Math 11 Midterm Exam #2 March 25, 2011

Answer Key

1. [12 Points] Compute each of the following **limits**. Justify your answers.

(a) $$\lim_{x\to 0} \frac{2\tan(5x)}{7x} = \frac{2}{7}\lim_{x\to 0} \frac{\sin(5x)}{x}\frac{1}{\cos(5x)} = \frac{2}{7}\lim_{x\to 0} \frac{5\sin(5x)}{5x}\frac{1}{\cos(5x)}$$

$$= \frac{10}{7}\lim_{x\to 0} \frac{\sin(5x)}{5x}\lim_{x\to 0}\frac{1}{\cos(5x)} = \frac{10}{7}\cdot 1\cdot 1 = \boxed{\frac{10}{7}}$$

(b) $$\lim_{x\to 0} \frac{7x^2-8x^3}{\sin^2(3x)} = \lim_{x\to 0} \frac{x^2(7-8x)}{\sin^2(3x)} = \lim_{x\to 0} \frac{x}{\sin(3x)}\frac{x}{\sin(3x)}(7-8x)$$

$$= \lim_{x\to 0} \frac{3x}{3\sin(3x)}\frac{3x}{3\sin(3x)}(7-8x) = \frac{1}{9}\lim_{x\to 0} \frac{3x}{\sin(3x)}\frac{3x}{\sin(3x)}(7-8x)$$

$$= \frac{1}{9}\lim_{x\to 0} \frac{3x}{\sin(3x)}\lim_{x\to 0} \frac{3x}{\sin(3x)}\lim_{x\to 0}(7-8x) = \frac{1}{9}\cdot 1\cdot 1\cdot 7 = \boxed{\frac{7}{9}}$$

(c) $$\lim_{x\to\infty} \frac{x^2-x+5}{3x^7+x^6-2011} = \lim_{x\to\infty} \frac{x^2-x+5}{3x^7+x^6-2011}\cdot\frac{\left(\frac{1}{x^7}\right)}{\left(\frac{1}{x^7}\right)} = \lim_{x\to\infty} \frac{\frac{1}{x^5}-\frac{1}{x^4}+\frac{5}{x^7}}{3+\frac{1}{x}-\frac{2011}{x^7}} = \boxed{0}$$

2. [18 Points] **Differentiate** each of the following functions. You **do not** need to simplify your answers. Please do not waste time simplifying your derivative.

(a) $f(x) = \cos^4\left(\sin\left(\frac{7}{x}\right)\right)$

$$f'(x) = \boxed{4\cos^3\left(\sin\left(\frac{7}{x}\right)\right)\cdot\left(-\sin\left(\sin\left(\frac{7}{x}\right)\right)\right)\cdot\cos\left(\frac{7}{x}\right)\cdot\left(-\frac{7}{x^2}\right)}$$

(b) $$f(x) = \frac{\left(1-\frac{1}{\sqrt{x}}\right)^9}{\sec(4x)}$$

$$f'(x) = \boxed{\frac{\sec(4x)9\left(1-\frac{1}{\sqrt{x}}\right)^8\left(\frac{1}{2x^{\frac{3}{2}}}\right)-\left(1-\frac{1}{\sqrt{x}}\right)^9\sec(4x)\tan(4x)4}{\sec^2(4x)}}$$

(c) $f(x) = \dfrac{1}{\left(\dfrac{1}{x^5} + \sqrt{x^2-4}\right)^{\frac{3}{8}}}$

$$f'(x) = \boxed{-\frac{3}{8}\left(\frac{1}{x^5} + \sqrt{x^2-4}\right)^{-\frac{11}{8}}\left(-\frac{5}{x^6} + \frac{1}{2\sqrt{x^2-4}}(2x)\right)}$$

3. [10 Points] Find the **absolute maximum** and **absolute minimum value(s)** of the function

$$F(x) = (x-1)^2(x-9)^2 \qquad \text{on the interval} \qquad [0,8].$$

$F'(x) = (x-1)^2 \cdot 2(x-9) + (x-9)^2 \cdot 2(x-1) = 2(x-1)(x-9)(2x-10)$. On the interval $[0,8]$, F' is always defined. Also, $F'(x) = 0$ happens only when $x = 1$, $x = 9$, and $x = 5$ (our critical numbers). Here $x = 9$ is outside of our interval of interest. Applying the closed interval method:

$F(1) = 0 \longleftarrow$ Absolute Minimum Value

$F(5) = 256 \longleftarrow$ Absolute Maximum Value

$F(0) = 81$

$F(8) = 49$.

So the absolute maximum value is 256 (attained at $x = 5$), and the absolute minimum value is 0 (attained at $x = 1$).

4. [25 Points] Let $f(x) = \dfrac{-x^2+x+2}{x^2-2x+1} = \dfrac{-x^2+x+2}{(x-1)^2}$.

For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve. **Hint:**

Take my word for it that (you do **not** have to compute these)

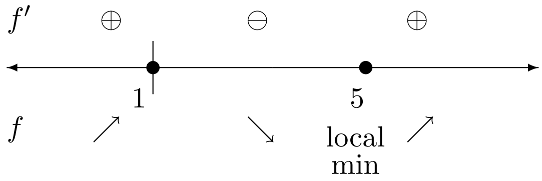
$$\boldsymbol{f'(x) = \frac{x-5}{(x-1)^3}} \quad \text{and} \quad \boldsymbol{f''(x) = \frac{-2x+14}{(x-1)^4}}.$$

- Domain: $f(x)$ has domain $\{x | x \neq 1\}$
- VA: Vertical asymptotes $x = 1$.
- HA: Horizontal asymptote is $y = -1$ for this f since $\lim\limits_{x\to\pm\infty} f(x) = -1$ because

$$\lim_{x\to\pm\infty} \frac{-x^2+x+2}{x^2-2x+1} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\pm\infty} \frac{-1+\frac{1}{x}+\frac{2}{x^2}}{1-\frac{2}{x}+\frac{1}{x^2}} = -1$$

- First Derivative Information:

We know $f'(x) = \dfrac{x-5}{(x-1)^3}$. The critical points occur where f' is undefined or zero. The former happens when $x = 1$, but $x = 1$ was not in the domain of the original function, so it isn't technically a critical number. The latter happens when $x = 5$. As a result, $x = 5$ is the critical number. Using sign testing/analysis for f',



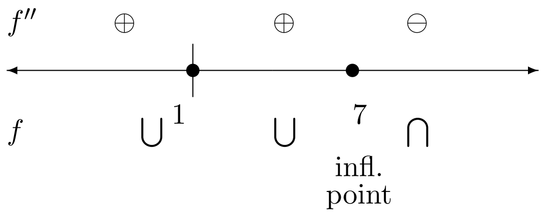
or our f' chart is

x	$(-\infty, 1)$	$(1, 5)$	$(5, \infty)$
$f'(x)$	$\oplus$	$\ominus$	$\oplus$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

So f is decreasing on $(1, 5)$ and increasing on $(-\infty, 1)$ and $(5, \infty)$. Moreover, f has a local min at $x = 5$ with $f(5) = -\dfrac{9}{8}$.

• Second Derivative Information:

Meanwhile, $f'' = \dfrac{-2x + 14}{(x-1)^4}$. $f'' = 0$ when $x = 7$. Using sign testing/analysis for f'',

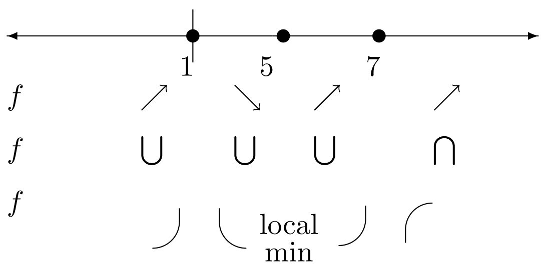


or our f'' chart is

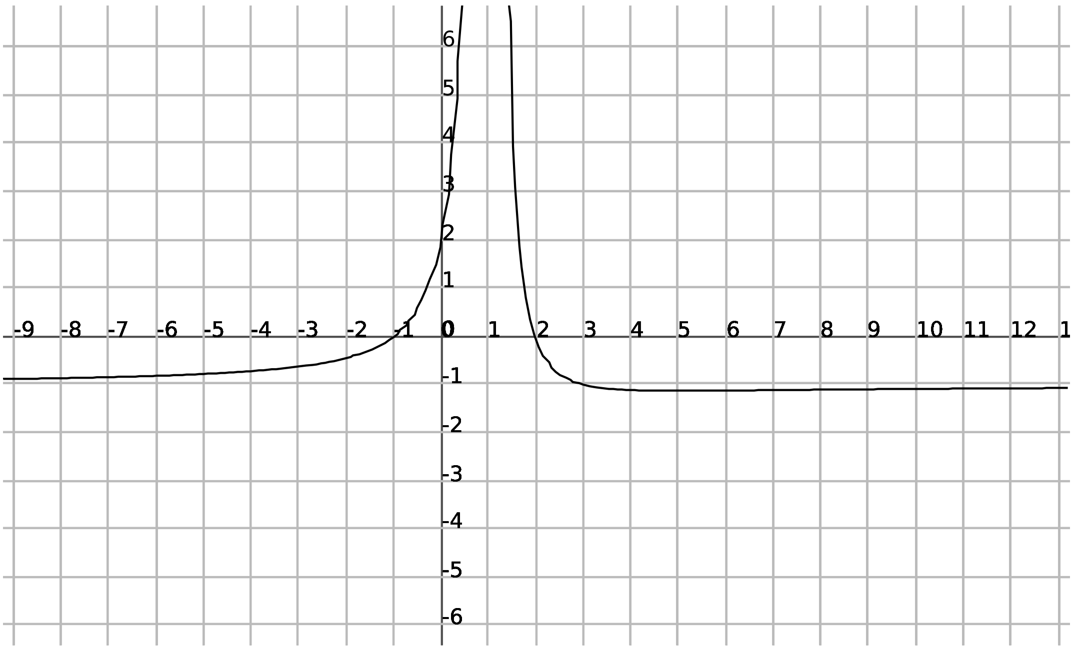
x	$(-\infty, 1)$	$(1, 7)$	$(7, \infty)$
$f''(x)$	$\oplus$	$\oplus$	$\ominus$
$f(x)$	$\cup$	$\cup$	$\cap$

So f is concave down on $(7, \infty))$ and concave up on $(-\infty, 1)$ and $(1, 7)$. There is an inflection point at $(7, -\frac{10}{9})$.

• Piece the first and second derivative information together:



• Sketch:



5. [10 Points] Consider the equation $\sin(xy) + \cos y + 7 = \sqrt{xy^3} + 9.$ Find $\dfrac{\mathrm{dy}}{\mathrm{dx}}$.

Differentiate both sides implicitly with respect to x:

$$\cos(xy)\left(x\frac{dy}{dx} + y\right) - \sin y\frac{dy}{dx} = \frac{1}{2\sqrt{xy^3}}\left(x3y^2\frac{dy}{dx} + y^3\right)$$

Distribute:

$$x\cos(xy)\frac{dy}{dx} + y\cos(xy) - \sin y\frac{dy}{dx} = \frac{3xy^2}{2\sqrt{xy^3}}\frac{dy}{dx} + \frac{y^3}{2\sqrt{xy^3}}$$

Solve:

$$x\cos(xy)\frac{dy}{dx} - \sin y\frac{dy}{dx} - \frac{3xy^2}{2\sqrt{xy^3}}\frac{dy}{dx} = -y\cos(xy) + \frac{y^3}{2\sqrt{xy^3}}$$

Factor:

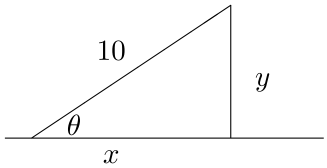
$$\left(x\cos(xy) - \sin y - \frac{3xy^2}{2\sqrt{xy^3}}\right)\frac{dy}{dx} = -y\cos(xy) + \frac{y^3}{2\sqrt{xy^3}}$$

Finally,

$$\frac{dy}{dx} = \boxed{\frac{-y\cos(xy) + \dfrac{y^3}{2\sqrt{xy^3}}}{x\cos(xy) - \sin y - \dfrac{3xy^2}{2\sqrt{xy^3}}}}$$

6. [15 Points] The top of a ten foot ladder is sliding down a vertical wall at the rate of one foot every second. Consider the angle formed by the bottom of the ladder and the ground. How fast is this angle changing when the top of the ladder is three feet above the ground?

• Diagram



• Variables

Let $x =$ distance between bottom of ladder and wall at time t

Let $y =$ distance between top of ladder and ground at time t

Let $\theta =$ angle formed by the ground and base of ladder at time t

Find $\dfrac{d\theta}{dt} =$? when $y = 3$ ft

and $\dfrac{dy}{dt} = -1\dfrac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

We have $\sin\theta = \dfrac{y}{10}$.

• Differentiate both sides w.r.t. time t.

$\dfrac{d}{dt}(\sin\theta) = \dfrac{d}{dt}\left(\dfrac{y}{10}\right) \implies \cos\theta\dfrac{d\theta}{dt} = \dfrac{1}{10}\dfrac{dy}{dt}$(Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

We're not given θ for this problem, but we can still compute $\cos\theta$ from trig. relations on the diagram's triangle with $\cos\theta = \frac{\text{adj}}{\text{hyp}}$. When $y = 3$, we can use the Pyth. Theorem to compute $x = \sqrt{(10)^2 - (3)^2} = \sqrt{91}$. Finally, $\cos\theta = \dfrac{\sqrt{91}}{10}$.

$\dfrac{\sqrt{91}}{10}\dfrac{d\theta}{dt} = \dfrac{1}{10}(-1)$

• Solve for the desired quantity:

$\dfrac{d\theta}{dt} = -\dfrac{1}{10} \cdot \dfrac{10}{\sqrt{91}} = -\dfrac{1}{\sqrt{91}}\dfrac{\text{rad}}{\text{sec}}$

• Answer the question that was asked: The angle is decreasing at a rate of $\dfrac{1}{\sqrt{91}}$ radians every second.

7. [10 Points] Let $f(x) = \tan^2 x + \cos(2x)$. Find $f'\left(\frac{\pi}{6}\right)$.

$f'(x) = 2\tan x \cdot \sec^2 x - 2\sin(2x)$

$f'\left(\frac{\pi}{6}\right) = 2\tan\left(\frac{\pi}{6}\right) \cdot \sec^2\left(\frac{\pi}{6}\right) - 2\sin\left(2\left(\frac{\pi}{6}\right)\right) = 2\tan\left(\frac{\pi}{6}\right) \cdot \sec^2\left(\frac{\pi}{6}\right) - 2\sin\left(\frac{\pi}{3}\right)$

$= 2\left(\dfrac{1}{\sqrt{3}}\right) \cdot \left(\dfrac{2}{\sqrt{3}}\right)^2 - 2\left(\dfrac{\sqrt{3}}{2}\right) = \boxed{\dfrac{8}{3\sqrt{3}} - \sqrt{3}} = \dfrac{8}{3\sqrt{3}} - \dfrac{9}{3\sqrt{3}} = \boxed{-\dfrac{1}{3\sqrt{3}}}$