Math11 Spring 2010 **Exam #1 Friday February 19, 2010** Professor D. Benedetto

Please carefully write all of your answers in your **Blue Book**. Justify all of your answers. There are **No Calculators** allowed.

1. (6 Points) State the **domain** for each of the following functions and justify your answers:

(a) $h(t) = |t-1|$ **(b)** $f(x) = \dfrac{5}{x-7}$

Domain $h(t) = |t-1|$ is all Real numbers $\mathbb{R}$, since the absolute value function is defined for all Real numbers.

Domain $f(x) = \{x | x \neq 7\}$ because f is undefined when you divide by a negative number.

2. (24 Points) Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 1} \frac{x^2-7x}{x^2-3x-11}$ **(c)** $\displaystyle\lim_{x\to 7} \frac{x-7}{|x-7|}$

(b) $\displaystyle\lim_{x\to 3} \frac{x^2-9}{\sqrt{x+1}-2}$ **(d)** $\displaystyle\lim_{x\to 1} \frac{f(x+1)-4}{x^2-x}$, where $f(x) = x^2$.

(a) $\displaystyle\lim_{x\to 1} \frac{x^2-7x}{x^2-3x-11} = \frac{-6}{-13} = \frac{6}{13}$ by the Direct Substitution Property.

(b) $\displaystyle\lim_{x\to 3} \frac{x^2-9}{\sqrt{x+1}-2} = \lim_{x\to 3} \frac{x^2-9}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3} \frac{(x-3)(x+3)(\sqrt{x+1}+2)}{(x+1)-4}$

$\displaystyle= \lim_{x\to 3} \frac{(x-3)(x+3)(\sqrt{x+1}+2)}{x-3} = \lim_{x\to 3}(x+3)(\sqrt{x+1}+2) = 6 \cdot 4 = 24$

(c) $\displaystyle\lim_{x\to 7} \frac{1}{|x-7|}$ DOES NOT EXIST because RHL$\neq$ LHL.

RHL: $\displaystyle\lim_{x\to 7^+} \frac{x-7}{|x-7|} = \lim_{x\to 7^+} \frac{x-7}{x-7} = \lim_{x\to 7^+} 1 = 1$

LHL: $\displaystyle\lim_{x\to 7^-} \frac{x-7}{|x-7|} = \lim_{x\to 7^-} \frac{x-7}{-(x-7)} = \lim_{x\to 7^-} -1 = -1$

(d) $\displaystyle\lim_{x\to 1} \frac{(x+1)^2-4}{x(x-1)} = \lim_{x\to 1} \frac{x^2+2x+1-4}{x(x-1)} = \lim_{x\to 1} \frac{x^2+2x-3}{x(x-1)} = \lim_{x\to 1} \frac{(x+3)(x-1)}{x(x-1)} = \lim_{x\to 1} \frac{x+3}{x} =$
$\displaystyle\frac{4}{1} = 4$

3. (15 Points) Prove that $\displaystyle\lim_{x\to 2} 3-4x = -5$ using the $\varepsilon - \delta$ definition of the limit.

Scratchwork: we want $|f(x) - L| = |(3-4x) - (-5)| < \varepsilon$

$$|f(x) - L| = |(3-4x) - (-5)| = |-4x+8| = |-4(x-2)| = |-4||x-2| = 4|x-2| \text{ (want } < \varepsilon)$$

$$4|x-2| < \varepsilon \text{ means } |x-2| < \frac{\varepsilon}{4}$$

$$\text{So choose } \delta = \frac{\varepsilon}{4} \text{ to restrict } 0 < |x-2| < \delta. \text{ That is } 0 < |x-2| < \frac{\varepsilon}{4}.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{4}$. Given x such that $0 < |x-2| < \delta$, then

$$|f(x) - L| = |(3-4x) - (-5)| = |-4x+8| = |-4(x-2)| = |-4||x-2| = 4|x-2| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

$\square$

4. (15 Points) Suppose that $f(x) = \frac{1}{x-7}$. Compute $f'(x)$ using the limit definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)-7} - \frac{1}{x-7}}{h} = \lim_{h\to 0} \frac{\left(\frac{(x-7)-(x+h-7)}{(x+h-7)(x-7)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{x-7-x-h+7}{(x+h-7)(x-7)}\right)}{h} = \lim_{h\to 0} \frac{\left(\frac{-h}{(x+h-7)(x-7)}\right)}{h} = \lim_{h\to 0} \frac{-h}{h(x+h-7)(x-7)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-7)(x-7)} = \frac{-1}{(x-7)^2}$$

5. (10 Points) Suppose that $f(x) = x^2 + x - 6$. Write the *equation* of the tangent line to the curve $y = f(x)$ when $x = 3$. Use the limit definition of the derivative when computing the derivative.

First, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^2 + (x+h) - 6) - (x^2 + x - 6)}{h}$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + x + h - 6 - x^2 - x + 6}{h} = \lim_{h\to 0} \frac{2xh + h^2 + h}{h} = \lim_{h\to 0} \frac{h(2x+h+1)}{h}$$

$$= \lim_{h\to 0} 2x + h + 1 = 2x + 1$$

Then the slope at $x = 3$ is given by $f'(3) = 7$. The point is given by $(3, f(3)) = (3, 6)$. Finally, the equation of the tangent line is given by $y - 6 = 7(x-3)$ or $y = 7x - 15$.

6. (10 Points) Suppose that f and g are functions, and

- $\lim_{x\to 5} f(x) = 4$
- $\lim_{x\to 5} g(x) = -7$
- $g(x)$ is continuous at $x = 5$.

Evaluate the following quantities and fully justify your answers. Do not just put down a value:

(a) $\lim_{x\to 5}(3f(x) - 2g(x)) = \lim_{x\to 5} 3f(x) - \lim_{x\to 5} 2g(x) = 3\lim_{x\to 5} f(x) - 2\lim_{x\to 5} g(x) = 3 \cdot 4 - 2 \cdot (-7) = 12 + 14 = 26$

These steps are valid by application of the Limit Laws.

(b) $g(5) = \lim_{x \to 5} g(x)$ by definition of continuity assumption for g at $x = 5$. We know $\lim_{x \to 5} g(x) = -7$ by assumption, so $g(5) = -7$

TURN PAPER OVER PLEASE!!

REMEMBER: ALL OF YOUR WORK GOES IN THE BLUE ANSWER BOOK

7. (20 Points) Consider the function defined by

$$f(x) = \begin{cases} -x - 1 & \text{if } x < 0 \\ x^2 - 1 & \text{if } 0 \le x \le 3 \\ 10 & \text{if } 3 < x < 7 \\ \dfrac{1}{x-7} & \text{if } x > 7 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$. Use this sketch to help answer the following questions: See me for the sketch.

(b) Compute
$$\begin{cases} \lim_{x \to 0^+} f(x) = -1 \\ \lim_{x \to 0^-} f(x) = -1 \\ \lim_{x \to 0} f(x) = -1 \text{ because RHL=LHL.} \end{cases}$$

(c) Compute
$$\begin{cases} \lim_{x \to 3^+} f(x) = 10 \\ \lim_{x \to 3^-} f(x) = 8 \\ \lim_{x \to 3} f(x) = \text{ DOES NOT EXIST because RHL} \neq \text{ LHL.} \end{cases}$$

(d) Compute
$$\begin{cases} \lim_{x \to 7^+} f(x) = +\infty \\ \lim_{x \to 7^-} f(x) = 10 \\ \lim_{x \to 7} f(x) = \text{ DOES NOT EXIST because RHL} \neq \text{ LHL.} \end{cases}$$

(e) State the value(s) at which f is discontinuous. Justify your answers using definitions or theorems discussed in class.

Despite the fact that $f(3) = 8$ is defined, f is discontinuous at $x = 3$ since $\lim_{x\to 3} f(x)$ Does Not Exist.

Also, $f(x)$ is discontinuous at $x = 7$ for two reasons, $f(7)$ is undefined, and $\lim_{x\to 7} f(x)$ DOES NOT EXIST

Just a side note that f is continuous at $x = 0$ because $\lim_{x\to 0} f(x) = -1 = f(0)$.

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BONUS PROBLEM: THIS IS OPTIONAL! Feel free to attempt the following bonus problem, but ONLY if you are completely done with the original part of the exam, problems 1-7.

Bonus 1: Let $f(x) = \sqrt{x^3 - 4x^2 + x - 7}$. Compute $f'(x)$ using the limit definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} - \sqrt{x^3 - 4x^2 + x - 7}}{h} =$$

$$\lim_{h\to 0} \frac{\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} - \sqrt{x^3 - 4x^2 + x - 7}}{h} \cdot \left(\frac{\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} + \sqrt{x^3 - 4x^2 + x - 7}}{\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} + \sqrt{x^3 - 4x^2 + x - 7}} \right)$$

$$= \lim_{h\to 0} \frac{((x+h)^3 - 4(x+h)^2 + (x+h) - 7) - (x^3 - 4x^2 + x - 7)}{h(\sqrt{(x+h)^3 - 4(x+h)^2 + x - 7} + \sqrt{x^3 - 4x^2 + x - 7})}$$

$$= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 4x^2 - 8xh - 4h^2 + x + h - 7) - (x^3 - 4x^2 + x - 7)}{h(\sqrt{(x+h)^3 - 4(x+h)^2 + x - 7} + \sqrt{x^3 - 4x^2 + x - 7})}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 4x^2 - 8xh - 4h^2 + x + h - 7 - x^3 + 4x^2 - x + 7}{h(\sqrt{(x+h)^3 - 4(x+h)^2 + x - 7} + \sqrt{x^3 - 4x^2 + x - 7})}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 8xh - 4h^2 + h}{h(\sqrt{(x+h)^3 - 4(x+h)^2 + x - 7} + \sqrt{x^3 - 4x^2 + x - 7})}$$

$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 8x - 4h + 1)}{h(\sqrt{(x+h)^3 - 4(x+h)^2 + x - 7} + \sqrt{x^3 - 4x^2 + x - 7})}$$

$$= \lim_{h\to 0} \frac{3x^2 + 3xh + h^2 - 8x - 4h + 1}{\sqrt{(x+h)^3 - 4(x+h)^2 + x - 7} + \sqrt{x^3 - 4x^2 + x - 7}}$$

$$= \frac{3x^2 - 8x + 1}{2\sqrt{x^3 - 4x^2 + x - 7}}$$