

Answer Key

1. [20 Points] **Differentiate** each of the following functions. You **do not** need to simplify your answers.

(a) $f(x) = \int_{\sec x}^7 \sqrt{\cos t + 7e^t} dt$

$$f'(x) = \frac{d}{dx} \int_{\sec x}^7 \sqrt{\cos t + 7e^t} dt = -\frac{d}{dx} \int_7^{\sec x} \sqrt{\cos t + 7e^t} dt = \boxed{-\sqrt{\cos(\sec(x)) + 7e^{\sec x}} \cdot \sec x \tan x}$$

(b) $f(x) = \tan(e^x + \sqrt{x}) + e^{\tan \sqrt{x}} + \sqrt{e^x + \tan x}$

$$f'(x) = \boxed{\sec^2(e^x + \sqrt{x}) \left(e^x + \frac{1}{2\sqrt{x}} \right) + e^{\tan \sqrt{x}} \sec^2 \sqrt{x} \left(\frac{1}{2\sqrt{x}} \right) + \frac{1}{2\sqrt{e^x + \tan x}} (e^x + \sec^2 x)}$$

(c) $f(x) = e^x + x^e + ex + e^e + e^{(e^x)} + (x^e)^e + e^{\frac{1}{x}} - \frac{1}{e^x}$

$$= e^x + x^e + ex + e^e + e^{(e^x)} + x^{e^2} + e^{\frac{1}{x}} - e^{-x}.$$

$$f'(x) = \boxed{e^x + ex^{e-1} + e + 0 + e^{(e^x)}e^x + e^2 x^{e^2-1} + e^{\frac{1}{x}} \left(-\frac{1}{x^2} \right) - e^{-x}(-1)}$$

2. [25 Points] Compute each of the following integrals. Simplify your answers.

(a) $\int \left(e^{7x} + \frac{1}{e^{4x}} \right)^2 dx = \int \left(e^{7x} + \frac{1}{e^{4x}} \right) \left(e^{7x} + \frac{1}{e^{4x}} \right) dx = \int e^{14x} + 2e^{3x} + e^{-8x} dx$

$$= \boxed{\frac{1}{14}e^{14x} + \frac{2}{3}e^{3x} - \frac{1}{8}e^{-8x} + C}$$

(b) $\int_0^1 \frac{e^x}{\sqrt{e^x + 8}} dx = \int_9^{e+8} \frac{1}{\sqrt{u}} du = \int_9^{e+8} u^{-\frac{1}{2}} du = 2\sqrt{u} \Big|_9^{e+8} = 2\sqrt{e+8} - 2\sqrt{9} = \boxed{2\sqrt{e+8} - 6}$

Here $\begin{cases} u = e^x + 8 \\ du = e^x dx \end{cases}$ and $\begin{cases} x = 0 \implies u = e^0 + 8 = 1 + 8 = 9 \\ x = 1 \implies u = e^1 + 8 = e + 8 \end{cases}$

(c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{\sin^3 x} dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{u^3} du = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} u^{-3} du = \frac{u^{-2}}{-2} \Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = -\frac{1}{2u^2} \Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$

$$= -\frac{1}{2\left(\frac{\sqrt{3}}{2}\right)^2} - \left(-\frac{1}{2\left(\frac{1}{2}\right)^2}\right) = -\frac{1}{2\left(\frac{3}{4}\right)} + \left(\frac{1}{2\left(\frac{1}{4}\right)}\right) = -\frac{2}{3} + 2 = \boxed{\frac{4}{3}}$$

Here $\begin{matrix} u = \sin x \\ du = \cos x dx \end{matrix}$ and $\begin{matrix} x = \frac{\pi}{6} \implies u = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\ x = \frac{\pi}{3} \implies u = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \end{matrix}$

$$\begin{aligned} \text{(d)} \quad \int x(x-1)^{\frac{5}{7}} dx &= \int (u+1)u^{\frac{5}{7}} du = \int u^{\frac{12}{7}} + u^{\frac{5}{7}} du = \frac{7}{19}u^{\frac{19}{7}} + \frac{7}{12}u^{\frac{12}{7}} + C \\ &= \boxed{\frac{7}{19}(x-1)^{\frac{19}{7}} + \frac{7}{12}(x-1)^{\frac{12}{7}} + C} \end{aligned}$$

Here $\begin{matrix} u = x-1 \implies x = u+1 \\ du = dx \end{matrix}$ Inverted Substitution

3. [10 Points] Find the function $f(x)$ that satisfies $f'(x) = \frac{e^{\sqrt{\tan x}} \sec^2 x}{\sqrt{\tan x}}$ and $f\left(\frac{\pi}{4}\right) = 1$.

$$f(x) = \int \frac{e^{\sqrt{\tan x}} \sec^2 x}{\sqrt{\tan x}} dx = 2 \int e^u du = 2e^u + C = 2e^{\sqrt{\tan x}} + C$$

now use the initial condition:

$$f\left(\frac{\pi}{4}\right) = 2e^{\sqrt{\tan\left(\frac{\pi}{4}\right)}} + C \stackrel{\text{set}}{=} 1 \implies 2e + C = 1 \implies C = 1 - 2e$$

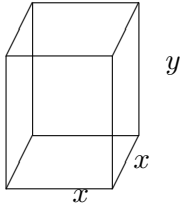
Finally, $f(x) = \boxed{2e^{\sqrt{\tan x}} + (1 - 2e)}$

Here $\begin{matrix} u = \sqrt{\tan x} \\ du = \frac{1}{2\sqrt{\tan x}} \sec^2 x dx \\ 2du = \frac{1}{\sqrt{\tan x}} \sec^2 x dx \end{matrix}$

4. [15 Points] You need to construct a box with a square base with a fixed volume of 24 cubic feet. The material for the bottom and top costs \$3 per square foot, and the material for the sides costs \$1 per square foot. What are the **dimensions** that minimize the cost required to build such a box? What is that **minimum cost**?

(Don't forget to state the common sense bounds, that is, the domain of the function that you are maximizing or minimizing.)

- Diagram:



- Variables:

Let x = length of side on base of box.

Let y = height of box.

Let C = Cost for amount of material (surface area).

Let V = volume of box.

- Equations:

$$\text{Volume } V = x^2y = 24. \text{ (Fixed!)} \implies y = \frac{24}{x^2}$$

Then the Cost of materials is given by

$$\begin{aligned} C &= \text{cost of base} + \text{cost of top} + \text{cost of 4 sides} \\ &= x^2(\$3) + x^2(\$3) + 4xy(\$1) \\ &= 6x^2 + 4xy \end{aligned}$$

Substitute,

$$C = 6x^2 + 4x \left(\frac{24}{x^2} \right) = 6x^2 + \frac{96}{x} \text{ (Minimize!)}$$

The (common-sense-bounds) domain of V is $\{x : 0 < x < \infty\}$.

- Minimize:

Next $C' = 12x - \frac{96}{x^2}$. Setting $C' = 0$ we solve $x^3 = 8$ for $x = 2$ as the critical number.

Sign-testing the critical number does indeed yield a minimum for the cost function.

$$\begin{array}{c} C' \quad \ominus \quad \oplus \\ \hline C \quad \searrow \quad \nearrow \\ \quad \quad \quad \text{MIN} \end{array}$$

$$\text{Here } x = 2 \implies y = \frac{24}{4} = 6 \text{ and } C(4) = 6(2)^2 + \frac{96}{2} = 24 + 48 = 72.$$

- Answer: As a result, the dimensions that minimize the cost are 2 feet by 2 feet by 6 feet. The minimum cost will be \$72.

5. [20 Points] Compute $\int_1^3 x^2 - 3x \, dx$ using each of the following **two** different methods:

(a) Fundamental Theorem of Calculus.

$$\begin{aligned} \int_1^3 x^2 - 3x \, dx &= \left. \frac{x^3}{3} - \frac{3x^2}{2} \right|_1^3 = \frac{3^3}{3} - \frac{3(3)^2}{2} - \left(\frac{1^3}{3} - \frac{3}{2} \right) = 9 - \frac{27}{2} - \frac{1}{3} + \frac{3}{2} = 9 - \frac{24}{2} - \frac{1}{3} = 9 - 12 - \frac{1}{3} \\ &= -3 - \frac{1}{3} = \boxed{-\frac{10}{3}} \end{aligned}$$

(b) Riemann Sums and the limit definition of the definite integral ***.

*** Recall $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, and $\sum_{i=1}^n 1 = n$

(b) Here $a = 1, b = 3, \Delta x = \frac{b-a}{n} = \frac{2}{n}$, and $x_i = a + i\Delta x = 1 + \frac{2i}{n}$

$$\begin{aligned}
 \int_1^3 x^2 - 3x \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \frac{2}{n} \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\left(1 + \left(\frac{2i}{n}\right)\right)^2 - 3\left(1 + \frac{2i}{n}\right) \right) \frac{2}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n 1 + \frac{4i}{n} + \frac{4i^2}{n^2} - 3 - \frac{6i}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \frac{4i^2}{n^2} - \frac{2i}{n} - 2 \\
 &= \lim_{n \rightarrow \infty} \left(\frac{2}{n} \sum_{i=1}^n \frac{4i^2}{n^2} - \frac{2}{n} \sum_{i=1}^n \frac{2i}{n} - \frac{2}{n} \sum_{i=1}^n 2 \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{8}{n^3} \sum_{i=1}^n i^2 - \frac{4}{n^2} \sum_{i=1}^n i - \frac{4}{n} \sum_{i=1}^n 1 \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2} \frac{n(n+1)}{2} - \frac{4}{n} \cdot n \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{8}{6} \left(\frac{n}{n}\right) \left(\frac{n+1}{n}\right) \left(\frac{2n+1}{n}\right) - \frac{4}{2} \left(\frac{n}{n}\right) \left(\frac{n+1}{n}\right) - 4 \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{8}{6} (1) \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) - \frac{4}{2} (1) \left(1 + \frac{1}{n}\right) - 4 \right) \\
 &= \frac{8}{3} - 2 - 4 \\
 &= \frac{8}{3} - 6 = \boxed{-\frac{10}{3}} \quad \text{Match}
 \end{aligned}$$

6. [10 Points] A moving object has velocity $v(t) = 2t - 6$ feet per second, at time t seconds. Compute the **Total Distance** travelled by this object from time $t = 0$ to $t = 4$ seconds.

Total Distance=

$$\begin{aligned} \int_0^4 |v(t)| dt &= \int_0^4 |2t - 6| dt = \int_0^3 |2t - 6| dt + \int_3^4 |2t - 6| dt \\ &= \int_0^3 -(2t - 6) dt + \int_3^4 2t - 6 dt = \int_0^3 6 - 2t dt + \int_3^4 2t - 6 dt \\ &= 6t - t^2 \Big|_0^3 + t^2 - 6t \Big|_3^4 = (18 - 9) - (0) + (16 - 24) - (9 - 18) \\ &= 9 - 8 + 9 = \boxed{10} \text{ feet} \end{aligned}$$

OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

OPTIONAL BONUS #1 Compute $\lim_{n \rightarrow \infty} \frac{e^{(1+\frac{1}{n})} + e^{(1+\frac{2}{n})} + e^{(1+\frac{3}{n})} + \dots + e^2}{n}$
 $= \lim_{n \rightarrow \infty} \sum_{i=1}^n e^{(1+\frac{i}{n})} \left(\frac{1}{n}\right) = \int_0^1 e^{x+1} dx = e^{x+1} \Big|_0^1 = \boxed{e^2 - e}$ Note: this limit/sum was the Riemann
 Sum for that definite integral. Or same as $\int_1^2 e^x dx$.

OPTIONAL BONUS #2 Compute $\int \sin^3 x dx = \int \sin^2 x \sin x dx$
 $= \int (1 - \cos^2 x) \sin x dx = - \int 1 - u^2 du = - \left(u - \frac{u^3}{3}\right) + C = - \left(\cos x - \frac{\cos^3 x}{3}\right) + C$
 $= \boxed{-\cos x + \frac{\cos^3 x}{3} + C}$

Here $\begin{array}{l} u = \cos x \\ du = -\sin x dx \\ -du = \sin x dx \end{array}$