Math 111, Section 01, Fall 2012 ANSWER KEY!!!

Worksheet 2, Thursday, September 13, 2012

Carefully compute the following limits. Be clear if the limit equals a value, **Does Not Exist**, or is $+\infty$ or $-\infty$. Always justify your reasoning and show all work.

1. $\lim_{x\to 2} \dfrac{x^2+6x+8}{x+2} = \lim_{x\to 2} \dfrac{(x+2)(x+4)}{x+2} = \lim_{x\to 2} x+4 = \boxed{6}$

   OR

   $\lim_{x\to 2} \dfrac{x^2+6x+8}{x+2} \overset{\text{DSP}}{=} \dfrac{24}{4} = \boxed{6}$

   Here: **DSP** stands for Direct Substitution Property

2. $\lim_{x\to 2} \dfrac{x^2+6x+8}{x-2} = \lim_{x\to 2} \dfrac{(x+2)(x+4)}{x-2}$ $\boxed{\text{DOES NOT EXIST}}$ since RHL $\neq$ LHL

   RHL: $\lim_{x\to 2^+} \dfrac{(x+2)(x+4)}{x-2} = \lim_{x\to 2^+} \dfrac{(4)(6)}{0^+} = +\infty$

   LHL: $\lim_{x\to 2^-} \dfrac{(x+2)(x+4)}{x-2} = \lim_{x\to 2^-} \dfrac{(4)(6)}{0^-} = -\infty$

   Here saying DOES NOT EXIST is not enough of an explanation. We want to clarify if the function is approaching $\pm\infty$ from both sides. Here the function is approaching $+\infty$ *from the right* and $-\infty$ *from the left*. Because the left and right hand limits are not equal, THEN we can declare that the two-sided limit DOES NOT EXIST.

3. $\lim_{x\to 2} \dfrac{x^2-6x+8}{x-2} = \lim_{x\to 2} \dfrac{(x-2)(x-4)}{x-2} = \lim_{x\to 2} x-4 = \boxed{-2}$

   A first attempt would be to directly substitute $x = 2$. Then you get $\dfrac{0}{0}$. This means that there is a factor of $x-2$ in both the numerator and the denominator. Here we used the factoring technique and rid the quotient of the factor causing the zero in the denominator. Then we were free to directly substitute.

4. $\lim_{t\to 0} \left(\dfrac{1}{t\sqrt{1+t}} - \dfrac{1}{t}\right) = \lim_{t\to 0} \left(\dfrac{1}{t\sqrt{1+t}} - \dfrac{\sqrt{1+t}}{t\sqrt{1+t}}\right) = \lim_{t\to 0} \left(\dfrac{1-\sqrt{1+t}}{t\sqrt{1+t}}\right)$

   $= \lim_{t\to 0} \left(\dfrac{1-\sqrt{1+t}}{t\sqrt{1+t}}\right) \cdot \left(\dfrac{1+\sqrt{1+t}}{1+\sqrt{1+t}}\right) = \lim_{t\to 0} \dfrac{1-(1+t)}{t\sqrt{1+t}\,(1+\sqrt{1+t})}$

   $= \lim_{t\to 0} \dfrac{1-1-t}{t\sqrt{1+t}\,(1+\sqrt{1+t})} = \lim_{t\to 0} \dfrac{-t}{t\sqrt{1+t}\,(1+\sqrt{1+t})}$

   $= \lim_{t\to 0} \dfrac{-1}{\sqrt{1+t}\,(1+\sqrt{1+t})} = \dfrac{-1}{(1)(1+1)} = \boxed{-\dfrac{1}{2}}$

   Note: Watch the common denominator here, as well as the conjugate trick.

5. $\lim_{t\to 1} \dfrac{t-1}{g(t^2)-3}$, where $g(t) = 2t+1$

$$\lim_{t\to 1}\frac{t-1}{g(t^2)-3}=\lim_{t\to 1}\frac{t-1}{(2t^2+1)-3}=\lim_{t\to 1}\frac{t-1}{2t^2-2}=\lim_{t\to 1}\frac{t-1}{2(t^2-1)}=\lim_{t\to 1}\frac{t-1}{2(t-1)(t+1)}$$
$$=\lim_{t\to 1}\frac{1}{2(t+1)}=\boxed{\frac{1}{4}}$$

Again, after evaluating the function g, we get $\frac{0}{0}$. Factor to cancel the common factor $t-1$ causing the zeroes. Then you are free to directly substitute.

6. $\lim_{x\to 0}\frac{x+1}{x^2(x+2)}=+\infty$ since RHL=LHL.

RHL: $\lim_{x\to 0}\frac{x+1}{x^2(x+2)}=\lim_{x\to 0^+}\frac{1}{(0^+)^2(2)}=+\infty=\lim_{x\to 0^+}\frac{1}{0^+(2)}=+\infty$

LHL: $\lim_{x\to 0}\frac{x+1}{x^2(x+2)}=\lim_{x\to 0^-}\frac{1}{(0^-)^2(2)}=+\infty=\lim_{x\to 0^-}\frac{1}{0^+(2)}=+\infty$

Here saying DOES NOT EXIST is not enough of an explanation. We want to clarify if the function is approaching $+\infty$ (or $-\infty$) from both sides. Here the function is approaching $+\infty$ *from the right* AND *from the left.* Because the left and right hand limits are behaving the same, THEN we can declare that the two-sided limit equals $+\infty$. Yes, the limit technically *does not exist*, but we can give a more in depth description. By stating that the limit is $+\infty$, we capture some sense of how the function is behaving near this value $x=0$ where the function seems to blow up.

7. $\lim_{x\to 2}\frac{x-2}{x^2-4x+12}\overset{\text{DSP}}{=}\frac{0}{8}=\boxed{0}$

8. $$\lim_{x\to 2}\frac{\frac{x}{x-1}-\frac{x+2}{x}}{x-2}\lim_{x\to 2}\frac{\left(\frac{x^2-(x-1)(x+2)}{x(x-1)}\right)}{x-2}=\lim_{x\to 2}\frac{\left(\frac{x^2-(x^2+x-2)}{x(x-1)}\right)}{x-2}$$
$$=\lim_{x\to 2}\frac{\left(\frac{-x+2}{x(x-1)}\right)}{x-2}=\lim_{x\to 2}\frac{-x+2}{x(x-1)(x-2)}=\lim_{x\to 2}\frac{-(x-2)}{x(x-1)(x-2)}=\lim_{x\to 2}\frac{-1}{x(x-1)}=\boxed{\frac{-1}{2}}$$

NOTE: This algebra should look familiar from last week's worksheet #1. Watch the factoring carefully.

9. $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{|x|}\right)$ $\boxed{\text{DOES NOT EXIST}}$ since RHL $\neq$ LHL

Two cases for the absolute value here. The case when $x\geq 0$ and $x<0$. That is, the Right Hand Limit and the Left Hand Limit.

RHL: $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)=\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x}\right)=\lim_{x\to 0^+}0=0$

LHL: $\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)=\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{(-x)}\right)=\lim_{x\to 0^-}\frac{2}{x}=\frac{2}{0^-}=-\infty$

Recall $|x|=\begin{cases} x & \text{if } x\geq 0\\ -x & \text{if } x<0\end{cases}$

10. $\lim_{x\to -4^-} \dfrac{|x+4|}{x+4} = \lim_{x\to -4^-} \dfrac{-(x+4)}{x+4} \lim_{x\to -4^-} -1 = \boxed{-1}$

Recall $|x+4| = \begin{cases} x+4 & \text{if } x+4 \geq 0 \\ -(x+4) & \text{if } x+4 < 0 \end{cases} = \begin{cases} x+4 & \text{if } x \geq -4 \\ -(x+4) & \text{if } x < -4 \end{cases}$

For this specific problem we use the bottom case in the piece-wise defined absolute value. In the *left hand limit* in this problem we have x approaching -4 *from the left* so that is the case $x < -4$.

11. $\lim_{x\to -1} \dfrac{|x|-1}{1-x^2} = \lim_{x\to -1} \dfrac{-x-1}{1-x^2} = \lim_{x\to -1} \dfrac{-(x+1)}{1-x^2} = \lim_{x\to -1} \dfrac{-(x+1)}{(1-x)(1+x)}$

$= \lim_{x\to -1} \dfrac{-1}{1-x} = \dfrac{-1}{1-(-1)} = \boxed{-\dfrac{1}{2}}$

Here we are in one specific case of the absolute value, since our limit is approaching $x = -1$. That is the case when $x < 0$. In that region, $|x| = -x$.

Recall $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

12. Let $g(x) = \sqrt{x}$. Compute $\lim_{s\to 1} \dfrac{g(s^2+8)-3}{s-1}$

$\lim_{s\to 1} \dfrac{\sqrt{s^2+8}-3}{s-1} = \lim_{s\to 1} \dfrac{\sqrt{s^2+8}-3}{s-1} \cdot \dfrac{\sqrt{s^2+8}+3}{\sqrt{s^2+8}+3} = \lim_{s\to 1} \dfrac{s^2+8-9}{(s-1)(\sqrt{s^2+8}+3)}$

$= \lim_{s\to 1} \dfrac{s^2-1}{(s-1)(\sqrt{s^2+8}+3)} = \lim_{s\to 1} \dfrac{(s-1)(s+1)}{(s-1)(\sqrt{s^2+8}+3)} = \lim_{s\to 1} \dfrac{s+1}{\sqrt{s^2+8}+3} = \dfrac{2}{6} = \boxed{\dfrac{1}{3}}$

Be careful using the conjugate trick here. You must multiply BOTH the numerator and the denominator by the same piece. Do not drop the introduced square root in the denominator. Once you factor and cancel the two $s-1$ terms that are approaching zero in the numerator and the denominator (when $s \to 1$), then you are free to directly substitute.

13. Let $f(x) = \dfrac{1}{x}$. Compute $\lim_{t\to 2} \dfrac{f(t-1)-2f(t)}{t^2-4}$

$\lim_{t\to 2} \dfrac{\left(\dfrac{1}{t-1} - \dfrac{2}{t}\right)}{t^2-4} = \lim_{t\to 2} \dfrac{\left(\dfrac{t-2(t-1)}{(t-1)t}\right)}{t^2-4} = \lim_{t\to 2} \dfrac{\left(\dfrac{-t+2}{(t-1)t}\right)}{t^2-4} = \lim_{t\to 2} \dfrac{-(t-2)}{(t-1)t(t-2)(t+2)}$

$= \lim_{t\to 2} \dfrac{-1}{(t-1)t(t+2)} = \dfrac{-1}{1\cdot 2\cdot 4} = \boxed{-\dfrac{1}{8}}$

Here, again, after carefully evaluating the function pieces and then simplifying and then factoring to cancel the term $t-2$ causing the zeroes, then you are free to directly substitute.

14. Consider the function defined by

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8-2x & \text{if } x < 4 \end{cases}$$

Determine whether $\lim_{x\to 4} f(x)$ exists? Why or why not?

See me for a piece-wise sketch. It appears that $f(x)$ is approaching 0 *from the right* of 4 and approaching 0 *from the left* of 4. That is,

RHL: $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = 0$

LHL: $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 8 - 2x = 0 = 0$

Finally, we conclude that $\lim_{x\to 4} f(x) = 0$ since RHL=LHL=0.