} = -\infty$

think $x = -3.999\ldots$

LHL: $\lim_{x\to -4^-} \frac{x+2}{x+4} = \frac{-2}{0^-} = +\infty$.

i. $\lim_{x\to 2} \frac{3-\sqrt{x+7}}{x^2-3x+2}$ $\quad 0/0$ $= \lim_{x\to 2} \frac{3-\sqrt{x+7}}{x^2-3x+2} \cdot \left(\frac{3+\sqrt{x+7}}{3+\sqrt{x+7}}\right) = \lim_{x\to 2} \frac{\overbrace{9-(x+7)}^{2-x}}{(x-2)(x-1)(3+\sqrt{x+7})}$

$= \lim_{x\to 2} \frac{-\cancel{(x-2)}}{\cancel{(x-2)}(x-1)(3+\sqrt{x+7})} = \frac{-1}{3+\sqrt{9}} = \frac{-1}{3+3} = \boxed{\frac{-1}{6}}$

$G(x) = (x-1)^2+3$

j. $\lim_{x\to 1} \dfrac{G(x+2)+x-8}{G(2x)-3x^2-3x+2} = \lim_{x\to 1} \dfrac{(\overbrace{x+2-1}^{x+1})^2+3 \quad +x-8}{(2x-1)^2+3 \quad -3x^2-3x+2}$

$\dfrac{0}{0}$

$= \lim_{x\to 1} \dfrac{x^2+2x+1+3+x-8}{4x^2-4x+1+3-3x^2-3x+2} = \lim_{x\to 1} \dfrac{x^2+3x-4}{x^2-7x+6}$

$= \lim_{x\to 1} \dfrac{(x+4)\cancel{(x-1)}}{(x-6)\cancel{(x-1)}} = \dfrac{5}{-5} = \boxed{-1}$

k. $\lim_{x\to 7} \dfrac{x-7}{|7-x|}$ $\dfrac{0}{0}$ DNE since RHL $\neq$ LHL

$|7-x| = \begin{cases} 7-x & \text{if } 7-x \geq 0 \quad (x \leq 7) \\ -(7-x) & \text{if } 7-x < 0 \quad (x > 7) \end{cases}$

RHL: $\lim_{x\to 7^+} \dfrac{x-7}{|7-x|} = \lim_{x\to 7^+} \dfrac{x-7}{-(7-x)} = \lim_{x\to 7^+} \dfrac{\cancel{x-7}}{\cancel{x-7}} = 1$

LHL: $\lim_{x\to 7^-} \dfrac{x-7}{|7-x|} = \lim_{x\to 7^-} \dfrac{x-7}{7-x} = \lim_{x\to 7^-} \dfrac{\cancel{x-7}}{-\cancel{(x-7)}} = -1$

l) $f(x) = x+3$

$\lim_{x\to 5} \dfrac{f(x^2)-28}{(f(x))^2-10x-14} = \lim_{x\to 5} \dfrac{x^2+3-28}{(x+3)^2-10x-14} = \lim_{x\to 5} \dfrac{x^2-25}{x^2+6x+9-10x-14}$

$= \lim_{x\to 5} \dfrac{x^2-25}{x^2-4x-5} \; \dfrac{0}{0} = \lim_{x\to 5} \dfrac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}(x+1)} = \dfrac{10}{6} = \boxed{\dfrac{5}{3}}$

2. Prove $\lim_{x\to 3} 1-2x=-5$

Scratchwork: $|f(x)-L| = |(1-2x)-(-5)| = |1-2x+5| = |6-2x| = |-2(x-3)|$
$= 2|x-3| \overset{\text{want}}{<} \varepsilon$
choose $|x-3| < \varepsilon/2$
$\Downarrow$
δ

Proof: Let $\varepsilon > 0$ be given.
Choose $\delta = \varepsilon/2$
Given x such that $0<|x-3|<\delta$
then

$$|f(x)-L| = |(1-2x)-(-5)| = |1-2x+5| = |6-2x| = |-2(x-3)|$$
$$= 2|x-3| < 2\cdot \varepsilon/2 = \varepsilon$$

$\square$

3 a. $f(x)=\sqrt{x^2-5x+3}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^2-5(x+h)+3} - \sqrt{x^2-5x+3}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2-5(x+h)+3} - \sqrt{x^2-5x+3}}{h} \cdot \left[\frac{\sqrt{(x+h)^2-5(x+h)+3} + \sqrt{x^2-5x+3}}{\sqrt{(x+h)^2-5(x+h)+3} + \sqrt{x^2-5x+3}}\right]$$

$$= \lim_{h\to 0} \frac{(x+h)^2-5(x+h)+3 - (x^2-5x+3)}{h\left[\sqrt{(x+h)^2-5(x+h)+3}+\sqrt{x^2-5x+3}\right]}$$ don't drop.

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{5x}-5h+\cancel{3}-\cancel{x^2}+\cancel{5x}-\cancel{3}}{h\left[\sqrt{(x+h)^2-5(x+h)+3}+\sqrt{x^2-5x+3}\right]}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h-5)}{\cancel{h}\left[\sqrt{(x+h)^2-5(x+h)+3}+\sqrt{x^2-5x+3}\right]}$$

$$= \lim_{h\to 0} \frac{2x+\overset{0}{h}-5}{\sqrt{(x+\underset{0}{h})^2-5(x+\underset{0}{h})+3}+\sqrt{x^2-5x+3}}$$

$$= \boxed{\frac{2x-5}{2\sqrt{x^2-5x+3}}}$$

Matches Chain Rule!

3b. $f(x) = \dfrac{1-3x}{x+2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(\frac{1-3(x+h)}{x+h+2}\right) - \left(\frac{1-3x}{x+2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{(x+2)\left[1-3x-3h\right] - (1-3x)(x+h+2)}{h\,(x+h+2)\,(x+2)}$$

$$= \lim_{h\to 0} \frac{x-3x^2-3xh+2-6x-6h-\left[x+h+2-3x^2-3xh-6x\right]}{h\,(x+h+2)\,(x+2)}$$

$$= \lim_{h\to 0} \frac{\cancel{x}-\cancel{3x^2}-\cancel{3xh}+\boxed{\cancel{2}}-\cancel{6x}\underbrace{-6h-h}_{-7h}-\cancel{x}-\boxed{\cancel{2}}+\cancel{3x^2}+\cancel{3xh}+\cancel{6x}}{h\,(x+h+2)\,(x+2)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-7)}{\cancel{h}(x+\overset{0}{\cancel{h}}+2)(x+2)}$$

$$= \boxed{\frac{-7}{(x+2)^2}}$$

Check Quotient Rule $\quad f'(x) = \dfrac{(x+2)(-3)-(1-3x)(1)}{(x+2)^2}$

$$= \frac{-\cancel{3x}-6-1+\cancel{3x}}{(x+2)^2}$$

$$= \frac{-7}{(x+2)^2}$$

4. $f(x) = 5 - 7x + 4x^2 - x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{5 - 7(x+h) + 4\underbrace{(x+h)^2}_{x^2+2xh+h^2} - \underbrace{(x+h)^3}_{x^3+3x^2h+3xh^2+h^3} - (5 - 7x + 4x^2 - x^3)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{5} - \cancel{7x} - 7h + \cancel{4x^2} + 8xh + 4h^2 - \cancel{x^3} - 3x^2h - 3xh^2 - h^3 - \cancel{5} + \cancel{7x} - \cancel{4x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-7 + 8x + \overset{0}{4h} - 3x^2 - \overset{0}{3xh} - \overset{0}{h^2})}{\cancel{h}}$$

$$= \boxed{-7 + 8x - 3x^2}$$

$f'(1) = -7 + 8 - 3 = -2$ slope   Point $(1, f(1))$
$(1, 1)$

a. $f(1) = 5 - 7 + 4 - 1 = 1$

Point Slope Form $y - 1 = -2(x - 1)$

$y - 1 = -2x + 2$

$\boxed{y = -2x + 3}$

b. $f'(x) = -7 + 8x - 3x^2 \overset{\text{set}}{=} 0$

$3x^2 - 8x + 7 = 0$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 4(3)(7)}}{6}$

Never any Horizontal Tangent Lines.

$= \frac{8 \pm \sqrt{64 - 84}}{6}$ No Solution Negative Discrim

5. $f(x)$ continuous at $x = -7$ means

$$\lim_{x \to -7} f(x) = f(-7)$$

6. a. $g(7) \overset{\text{by continuity of } g \text{ at } x=7.}{=} \lim_{x \to 7} g(x) = \boxed{3}$

b. $g \circ f(3) = g(f(3)) = g(2) \overset{\text{by continuity of } g \text{ at } x=2.}{=} \lim_{x \to 2} g(x) = \boxed{6}$

c. $f \circ g(7) = f(\underbrace{g(7)}_{\text{part a}}) = f(3) = \boxed{2}$

d. f is not continuous at $x = 3$ since $\lim_{x \to 3} f(x) = 5 \neq f(3) = 2$.

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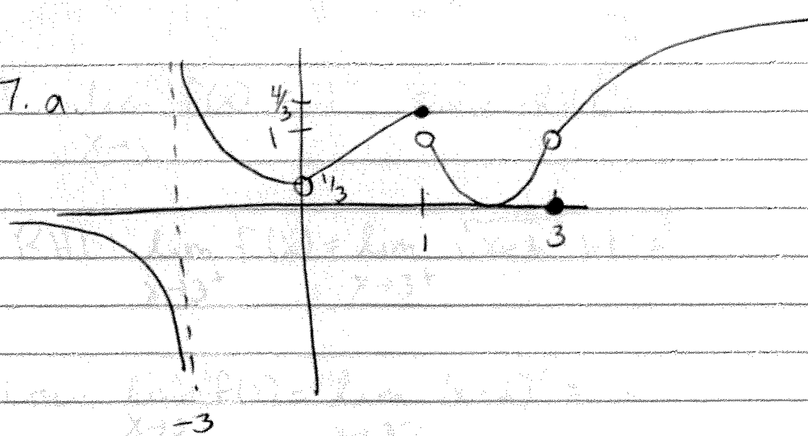
7. a.



b. Domain $\{x : x \neq -3, 0\}$

c. $\lim\limits_{x\to -3} f(x)$ $\boxed{\text{DNE}}$ since RHL $\neq$ LHL.

RHL: $\lim\limits_{x\to -3^+} f(x) = \lim\limits_{x\to -3^+} \frac{1}{x+3} = \frac{1}{0^+} = +\infty$

LHL: $\lim\limits_{x\to -3^-} f(x) = \lim\limits_{x\to -3^-} \frac{1}{x+3} = \frac{1}{0^-} = -\infty$.

d. $\lim\limits_{x\to 0} f(x) = \boxed{\frac{1}{3}}$ since RHL = LHL.

RHL: $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x + \frac{1}{3} = \frac{1}{3}$

LHL: $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{1}{x+3} = \frac{1}{3}$

e. $\lim\limits_{x\to 1} f(x)$ DNE since LHL $\neq$ RHL

RHL: $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (x-2)^2 = 1$

LHL: $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x + \frac{1}{3} = \frac{4}{3}$

f. $\lim_{x\to 3} f(x) = 1$ since RHL = LHL

$$\begin{cases} \text{RHL: } \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x-3} + 1 = 1 \\ \text{LHL: } \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x-2)^2 = 1 \end{cases}$$

g. f(x) is <u>discontinuous</u> @ $x=-3$ because $\lim_{x\to -3} f(x)$ DNE (See Question c

OR $f(-3)$ undefined.

@ $x=0$ since $f(0)$ undefined.

@ $x=1$ since $\lim_{x\to 1} f(x)$ DNE (See Question e)

@ $x=3$ since $\underbrace{\lim_{x\to 3} f(x)}_{1} \neq \underbrace{f(3)}_{0}$