Math 111, Section 01, Fall 2012

Worksheet 3, Thursday, September 20, 2012 ANSWER KEY!!!

1. Compute the following limits. Be clear if they equal a value, or $+\infty$, $-\infty$, or DNE.

(a) $\lim\limits_{x\to 2} \dfrac{x^2-9x+14}{x^2-4x+4} = \lim\limits_{x\to 2} \dfrac{(x-7)(x-2)}{(x-2)^2} = \lim\limits_{x\to 2} \dfrac{x-7}{x-2}$

$\boxed{\text{DOES NOT EXIST}}$ since RHL $\neq$ LHL

RHL: $\lim\limits_{x\to 2^+} \dfrac{x-7}{x-2} = \dfrac{-5}{0^+} = -\infty$

LHL: $\lim\limits_{x\to 2^-} \dfrac{x-7}{x-2} = \dfrac{-5}{0^-} = +\infty$

Tip: This is a complicated problem where you have $\left(\dfrac{0}{0}\right)$ upon initial DSP. You are able to factor and cancel only ONE copy of $x-2$ in the denominator. As a result you are left with an infinite sign analysis $\left(\dfrac{-5}{0}\right)$, with RHL and LHL. A sort of combo problem if you will....

(b) $\lim\limits_{x\to 2} \dfrac{3-\sqrt{x+1}}{x-8} \overset{\text{L.L.}}{=} \dfrac{3-\sqrt{2+1}}{2-8} = \boxed{\dfrac{3-\sqrt{3}}{-6}}$ Don't rush to use a conjugate trick here, since you don't have $\left(\dfrac{0}{0}\right)$.

(c) $\lim\limits_{x\to 8} \dfrac{3-\sqrt{x+1}}{x-8} \cdot \dfrac{3+\sqrt{x+1}}{3+\sqrt{x+1}} = \lim\limits_{x\to 8} \dfrac{9-(x+1)}{(x-8)(3+\sqrt{x+1})}$

$= \lim\limits_{x\to 8} \dfrac{8-x}{(x-8)(3+\sqrt{x+1})} = \lim\limits_{x\to 8} \dfrac{-(x-8)}{(x-8)(3+\sqrt{x+1})}$

$= \lim\limits_{x\to 8} \dfrac{-1}{3+\sqrt{x+1}} = \dfrac{-1}{3+\sqrt{8+1}} = \dfrac{-1}{3+\sqrt{9}} = \dfrac{-1}{3+3} = \boxed{\dfrac{-1}{6}}$

(d) $\lim\limits_{x\to 2} \dfrac{x^2-6+|x-4|}{3x-6} = \lim\limits_{x\to 2} \dfrac{x^2-6-(x-4)}{3x-6} = \lim\limits_{x\to 2} \dfrac{x^2-x-2}{3(x-2)}$

$= \lim\limits_{x\to 2} \dfrac{(x-2)(x+1)}{3(x-2)} = \lim\limits_{x\to 2} \dfrac{x+1}{3} = \lim\limits_{x\to 2} \dfrac{3}{3} = \boxed{1}$

$$|x-4| = \begin{cases} x-4 & \text{if } x-4 \geq 0 \\ -(x-4) & \text{if } x-4 < 0 \end{cases} = \begin{cases} x-4 & \text{if } x \geq 4 \\ -(x-4) & \text{if } x < 4 \end{cases}$$

Note: In this example, the limit is approaching $a = 2$. That is the specific case when $2 < 4$, so then $|x-4| = -(x-4)$. Look at the definition of the absolute value function there.

(e) $\lim_{x\to 2} \frac{x-2}{|2-x|}$ $\boxed{\text{DOES NOT EXIST}}$ since RHL $\neq$ LHL

RHL: $\lim_{x\to 2^+} \frac{x-2}{|2-x|} = \lim_{x\to 2^+} \frac{x-2}{-(2-x)} = \lim_{x\to 2^+} \frac{x-2}{x-2} = \lim_{x\to 2^+} 1 = \boxed{1}$

LHL: $\lim_{x\to 2^-} \frac{x-2}{|2-x|} = \lim_{x\to 2^-} \frac{x-2}{2-x} = \lim_{x\to 2^-} \frac{x-2}{-(x-2)} = \lim_{x\to 2^-} -1 = \boxed{-1}$

$$|2-x| = \begin{cases} 2-x & \text{if } 2-x \geq 0 \\ -(2-x) & \text{if } 2-x < 0 \end{cases} = \begin{cases} 2-x & \text{if } x \leq 2 \\ -(2-x) & \text{if } x > 2 \end{cases}$$

Note: In this example, watch the signs of the absolute value. It is $|2-x|$ **NOT** $|x-2|$ so watch the order carefully. Here the left and right hand limits may seem backwards in sign.

2. Write out the rigorous $\epsilon - \delta$ **Definition of the Limit** $\lim_{x\to a} f(x) = L$.

$\lim_{x\to a} f(x) = L$ means $\boxed{\text{For every } \varepsilon > 0, \text{ there exists a corresponding } \delta > 0 \text{ such that } \textbf{if } 0 < |x-a| < \delta, \textbf{ then } |f(x) - L| < \varepsilon.}$

The idea is the following: L is the limit for the function $f(x)$ near $x = a$ when the distance between the function values $f(x)$ and the limit L can be made arbitrarily tiny when there are x values that are sufficiently close to a. Think *given a horizontal ε zone, can you pick off the restricted vertical δ zone.* Can you draw the picture?

3. Give an ε-δ proof that $\lim_{x\to 1} 10 - 7x = 3$.

Scratchwork: we want $|f(x) - L| = |(10-7x) - 3| < \varepsilon$

$$|f(x) - L| = |(10-7x) - 3| = |7-7x| = |-7(x-1)| = |-7|\,|x-1| = 7|x-1| \text{ (want } < \varepsilon)$$

$$7|x-1| < \varepsilon \text{ means } |x-1| < \frac{\varepsilon}{7}$$

$$\text{So choose } \delta = \frac{\varepsilon}{7}$$

Proof: **Let** $\varepsilon > 0$ be given. **Choose** $\delta = \frac{\varepsilon}{7}$. **Given** x such that $0 < |x-1| < \delta$, then

$$|f(x) - L| = |(10-7x) - 3| = |7-7x| = |-7(x-1)| = |-7|\,|x-1| = 7|x-1| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon.$$

$\square$

Tip: Please be formal with writing the proof. Memorize the words and details involved. Make nice connections and conclusions carefully. A solid proof leaves no unclear details to the reader. Convince yourself, by looking at the definition above, that you have truly proven that 3 is the limit.

4. Give an ε-δ proof that $\lim_{x\to 6} 4 - \frac{3x}{2} = -5$.

Scratchwork: we want $|f(x) - L| = \left|\left(4 - \frac{3}{2}x\right) - (-5)\right| < \varepsilon$

$$|f(x) - L| = \left|\left(4 - \frac{3}{2}x\right) - (-5)\right| = \left|4 - \frac{3}{2}x + 5\right| = \left|9 - \frac{3}{2}x\right| = \left|-\frac{3}{2}(x-6)\right| =$$

$$\left|-\frac{3}{2}\right| |x-6| = \frac{3}{2}|x-6| \text{ (want } < \varepsilon)$$

$$\frac{3}{2}|x-6| < \varepsilon \text{ means } |x-6| < \frac{2}{3}\varepsilon$$

$$\text{So choose } \delta = \frac{2}{3}\varepsilon.$$

Proof: **Let** $\varepsilon > 0$ be given. **Choose** $\delta = \frac{2}{3}\varepsilon$. **Given** x such that $0 < |x-6| < \delta$, then

$$|f(x) - L| = \left|\left(4 - \frac{3}{2}x\right) - (-5)\right| = \left|4 - \frac{3}{2}x + 5\right| = \left|9 - \frac{3}{2}x\right|$$

$$= \left|-\frac{3}{2}(x-6)\right| = \left|-\frac{3}{2}\right| |x-6| = \frac{3}{2}|x-6| < \frac{3}{2}\cdot\frac{2}{3}\varepsilon = \varepsilon.$$

$\square$

Tip: Here you have proven that the distance from $f(x)$ to the Limit L is **small** for every ε... that means essentially it's the LIMIT!!!! That is, there are function values piling up on L.

5. Let $f(x)$ be a function with the property $\lim_{x\to 2} f(x) = 5$.

(a) Discuss what you can conclude about your function $f(x)$.

Since the two-sided limit here is 5, then we can conclude the function approaches the limit value of 5 from *both* the left and right sides of 2. That is

RHL: $\lim_{x\to 2+} f(x) = 5$

LHL: $\lim_{x\to 2-} f(x) = 5$.

(b) Discuss what you know about $f(2)$. Explain your reasoning.

We know nothing about $f(2)$. It might be defined or undefined. We know something about the limiting value near 2, as in part a, but nothing about the output value $f(2)$ itself. Remember, LIMITS do not care about what happens AT the $a = 2$ value.

6. Consider the function $f(x)$ that is continuous at $x = 3$. Assume that $f(3) = 4$.

(a) Write the definition for $f(x)$ being continuous at $x = 3$.

$f(x)$ is continuous at $x = 3$ means $\boxed{\lim_{x\to 3} f(x) = f(3)}$.

(b) Discuss what you know about $\lim_{x\to 3} f(x) =??$.

Since we assumed that $f(x)$ is continuous at $x = 3$, then we know from part (a) that $\lim_{x\to 3} f(x) = f(3)$. Since it was assumed that $f(3) = 4$, then piecing these equalities together we have $\lim_{x\to 3} f(x) = f(3) = \boxed{4}$.

7. Let $h(x) = \begin{cases} \dfrac{8}{x+2} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ \dfrac{1}{2}x - 4 & \text{if } 0 < x < 16 \\ 0 & \text{if } x = 16 \\ \sqrt{x} & \text{if } x > 16 \end{cases}$

Answer the following questions:

(a) Sketch the graph of $h(x)$. State the Domain of $h(x)$.

See me for a sketch. Domain $h = \{x : x \neq -2\}$.

(b) Compute $\lim_{x\to 16} h(x) = \boxed{4}$ since RHL=LHL.

$$\begin{cases} \text{LHL}: \lim_{x\to 16^-} h(x) = \lim_{x\to 16^-} \frac{1}{2}x - 4 = 4 \\ \text{RHL}: \lim_{x\to 16^+} h(x) = \lim_{x\to 16^+} \sqrt{x} = 4 \end{cases}$$

(c) Compute $\lim_{x\to 0} h(x)$ $\boxed{\text{DOES NOT EXIST}}$ since RHL $\neq$ LHL.

$$\begin{cases} \text{LHL}: \lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} \frac{8}{x+2} = 4 \\ \text{RHL}: \lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} \frac{1}{2}x - 4 = -4 \end{cases}$$

(d) Compute $\lim_{x\to -2} h(x)$. $\boxed{\text{DOES NOT EXIST}}$ since RHL $\neq$ LHL.

$$\begin{cases} \text{LHL}: \lim_{x\to -2^-} h(x) = \lim_{x\to -2^-} \frac{8}{x+2} = \frac{8}{0^-} = -\infty \\ \text{RHL}: \lim_{x\to -2^+} h(x) = \lim_{x\to -2^+} \frac{8}{x+2} = \frac{8}{0^+} = +\infty \end{cases}$$

(e) State the x-values at which $h(x)$ is discontinuous. Justify your statements.

- $h(x)$ is discontinuous at $x = -2$ **because** $h(-2)$ is undefined **OR** because $\lim_{x\to -2} h(x)$. $\boxed{\text{DOES NOT EXIST}}$. (Remember you only need to give one reason here.)
- $h(x)$ is discontinuous at $x = 0$ **because**, despite $h(0) = 2$ is defined, $\lim_{x\to 0} h(x)$. $\boxed{\text{DOES NOT EXIST}}$.
- $h(x)$ is discontinuous at $x = 16$ **because**, despite $h(16) = 0$ is defined and $\lim_{x\to 16} h(x) = 4$ exists, those two are not equal. That is, $\lim_{x\to 16} h(x) \neq h(16)$.

8. Write out the **Limit Definition of the Derivative** $f'(x)$.

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

9. For each of the following functions, find $f'(x)$ using the limit definition of the derivative.

(a) $f(x) = x^4$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h\to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h}$$

$$= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 = \boxed{4x^3}$$

(b) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

(c) $f(x) = \dfrac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\left(\frac{x-(x+h)}{(x+h)x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{x-x-h}{h(x+h)x} = \lim_{h\to 0} \frac{-h}{h(x+h)x} = \lim_{h\to 0} \frac{-1}{(x+h)x} = \boxed{\frac{-1}{x^2}}$$

(d) $f(x) = \dfrac{x+1}{x-1}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{(x-1)(x+h+1)-(x+h-1)(x+1)}{(x+h-1)(x-1)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{x^2+xh+x-x-h-1-(x^2+xh-x+x+h-1)}{(x+h-1)(x-1)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{x^2+xh+x-x-h-1-x^2-xh+x-x-h+1}{(x+h-1)(x-1)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{-2h}{h(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)} = \boxed{-\frac{2}{(x-1)^2}}$$

(e) $f(x) = \dfrac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} = \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \cdot \left(\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right)$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0} \frac{-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{(\sqrt{x})^2 2\sqrt{x}} = \boxed{\frac{-1}{2x^{\frac{3}{2}}}}$$

note: This is a combination problem, common denominator PLUS conjugate trick.

Turn in solutions for your group.