Extra Examples of Riemann Sums-Chapter 5

Professor Danielle Benedetto -Math 11

1. Evaluate $\int_0^6 -x^2 \, dx$ using Riemann Sums.

   Here $a = 0, b = 6, \Delta x = \frac{6-0}{n} = \frac{6}{n}$ and $x_i = 0 + i\left(\frac{6}{n}\right) = \frac{6i}{n}$.

$$\begin{aligned}
\int_0^6 -x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{6i}{n}\right)\frac{6}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-\left(\frac{6i}{n}\right)^2\right)\frac{6}{n} \\
&= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \frac{-36i^2}{n^2} \\
&= \lim_{n\to\infty} \left(\frac{-216}{n^3} \sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \left(\frac{-216}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right) \text{ using } (**) \\
&= \lim_{n\to\infty} \left(\frac{-216}{6} \cdot \frac{n(n+1)(2n+1)}{n^3}\right) \\
&= \lim_{n\to\infty} \left(\frac{-216}{6} \cdot \frac{n(n+1)(2n+1)}{n \cdot n \cdot n}\right) \\
&= \lim_{n\to\infty} \left(\frac{-216}{6} \cdot \left(\frac{n}{n}\right) \cdot \left(\frac{n+1}{n}\right) \cdot \left(\frac{2n+1}{n}\right)\right) \\
&= \lim_{n\to\infty} \left(\frac{-216}{6} \cdot 1 \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right)\right) \\
&= \frac{-216}{6} \cdot 1 \cdot 2 \\
&= \frac{-216}{3} = \boxed{-72}
\end{aligned}$$

   Recall:

$$(*) \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$(**) \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(***) \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

2. Evaluate $\int_0^3 x^3 - 2x\ dx$ using Riemann Sums.

Here $a = 0, b = 3, \Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}$ and $x_i = 0 + i\left(\dfrac{3}{n}\right) = \dfrac{3i}{n}$.

$$\begin{aligned}
\int_0^3 x^3 - 2x\ dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{3i}{n}\right)^3 - 2\left(\frac{3i}{n}\right)\right)\frac{3}{n} \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(\frac{27i^3}{n^3} - \frac{6i}{n}\right) \\
&= \lim_{n\to\infty}\left(\frac{3}{n}\sum_{i=1}^{n}\frac{27i^3}{n^3} - \frac{3}{n}\sum_{i=1}^{n}\frac{6i}{n}\right) \\
&= \lim_{n\to\infty}\left(\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{18}{n^2}\sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty}\left(\frac{81}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{18}{n^2}\cdot\frac{n(n+1)}{2}\right) \text{using } (***), (*) \\
&= \lim_{n\to\infty}\left(\frac{81}{4}\cdot\frac{n^2(n+1)^2}{n^4} - \frac{18}{2}\cdot\frac{n(n+1)}{n^2}\right) \\
&= \lim_{n\to\infty}\left(\frac{81}{4}\frac{n^2(n+1)^2}{n^2\cdot n^2} - 9\cdot\frac{n(n+1)}{n\cdot n}\right) \\
&= \lim_{n\to\infty}\left(\frac{81}{4}\left(\frac{n^2}{n^2}\right)\cdot\left(\frac{(n+1)^2}{n^2}\right) - 9\cdot\left(\frac{n}{n}\right)\cdot\left(\frac{n+1}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{81}{4}\cdot 1\cdot\left(\frac{n+1}{n}\right)^2 - 9\cdot 1\cdot\left(\frac{n+1}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{81}{4}\cdot 1\cdot\left(1+\frac{1}{n}\right)^2 - 9\cdot 1\cdot\left(1+\frac{1}{n}\right)\right) \\
&= \frac{81}{4}\cdot 1 - 9\cdot 1 = \boxed{\frac{45}{4}}
\end{aligned}$$

3. Evaluate $\int_1^5 100 - 3x^2 \, dx$ using Riemann Sums.

Here $a = 1, b = 5, \Delta x = \dfrac{5-1}{n} = \dfrac{4}{n}$ and $x_i = 1 + i\left(\dfrac{4}{n}\right) = 1 + \dfrac{4i}{n}$.

$$\begin{aligned}
\int_1^5 100 - 3x^2 \, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{4i}{n}\right)\frac{4}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(100 - 3\left(1 + \frac{4i}{n}\right)^2\right)\frac{4}{n} \\
&= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^{n} \left(100 - 3\left(1 + \frac{8i}{n} + \frac{16i^2}{n^2}\right)\right) \\
&= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^{n} \left(97 - \frac{24i}{n} - \frac{48i^2}{n^2}\right) \\
&= \lim_{n\to\infty} \left(\frac{4}{n}\sum_{i=1}^{n} 97 - \frac{4}{n}\sum_{i=1}^{n}\frac{24i}{n} - \frac{4}{n}\sum_{i=1}^{n}\frac{48i^2}{n^2}\right) \\
&= \lim_{n\to\infty} \left(\frac{388}{n}\sum_{i=1}^{n} 1 - \frac{96}{n^2}\sum_{i=1}^{n} i - \frac{192}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \left(\frac{388}{n}\cdot n - \frac{96}{n^2}\cdot\frac{n(n+1)}{2} - \frac{192}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \text{using } (*), (**) \\
&= \lim_{n\to\infty} \left(388\cdot\frac{n}{n} - 48\cdot\frac{n(n+1)}{n^2} - 32\cdot\frac{n(n+1)(2n+1)}{n^3}\right) \\
&= \lim_{n\to\infty} \left(388\cdot 1 - 48\cdot\frac{n(n+1)}{n\cdot n} - 32\cdot\frac{n(n+1)(2n+1)}{n\cdot n\cdot n}\right) \\
&= \lim_{n\to\infty} \left(388\cdot 1 - 48\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) - 32\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right) \\
&= \lim_{n\to\infty} \left(388 - 48\cdot 1\cdot\left(1 + \frac{1}{n}\right) - 32\cdot 1\cdot\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right) \\
&= 388 - 48 - 64 \\
&= \boxed{276}
\end{aligned}$$

4. Evaluate $\int_{-2}^{2} x^3\ dx$ using Riemann Sums.

Here $a = -2, b = 2, \Delta x = \dfrac{2-(-2)}{n} = \dfrac{4}{n}$ and $x_i = -2 + i\left(\dfrac{4}{n}\right) = -2 + \dfrac{4i}{n}$.

$$
\begin{aligned}
\int_{-2}^{2} x^3 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2+\frac{4i}{n}\right)\frac{4}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-2+\frac{4i}{n}\right)^3\frac{4}{n} \\
&= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^{n} \left(-8+\frac{48i}{n}-\frac{96i^2}{n^2}+\frac{64i^3}{n^3}\right) \\
&= \lim_{n\to\infty} \left(\frac{4}{n}\sum_{i=1}^{n} -8+\frac{4}{n}\sum_{i=1}^{n}\frac{48i}{n}-\frac{4}{n}\sum_{i=1}^{n}\frac{96i^2}{n^2}+\frac{4}{n}\sum_{i=1}^{n}\frac{64i^3}{n^3}\right) \\
&= \lim_{n\to\infty} \left(\frac{-32}{n}\sum_{i=1}^{n} 1+\frac{192}{n^2}\sum_{i=1}^{n} i-\frac{384}{n^3}\sum_{i=1}^{n} i^2+\frac{256}{n^4}\sum_{i=1}^{n} i^3\right) \\
&= \lim_{n\to\infty} \left(\frac{-32}{n}\cdot n+\frac{192}{n^2}\frac{n(n+1)}{2}-\frac{384}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{256}{n^4}\left(\frac{n(n+1)}{2}\right)^2\right) \\
&\qquad \text{using } (*), (**), (***) \\
&= \lim_{n\to\infty} \left(-32+\frac{192}{2}\frac{n(n+1)}{n^2}-\frac{384}{6}\frac{n(n+1)(2n+1)}{n^3}+\frac{256}{4}\frac{n^2(n+1)^2}{n^4}\right) \\
&= \lim_{n\to\infty} \left(-32+96\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)-64\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)+64\left(\frac{n}{n}\right)^2\left(\frac{n+1}{n}\right)^2\right) \\
&= \lim_{n\to\infty} \left(-32+96(1)\left(1+\frac{1}{n}\right)-64(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+64(1)^2\left(1+\frac{1}{n}\right)^2\right) \\
&= -32+96-128+64 \\
&= \boxed{0}
\end{aligned}
$$

Note: $\left(-2+\dfrac{4i}{n}\right)^3 = -8+\dfrac{48i}{n}-\dfrac{96i^2}{n^2}+\dfrac{64i^3}{n^3}$