Answer Key for Review Packet for Exam #1

Math 111- D. Benedetto, updated Spring 2012

Limit Practice Problems

Evaluate the following limits, including if the limit Does Not Exist, or is $+\infty$ or $-\infty$. Always justify your work:

1. $\lim_{w\to 0^+} \frac{16}{w} = \frac{16}{0^+} = +\infty$

2. $\lim_{w\to 0^-} \frac{16}{w} = \frac{16}{0^-} = -\infty$

3. $\lim_{w\to 0} \frac{16}{w} =$ DOES NOT EXIST since RHL $\neq$ LHL

4. $\lim_{t\to 2^+} \frac{3-t}{t-2} = \frac{3-2}{0^+} = \frac{1}{0^+} = +\infty$

5. $\lim_{t\to 2^-} \frac{3-t}{t-2} = \frac{3-2}{0^-} = \frac{1}{0^-} = -\infty$

6. $\lim_{t\to 2} \frac{3-t}{t-2} =$ DOES NOT EXIST since RHL $\neq$ LHL

7. $\lim_{t\to 2^+} \frac{3-t}{(t-2)^2} = \frac{3-2}{(0^+)^2} = \frac{1}{0^+} = +\infty$

8. $\lim_{t\to 2^-} \frac{3-t}{(t-2)^2} = \frac{3-2}{(0^-)^2} = \frac{1}{0^+} = +\infty$

9. $\lim_{t\to 2} \frac{3-t}{(t-2)^2} = +\infty$ since RHL = LHL

10. $\lim_{x\to 4^+} \frac{(x+2)^2}{x^2-3x-4} = \lim_{x\to 4^+} \frac{(x+2)^2}{(x-4)(x+1)} = \frac{(4+2)^2}{0^+(4+1)} = \frac{36}{0^+(5)} = +\infty$

11. $\lim_{x\to 4^-} \frac{(x+2)^2}{x^2-3x-4} = \lim_{x\to 4^-} \frac{(x+2)^2}{(x-4)(x+1)} = \frac{(4+2)^2}{0^-(4+1)} = \frac{36}{0^-(5)} = -\infty$

12. $\lim_{x\to 4} \frac{(x+2)^2}{x^2-3x-4} =$ DOES NOT EXIST since RHL $\neq$ LHL

13. $\lim_{x\to 4^+} \frac{x-4}{x^2-3x-4} = \lim_{x\to 4^+} \frac{x-4}{(x-4)(x+1)} = \lim_{x\to 4^+} \frac{1}{x+1} \overset{\text{DSP}}{=} \frac{1}{5}$

14. $\lim_{x\to 4^-} \frac{x-4}{x^2-3x-4} = \lim_{x\to 4^-} \frac{x-4}{(x-4)(x+1)} = \lim_{x\to 4^-} \frac{1}{x+1} \overset{\text{DSP}}{=} \frac{1}{5}$

15. $\lim_{x\to 4} \frac{x-4}{x^2-3x-4} = \lim_{x\to 4} \frac{x-4}{(x-4)(x+1)} = \lim_{x\to 4} \frac{1}{x+1} \overset{\text{DSP}}{=} \frac{1}{5}$

    Note: RHL=LHL here, but we didn't necessarily need them, since we could just factor and cancel terms.

16. $\lim_{x\to 4^+} \frac{x^2-2x-8}{x^2-3x-4} = \lim_{x\to 4^+} \frac{(x-4)(x+2)}{(x-4)(x+1)} = \lim_{x\to 4^+} \frac{x+2}{x+1} \overset{\text{DPS}}{=} \frac{4+2}{4+1} = \frac{6}{5}$

17. $\lim_{x\to 4^-} \frac{x^2-2x-8}{x^2-3x-4} = \lim_{x\to 4^-} \frac{(x-4)(x+2)}{(x-4)(x+1)} = \lim_{x\to 4^-} \frac{x+2}{x+1} \overset{\text{DSP}}{=} \frac{4+2}{4+1} = \frac{6}{5}$

18. $\lim_{x\to 4} \frac{x^2-2x-8}{x^2-3x-4} = \lim_{x\to 4} \frac{(x-4)(x+2)}{(x-4)(x+1)} = \lim_{x\to 4} \frac{x+2}{x+1} \overset{\text{DSP}}{=} \frac{4+2}{4+1} = \frac{6}{5}$

    Note: RHL=LHL here, but we didn't necessarily need them, since we could just factor and cancel terms.

19. $\lim_{x\to 6} \frac{x^2-4x-12}{x^2-3x-18} = \lim_{x\to 6} \frac{(x-6)(x+2)}{(x-6)(x+3)} = \lim_{x\to 6} \frac{x+2}{x+3} \overset{\text{DSP}}{=} \frac{6+2}{6+3} = \frac{8}{9}$

20. $\lim_{x\to 1} \frac{x^2-4x-12}{x^2-3x-18} = \frac{1-4-12}{1-3-18} \overset{\text{DSP}}{=} \frac{-15}{-20} = \frac{3}{4}$

21. $\lim_{x\to 0} \frac{x^2-4x-12}{x^2-3x-18} \overset{\text{DSP}}{=} \frac{-12}{-18} = \frac{2}{3}$

22. $\lim_{x\to -3} \frac{x^2-4x-12}{x^2-3x-18} = \lim_{x\to -3} \frac{(x-6)(x+2)}{(x-6)(x+3)} = \lim_{x\to -3} \frac{x+2}{x+3}$ DOES NOT EXIST since RHL $\neq$ LHL

    RHL: $\lim_{x\to -3^+} \frac{x+2}{x+3} = \frac{-1}{0^+} = -\infty$

    LHL: $\lim_{x\to -3^-} \frac{x+2}{x+3} = \frac{-1}{0^-} = +\infty$

23. $\lim_{x\to -2} \frac{x^2-4x-12}{x^2-3x-18} \overset{\text{DSP}}{=} \frac{4+8-12}{4+6-18} = \frac{0}{-8} = 0$

24. $\lim_{x\to 0+} \frac{x^2-4x-12}{x^2-7x} = \lim_{x\to 0+} \frac{x^2-4x-12}{x(x-7)} = \frac{-12}{0^+(-7)} = +\infty$

25. $\lim_{x\to 0-} \frac{x^2-4x-12}{x^2-7x} = \lim_{x\to 0-} \frac{x^2-4x-12}{x(x-7)} = \frac{-12}{0^-(-7)} = -\infty$

26. $\lim_{x\to 0} \frac{x^2-4x-12}{x^2-7x} =$ DOES NOT EXIST since RHL $\neq$ LHL

27. $\lim_{x\to 0} \frac{x^2-4x}{x^2-7x} = \lim_{x\to 0} \frac{x(x-4)}{x(x-7)} = \lim_{x\to 0} \frac{x-4}{x-7} \overset{\text{DSP}}{=} \frac{-4}{-7} = \frac{4}{7}$

28. $\lim_{x\to 3^+} \frac{x^2-9}{|x-3|} = \lim_{x\to 3^+} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3^+} x+3 \overset{\text{DSP}}{=} 6$

29. $\lim_{x\to 3^-} \frac{x^2-9}{|x-3|} = \lim_{x\to 3^-} \frac{(x-3)(x+3)}{-(x-3)} = \lim_{x\to 3^-} -(x+3) \overset{\text{DSP}}{=} -6$

30. $\lim_{x\to 0^+} \frac{x^3+2009x^2+2000x}{|x|} = \lim_{x\to 0^+} \frac{x^3+2009x^2+2000x}{x} = \lim_{x\to 0^+} \frac{x(x^2+2009x+2000)}{x}$

    $= \lim_{x\to 0^+} x^2+2009x+2000 \overset{\text{DSP}}{=} 2000$

31. $\lim_{x\to 0^-} \dfrac{x^3+2009x^2+2000x}{|x|} = \lim_{x\to 0^-} \dfrac{x^3+2009x^2+2000x}{-x} = \lim_{x\to 0^-} \dfrac{x(x^2+2009x+2000)}{-x}$

$= \lim_{x\to 0^-} -(x^2+2009x+2000) \overset{\text{DSP}}{=} -2000$

32. $\lim_{x\to(-5)^+} \dfrac{x^2+6x+5}{|x+5|} = \lim_{x\to(-5)^+} \dfrac{(x+5)(x+1)}{x+5} = \lim_{x\to(-5)^+} x+1 \overset{\text{DSP}}{=} -4$

33. $\lim_{x\to(-5)^-} \dfrac{x^2+6x+5}{|x+5|} = \lim_{x\to(-5)^-} \dfrac{(x+5)(x+1)}{-(x+5)} = \lim_{x\to(-5)^-} -(x+1) \overset{\text{DSP}}{=} 4$

34. $\lim_{x\to -5} \dfrac{x^2+6x+5}{|x+5|}$ DNE since RHL$\neq$LHL

35. $\lim_{t\to 1} \dfrac{t^2-1}{t^2-11t+10} = \lim_{t\to 1} \dfrac{(t-1)(t+1)}{(t-10)(t-1)} = \lim_{t\to 1} \dfrac{t+1}{t-10} \overset{\text{DSP}}{=} \dfrac{1+1}{1-10} = -\dfrac{2}{9}$

36. $\lim_{t\to 1} \dfrac{t^2}{t^2+t-1} \overset{\text{DSP}}{=} \dfrac{1}{1+1-1} = 1$

37. $\lim_{t\to -1} \dfrac{2009(t^2+6t+5)}{t^2+t} = \lim_{t\to -1} \dfrac{2009(t+5)(t+1)}{t(t+1)} = \lim_{t\to -1} \dfrac{2009(t+5)}{t} \overset{\text{DSP}}{=} \dfrac{2009(4)}{-1} = -8036$

38. $\lim_{x\to 9} \dfrac{x^2-10x+9}{x^2+x-90} = \lim_{x\to 9} \dfrac{(x-9)(x-1)}{(x+10)(x-9)} = \lim_{x\to 9} \dfrac{x-1}{x+10} \overset{\text{DSP}}{=} \dfrac{9-1}{9+10} = \dfrac{8}{19}$

39. $\lim_{t\to 1} t^{500}+t^{400}+t^{300}+t^{200}+t^{100} \overset{\text{DSP}}{=} 5$

40. $\lim_{x\to 3} \dfrac{x^2-x-6}{x^2-2x-3} = \lim_{x\to 3} \dfrac{(x-3)(x+2)}{(x-3)(x+1)} = \lim_{x\to 3} \dfrac{x+2}{x+1} \overset{\text{DSP}}{=} \lim_{x\to 3} \dfrac{3+2}{3+1} = \lim_{x\to 3} \dfrac{5}{4}$

41. $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2} \cdot \dfrac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \lim_{x\to 1} \dfrac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4} = \lim_{x\to 1} \dfrac{(x-1)(\sqrt{x+3}+2)}{x-1}$

$= \lim_{x\to 1} \sqrt{x+3}+2 \overset{\text{L.L.}}{=} 4$

42. $\lim_{x\to 9} \dfrac{9x-x^2}{3-\sqrt{x}} \cdot \dfrac{3+\sqrt{x}}{3+\sqrt{x}} = \lim_{x\to 9} \dfrac{x(9-x)(3+\sqrt{x})}{9-x} = \lim_{x\to 9} x(3+\sqrt{x}) \overset{\text{L.L.}}{=} 9(3+3) = 54$

43. $\lim_{x\to 1} \dfrac{\sqrt{x^2+8}-3}{x-1} \cdot \dfrac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3} = \lim_{x\to 1} \dfrac{(x^2+8)-9}{(x-1)(\sqrt{x^2+8}+3)} = \lim_{x\to 1} \dfrac{x^2-1}{(x-1)(\sqrt{x^2+8}+3)}$

$= \lim_{x\to 1} \dfrac{(x-1)(x+1)}{(x-1)(\sqrt{x^2+8}+3)} = \lim_{x\to 1} \dfrac{x+1}{\sqrt{x^2+8}+3)} \overset{\text{L.L.}}{=} \dfrac{2}{6} = \dfrac{1}{3}$

44. $\lim_{t\to 1} \dfrac{t+4}{t^2+6t} \overset{\text{DSP}}{=} \dfrac{1+4}{1+6} = \dfrac{5}{7}$

45. $\lim_{w\to 0} \dfrac{2}{w+6} \overset{\text{DSP}}{=} \dfrac{2}{0+6} = \dfrac{1}{3}$

46. $\lim_{w\to 6} \dfrac{2}{w+6} \overset{\text{DSP}}{=} \dfrac{2}{6+6} = \dfrac{1}{6}$

47. $\lim_{x\to -5} x^2 - 3x + 6 \overset{\text{DSP}}{=} 25 + 15 + 6 = 46$

48. $\lim_{w\to -2} \dfrac{w+2}{w^2 - 3w + 2} \overset{\text{DSP}}{=} \dfrac{0}{12} = 0$

49. $\lim_{x\to 2} \dfrac{3}{x-2}$ DOES NOT EXIST since RHL $\neq$ LHL

RHL: $\lim_{x\to 2^+} \dfrac{3}{x-2} = \dfrac{3}{0^+} = +\infty$

LHL: $\lim_{x\to 2^-} \dfrac{3}{x-2} = \dfrac{3}{0^-} = -\infty$

50. $\lim_{x\to -1} \dfrac{5}{1-x} \overset{\text{DSP}}{=} \dfrac{5}{2}$

51. $\lim_{x\to 1} \dfrac{5x}{1-x}$ DOES NOT EXIST since RHL $\neq$ LHL

RHL: $\lim_{x\to 1^+} \dfrac{5x}{1-x} = \dfrac{5}{0^-} = -\infty$

LHL: $\lim_{x\to 1^-} \dfrac{5x}{1-x} = \dfrac{5}{0^+} = +\infty$

52. $\lim_{x\to 5^+} \dfrac{6x}{5-x} = \dfrac{30}{0^-} = -\infty$

53. $\lim_{x\to 5^-} \dfrac{6x}{5-x} = \dfrac{30}{0^+} = +\infty$

54. $\lim_{x\to 5} \dfrac{6x}{5-x}$ DOES NOT EXIST since RHL $\neq$ LHL

55. $\lim_{x\to -4} \dfrac{x^2 - 3x - 28}{x^2 + 4x} = \lim_{x\to -4} \dfrac{(x-7)(x+4)}{x(x+4)} = \lim_{x\to -4} \dfrac{x-7}{x} \overset{\text{DSP}}{=} \dfrac{-4-7}{-4} = \dfrac{11}{4}$

56. $\lim_{x\to 0} \dfrac{x^2 - 3x - 28}{x^2 + 4x} = \lim_{x\to 0} \dfrac{(x-7)(x+4)}{x(x+4)} = \lim_{x\to 0} \dfrac{x-7}{x} =$ DOES NOT EXIST since RHL $\neq$ LHL

RHL: $\lim_{x\to 0^+} \dfrac{x-7}{x} = \dfrac{-7}{0^+} = -\infty$

LHL: $\lim_{x\to 0^-} \dfrac{x-7}{x} = \dfrac{-7}{0^-} = +\infty$

57. $\lim_{x\to 3^-} \dfrac{-4}{x-3} = \dfrac{-4}{0^-} = +\infty$

58. $\lim_{x\to 3} \dfrac{-4}{x-3}$ DOES NOT EXIST since RHL $\neq$ LHL

RHL: $\lim_{x\to 3^+} \dfrac{-4}{x-3} = \dfrac{-4}{0^+} = -\infty$

LHL: $+\infty$ see #57 above

59. $\lim_{x\to 3^+} \dfrac{-4}{3-x} = \dfrac{-4}{0^-} = +\infty$

60. $\lim_{x\to 3} \frac{-4}{3-x}$ DOES NOT EXIST since RHL $\neq$ LHL

RHL: $+\infty$ see #59 above

LHL: $\lim_{x\to 3^-} \frac{-4}{3-x} = \frac{-4}{0^+} = -\infty$

61. $\lim_{x\to 1^+} |x-1| - 3 = \lim_{x\to 1^+} (x-1) - 3 \overset{\text{DSP}}{=} -3$

62. $\lim_{x\to 1^-} |x-1| - 3 = \lim_{x\to 1^-} -(x-1) - 3 \overset{\text{DSP}}{=} -3$

63. $\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1^+} x+1 \overset{\text{DSP}}{=} 2$

64. $\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^-} \frac{(x+1)(x-1)}{-(x-1)} = \lim_{x\to 1^-} -(x+1) \overset{\text{DSP}}{=} -2$

65. $\lim_{x\to 1} \frac{|1-x|}{(1-x)^2} = +\infty$ since RHL= LHL

RHL: $\lim_{x\to 1^+} \frac{|1-x|}{(1-x)^2} = \lim_{x\to 1^+} \frac{-(1-x)}{(1-x)^2} = \lim_{x\to 1^+} \frac{-1}{1-x} = \lim_{x\to 1^+} \frac{-1}{0^-} = +\infty$

LHL: $\lim_{x\to 1^-} \frac{|1-x|}{(1-x)^2} = \lim_{x\to 1^-} \frac{1-x}{(1-x)^2} = \lim_{x\to 1^-} \frac{1}{1-x} = \lim_{x\to 1^-} \frac{1}{0^+} = +\infty$

66. $\lim_{x\to 2} \frac{x^2-4}{|x-2|}$ DOES NOT EXIST since RHL $\neq$ LHL

RHL: $\lim_{x\to 2^+} \frac{x^2-4}{|x-2|} = \lim_{x\to 2^+} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^+} x+2 \overset{\text{DSP}}{=} 4$

LHL: $\lim_{x\to 2^-} \frac{x^2-4}{|x-2|} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{-(x-2)} = \lim_{x\to 2^-} -(x+2) \overset{\text{DSP}}{=} -4$

67. $\lim_{x\to 7^-} \frac{7-x}{|x-7|} = \lim_{x\to 7^-} \frac{7-x}{-(x-7)} = \lim_{x\to 7^-} \frac{7-x}{7-x} = \lim_{x\to 7^-} 1 \overset{\text{L.L.}}{=} 1$

68. $\lim_{x\to 0^-} \frac{x}{x-|x|} = \lim_{x\to 0^-} \frac{x}{x-(-x)} = \lim_{x\to 0^-} \frac{x}{2x} = \lim_{x\to 0^-} \frac{1}{2} \overset{\text{L.L.}}{=} \frac{1}{2}$

69. $\lim_{x\to 2^+} \frac{2-x}{|x-2|} = \lim_{x\to 2^+} \frac{2-x}{x-2} = \lim_{x\to 2^+} \frac{-(x-2)}{x-2} = \lim_{x\to 2^+} -1 \overset{\text{L.L.}}{=} -1$

70. $\lim_{x\to 3} \frac{\sqrt{x+6}-3}{x^2-x-6} = \lim_{x\to 3} \frac{\sqrt{x+6}-3}{x^2-x-6} = \lim_{x\to 3} \frac{x+6-9}{(x-3)(x+2)(\sqrt{x+6}+3)}$

$= \lim_{x\to 3} \frac{x-3}{(x-3)(x+2)(\sqrt{x+6}+3)} = \lim_{x\to 3} \frac{1}{(x+2)(\sqrt{x+6}+3)} = \frac{1}{5(3+3)} = \boxed{\frac{1}{30}}$

71. $\lim_{x\to 7} \frac{\frac{1}{7}-\frac{1}{x}}{x-7} = \lim_{x\to 7} \frac{\frac{x-7}{7x}}{x-7} = \lim_{x\to 7} \frac{x-7}{(7x)(x-7)} = \lim_{x\to 7} \frac{1}{7x} = \boxed{\frac{1}{49}}$

72. $\displaystyle\lim_{x\to-6}\frac{\frac{1}{2-x}-\frac{1}{8}}{x+6}=\lim_{x\to-6}\frac{\frac{8-(2-x)}{(2-x)(8)}}{x+6}=\lim_{x\to-6}\frac{\frac{6+x)}{(2-x)(8)}}{x+6}$

$$=\lim_{x\to-6}\frac{6+x}{(2-x)(8)(x+6)}=\lim_{x\to-6}\frac{1}{(2-x)(8)}=\boxed{\frac{1}{64}}$$

73. $\displaystyle\lim_{x\to3}\frac{\sqrt{x+1}-2}{3-x}=\lim_{x\to3}\frac{\sqrt{x+1}-2}{3-x}\cdot\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right)=\lim_{x\to3}\frac{x+1-4}{(3-x)(\sqrt{x+1}+2)}$

$$=\lim_{x\to3}\frac{x-3}{(3-x)(\sqrt{x+1}+2)}=\lim_{x\to3}\frac{-1}{\sqrt{x+1}+2}=\frac{-1}{2+2}=\boxed{-\frac{1}{4}}$$

74. $\displaystyle\lim_{x\to7}\frac{x^2-49}{2-\sqrt{x-3}}=\lim_{x\to7}\frac{x^2-49}{2-\sqrt{x-3}}\cdot\left(\frac{2+\sqrt{x-3}}{2+\sqrt{x-3}}\right)=\lim_{x\to7}\frac{(x^2-49)(2+\sqrt{x-3})}{4-(x-3)}$

$$=\lim_{x\to7}\frac{(x-7)(x+7)(2+\sqrt{x-3})}{7-x}=\lim_{x\to7}-(x+7)(2+\sqrt{x-3})=\lim_{x\to7}-(14)(2+2)=\boxed{-56}$$

75. $\displaystyle\lim_{x\to5}\frac{\frac{1}{\sqrt{x+20}}-\frac{1}{5}}{x-5}=\lim_{x\to5}\frac{\frac{5-\sqrt{x+20}}{5(\sqrt{x+20})}}{x-5}=\lim_{x\to5}\frac{5-\sqrt{x+20}}{5(\sqrt{x+20})(x-5)}\cdot\left(\frac{5+\sqrt{x+20}}{5+\sqrt{x+20}}\right)$

$$=\lim_{x\to5}\frac{25-(x+20)}{5(\sqrt{x+20})(x-5)(5+\sqrt{x+20})}=\lim_{x\to5}\frac{5-x}{5(\sqrt{x+20})(x-5)(5+\sqrt{x+20})}$$

$$=\lim_{x\to5}\frac{-1}{5(\sqrt{x+20})(5+\sqrt{x+20})}=\frac{-1}{5(5)(5+5)}=\boxed{\frac{-1}{250}}$$

Limit Proofs: Use the $\varepsilon-\delta$ definition for limits to prove each of the following:

76. $\displaystyle\lim_{x\to2}7x-6=8.$

Scratchwork: we want $|f(x)-L|=|(7x-6)-8|<\varepsilon$

$$|f(x)-L|=|(7x-6)-8|=|7x-14|=|7(x-2)|=|7||x-2|=7|x-2|\ (\text{want}<\varepsilon)$$

$$7|x-2|<\varepsilon\text{ means }|x-2|<\frac{\varepsilon}{7}$$

$$\text{So choose }\delta=\frac{\varepsilon}{7}\text{ to restrict }0<|x-2|<\delta.\text{ That is }0<|x-2|<\frac{\varepsilon}{7}.$$

Proof: Let $\varepsilon>0$ be given. Choose $\delta=\frac{\varepsilon}{7}$. Given x such that $0<|x-2|<\delta$, then

$$|f(x)-L|=|(7x-6)-8|=|7x-14|=|7(x-2)|=|7||x-2|=7|x-2|<7\cdot\frac{\varepsilon}{7}=\varepsilon.$$

□

77. $\displaystyle\lim_{x\to-7}2-\frac{3}{7}x=5.$

Scratchwork: we want $|f(x)-L|=\left|\left(2-\frac{3}{7}x\right)-5\right|<\varepsilon$

$$|f(x)-L| = \left|\left(2-\frac{3}{7}x\right)-5\right| = \left|-\frac{3}{7}x-3\right| = \left|-\frac{3}{7}(x+7)\right| = \left|-\frac{3}{7}\right||x-(-7)| = \frac{3}{7}|x-(-7)|$$

$$\text{(want } < \varepsilon)$$

$$\frac{3}{7}|x-(-7)| < \varepsilon \text{ means } |x-(-7)| < \frac{7}{3}\varepsilon$$

So choose $\delta = \frac{7}{3}\varepsilon$ to restrict $0 < |x-(-7)| < \delta$. That is $0 < |x-(-7)| < \frac{7}{3}\varepsilon$.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{7}{3}\varepsilon$. Given x such that $0 < |x-(-7)| < \delta$, then

$$|f(x)-L| = \left|\left(2-\frac{3}{7}x\right)-5\right| = \left|-\frac{3}{7}x-3\right| = \left|-\frac{3}{7}(x+7)\right| = \left|-\frac{3}{7}\right||x-(-7)| = \frac{3}{7}|x-(-7)|$$

$$< \frac{3}{7}\cdot\frac{7}{3}\varepsilon = \varepsilon.$$

□

For these next proofs I will not show every detail...double check your scratchwork here, and then follow the formal proofs as with the other similar problems.

78. $\lim_{x\to -2} 2x+1 = -3.$

Scratchwork: we want $|f(x)-L| = |(2x+1)-(-3)| < \varepsilon$

$$|f(x)-L| = |(2x+1)-(-3)| = |2x+4| = |2(x+2)| = |2||x+2| = 2|x+2| = 2|x-(-2)|$$

$$\text{(want } < \varepsilon)$$

$$2|x-(-2)| < \varepsilon \text{ means } |x-(-2)| < \frac{\varepsilon}{2}$$

So choose $\delta = \frac{\varepsilon}{2}$ to restrict $0 < |x-(-2)| < \delta$. That is $0 < |x-(-2)| < \frac{\varepsilon}{2}$.

Proof: Follow proofs from previous problems to fill in the details...

□

79. $\lim_{x\to 3} 1-4x = -11.$

Scratchwork: we want $|f(x)-L| = |(1-4x)-(-11)| < \varepsilon$

$$|f(x)-L| = |(1-4x)-(-11)| = |-4x+12| = |-4(x-3)| = |-4||x-3| = 4|x-3|$$

$$\text{(want } < \varepsilon)$$

$$4|x-3| < \varepsilon \text{ means } |x-3| < \frac{\varepsilon}{4}$$

So choose $\delta = \frac{\varepsilon}{4}$ to restrict $0 < |x-3| < \delta$. That is $0 < |x-3| < \frac{\varepsilon}{4}$.

Proof: Follow proofs from previous problems to fill in the details...

$\square$

80. $\lim_{t\to 2} 5 - 4t = -3.$

Scratchwork: we want $|f(t) - L| = |(5-4t) - (-3)| < \varepsilon$

$$|f(t) - L| = |(5-4t) - (-3)| = |-4t+8| = |-4(t-2)| = |-4||t-2| = 4|t-2|$$
$$\text{(want} < \varepsilon\text{)}$$
$$4|t-2| < \varepsilon \text{ means } |t-2| < \frac{\varepsilon}{4}$$
$$\text{So choose } \delta = \frac{\varepsilon}{4} \text{ to restrict } 0 < |t-2| < \delta. \text{ That is } 0 < |t-2| < \frac{\varepsilon}{4}.$$

Proof: Follow proofs from previous problems to fill in the details...

$\square$

81. $\lim_{x\to -1} 4 - 3x = 7.$

Scratchwork: we want $|f(x) - L| = |(4-3x) - 7| < \varepsilon$

$$|f(x) - L| = |(4-3x) - 7| = |-3x-3| = |-3(x+1)| = |-3||x+1| = 3|x+1| = 3|x-(-1)|$$
$$\text{(want} < \varepsilon\text{)}$$
$$3|x-(-1)| < \varepsilon \text{ means } |x-(-1)| < \frac{\varepsilon}{3}$$
$$\text{So choose } \delta = \frac{\varepsilon}{3} \text{ to restrict } 0 < |x-(-1)| < \delta. \text{ That is } 0 < |x-(-1)| < \frac{\varepsilon}{3}.$$

Proof: Follow proofs from previous problems to fill in the details...

$\square$

82. $\lim_{t\to 2} -2t - 5 = -9.$

Scratchwork: we want $|f(t) - L| = |(-2t-5) - (-9)| < \varepsilon$

$$|f(t) - L| = |(-2t-5) - (-9)| = |-2t+4| = |-2(t-2)| = |-2||t-2| = 2|t-2|$$
$$\text{(want} < \varepsilon\text{)}$$
$$2|t-2| < \varepsilon \text{ means } |t-2| < \frac{\varepsilon}{2}$$
$$\text{So choose } \delta = \frac{\varepsilon}{2} \text{ to restrict } 0 < |t-2| < \delta. \text{ That is } 0 < |t-2| < \frac{\varepsilon}{2}.$$

Proof: Follow proofs from previous problems to fill in the details...

$\square$

83. $\lim_{x\to 4} -3x + 17 = 5.$

Scratchwork: we want $|f(x) - L| = |(-3x + 17) - 5| < \varepsilon$

$$|f(x) - L| = |(-3x + 17) - 5| = | - 3x + 12| = | - 3(x - 4)| = | - 3||x - 4| = 3|x - 4|$$

$$\text{(want } < \varepsilon)$$

$$3|x - 4| < \varepsilon \text{ means } |x - 4| < \frac{\varepsilon}{3}$$

$$\text{So choose } \delta = \frac{\varepsilon}{3} \text{ to restrict } 0 < |x - 4| < \delta. \text{ That is } 0 < |x - 4| < \frac{\varepsilon}{3}.$$

Proof: Follow proofs from previous problems to fill in the details...

84. $\lim_{x\to -3} 1 - 5x = 16.$

Scratchwork: we want $|f(x) - L| = |(1 - 5x) - 16| < \varepsilon$

$$|f(x) - L| = |(-1 - 5x) - 16| = | - 5x - 15| = | - 5(x + 3)| = | - 5||x - (-3)| = 5|x - (-3)|$$

$$\text{(want } < \varepsilon)$$

$$5|x - (-3)| < \varepsilon \text{ means } |x - (-3)| < \frac{\varepsilon}{5}$$

$$\text{So choose } \delta = \frac{\varepsilon}{5} \text{ to restrict } 0 < |x - (-3)| < \delta. \text{ That is } 0 < |x - (-3)| < \frac{\varepsilon}{5}.$$

Proof: Follow proofs from previous problems to fill in the details...

85. $\lim_{x\to -14} \frac{4}{7}x + 3 = -5.$

Scratchwork: we want $|f(x) - L| = \left|(\frac{4}{7}x + 3) - (-5)\right| < \varepsilon$

$$|f(x) - L| = \left|(\frac{4}{7}x + 3) - (-5)\right| = \left|\frac{4}{7}x + 8\right| = \left|(\frac{4}{7}(x + 14)\right| = \left|\frac{4}{7}\right| |x - (-14)| = \frac{4}{7}|x - (-14)|$$

$$\text{(want } < \varepsilon)$$

$$\frac{4}{7}|x - (-14)| < \varepsilon \text{ means } |x - (-14)| < \frac{7}{4}\varepsilon$$

$$\text{So choose } \delta = \frac{7}{4}\varepsilon \text{ to restrict } 0 < |x - (-14)| < \delta. \text{ That is } 0 < |x - (-14)| < \frac{7}{4}\varepsilon.$$

Proof: Follow proofs from previous problems to fill in the details...

Tangent Lines Please use the limit definition for the derivative when computing the derivatives in this section.

86. Find an equation for the tangent line to the graph of $f(x) = x - 2x^2$ at the point $(1, -1)$

First compute the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{((x+h) - 2(x+h)^2) - (x - 2x^2)}{h}$$

$$= \lim_{h\to 0} \frac{x + h - 2x^2 - 4xh - 2h^2 - x + 2x^2}{h} = \lim_{h\to 0} \frac{h - 4xh - 2h^2}{h} = \lim_{h\to 0} \frac{h(1 - 4x - 2h)}{h}$$

$$= \lim_{h\to 0} 1 - 4x - 2h = 1 - 4x$$

Note: $f'(1) = 1 - 4(1) = -3$, so using *point slope form*, the equation of the tangent line through the point $(1, -1)$ with slope -3 is given by

$y - (-1) = -3(x - 1)$ or $\boxed{y = -3x + 2}$.

87. Find an equation for the tangent line to the graph of $f(x) = \sqrt{x}$ at $x = 4$

First compute the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Note: $f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$. The point is $(4, f(4)) = (4, \sqrt{4}) = (4, 2)$. Therefore, using *point slope form*, the equation of the tangent line throught the point $(4, 2)$ with slope $\dfrac{1}{4}$ is given by

$y - 2 = \dfrac{1}{4}(x - 4)$ or $\boxed{y = \dfrac{1}{4}x + 1}$.

88. At which point(s) does the graph of $f(x) = -x^2 + 13$ have a horizontal tangent line?

First compute the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(-(x+h)^2 + 13) - (-x^2 + 13)}{h}$$

$$= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 + 13 + x^2 - 13}{h} = \lim_{h\to 0} \frac{-2xh - h^2}{h} = \lim_{h\to 0} \frac{h(-2x - h)}{h} = \lim_{h\to 0} -2x - h$$

$$= -2x$$

Note: Set $f'(x) = 0$ and solve $f'(x) = -2x = 0 \Rightarrow x = 0$ so the point is $(0, f(0)) = \boxed{(0, 13)}$.

89. At which point(s) of the graph of $f(x) = -x^3 + 13$ is the slope of the tangent line equal to -27? What's the picture representing this problem?

First compute the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(-(x+h)^3 + 13) - (-x^3 + 13)}{h}$$

$$= \lim_{h\to 0} \frac{-x^3 - 3x^2h - 3xh^2 - h^3 + 13 + x^3 - 13}{h} = \lim_{h\to 0} \frac{-3x^2h - 3xh^2 - h^3}{h} = \lim_{h\to 0} \frac{h(-3x^2 - 3xh - h^2)}{h}$$

$$= \lim_{h\to 0} -3x^2 - 3xh - h^2 = -3x^2$$

Note: Set $f'(x) = -27$ and solve $f'(x) = -3x^2 = -27 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$ so the points are $(3, f(3)) = \boxed{(3, -14)}$ and $(-3, f(-3)) = \boxed{(-3, 40)}$.

90. There are two points on the graph of the curve $y = -x^2 + 7$ whose tangent line to the graph at those points passes through the point $(0, 11)$. Find those two points.

CHALLENGE!!

First compute the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(-(x+h)^2 + 7) - (-x^2 + 7)}{h}$$

$$= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 + 7 + x^2 - 7}{h} = \lim_{h\to 0} \frac{-2xh - h^2}{h} = \lim_{h\to 0} \frac{h(-2x - h)}{h} = \lim_{h\to 0} -2x - h$$

$$= -2x$$

Let a point on the graph be given by $(a, f(a)) = (a, -a^2+7)$. The slope of the tangent line at this point $(a, -a^2+7)$ is given by $f'(a) = -2a$. The tangent line to this curve through the point $(a, -a^2+7)$ with slope $-2a$ is given by $y-(-a^2+7) = -2a(x-a)$ or $y+a^2-7 = -2ax+2a^2$. For this tangent line to pass through the exterior point $(0, 11)$, that means the point $(0, 11)$ satisfies the equation of the tangent line. Then, $11 + a^2 - 7 = 0 + 2a^2$ or $a^2 = 4 \Rightarrow a = \pm 2$. So the two points of interest here are $(2, f(2)) = \boxed{(2, 3)}$ and $(-2, f(-2)) = \boxed{(-2, 3)}$.

91. Find the equation of the line passing through $(2, 3)$ which is perpendicular to the tangent to the curve $y = x^3 - 3x + 1$ at the point $(2, 3)$.

First we will find the slope of the tangent line to this curve when $x = 2$. Then we will take minus the reciprical of that slope to finish the problem.

First compute the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^3 - 3(x+h) + 1) - (x^3 - 3x + 1)}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h + 1 - x^3 + 3x - 1}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h}$$

$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 - 3 = 3x^2 - 3$$

Thus, $f'(2) = 9$, so the line perpendicular to that would have slope equal to $-\frac{1}{9}$. The equation of the line through the point $(2, 3)$ with slope $-\frac{1}{9}$ is given by *point slope form* as $y - 3 = -\frac{1}{9}(x - 2)$. So, $\boxed{y = -\frac{1}{9}x + \frac{29}{9}}$.

92. Find the equation of the tangent line to the curve $y = x^3 + x$ at the point(s) where the slope equals 4.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^3 + (x+h)) - (x^3 + x)}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + x + h - x^3 - x}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + h}{h}$$

$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 + 1)}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 + 1 = 3x^2 + 1$$

Set $f'(x) = 3x^2 + 1 = 4$ and solve for $x = \pm 1$. Therefore, the points where slope is equal to 4 are $(1, (f(1)) = (1, 2)$ and $(-1, f(-1)) = (-1, -2)$.

The equation of the tangent line to the curve, at the point $(1, 2)$ with slope equaling 4, is given by $y - 2 = 4(x - 1)$ or $\boxed{y = 4x - 2}$.

Finally, the equation of the tangent line to the curve, at the point $(-1, -2)$ with slope equaling 4, is given by $y - (-2) = 4(x - (-1))$ or $\boxed{y = 4x + 2}$.

Derivatives Use the limit definition of the derivative to calculate the derivative for each of the following functions:

93. $f(x) = 3 - 9x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(3 - 9(x+h)^2) - (3 - 9x^2)}{h} = \lim_{h\to 0} \frac{3 - 9x^2 - 18xh - 9h^2 - 3 + 9x^2}{h}$$
$$= \lim_{h\to 0} \frac{-18xh - 9h^2}{h} = \lim_{h\to 0} \frac{h(-18x - 9h)}{h} = \lim_{h\to 0} -18x - 9h = \boxed{-18x}$$

94. $f(x) = -4x - x^2 - 3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(-4(x+h) - (x+h)^2 - 3) - (-4x - x^2 - 3)}{h}$$
$$= \lim_{h\to 0} \frac{-4x - 4h - x^2 - 2xh - h^2 - 3 + 4x + x^2 + 3}{h} = \lim_{h\to 0} \frac{-4h - 2xh - h^2}{h} = \lim_{h\to 0} \frac{h(-4 - 2x - h)}{h}$$
$$= \lim_{h\to 0} -4 - 2x - h = \boxed{-4 - 2x}$$

95. $f(x) = \dfrac{-3}{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{-3}{x+h} - \left(\frac{-3}{x}\right)}{h} = \lim_{h\to 0} \frac{\frac{-3x + 3(x+h)}{(x+h)x}}{h}$$
$$= \lim_{h\to 0} \frac{-3x + 3x + 3h}{h(x+h)x} = \lim_{h\to 0} \frac{3h}{h(x+h)x} = \lim_{h\to 0} \frac{3}{(x+h)x} = \boxed{\frac{3}{x^2}}$$

96. $f(x) = -9x^2 + 3$ Repeat...See #91 above.

97. $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = \boxed{3x^2}$$

98. $f(x) = x^2 - 4x + 3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 4(x+h) + 3 - (x^2 - 4x + 3)}{h}$$
$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h + 3 - x^2 + 4x - 3}{h} = \lim_{h\to 0} \frac{2xh + h^2 - 4h}{h} == \lim_{h\to 0} \frac{h(2x + h - 4)}{h} =$$
$$\lim_{h\to 0} 2x + h - 4 = \boxed{2x - 4}$$

99. $f(x) = \dfrac{1}{x^2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\left(\dfrac{x^2-(x+h)^2}{(x+h)^2x^2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 - x^2 - 2xh - h^2}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{-2xh - h^2}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{h(-2x-h)}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{-2x-h}{(x+h)^2x^2} = \frac{-2x}{x^4}$$

$$\boxed{= \frac{-2}{x^3}}$$

100. $f(x) = \sqrt{x-7}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)-7} - \sqrt{x-7}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)-7} - \sqrt{x-7}}{h} \cdot \frac{\sqrt{(x+h)-7} + \sqrt{x-7}}{\sqrt{(x+h)-7} + \sqrt{x-7}} = \lim_{h\to 0} \frac{(x+h-7)-(x-7)}{h(\sqrt{(x+h)-7} + \sqrt{x-7})} =$$

$$\lim_{h\to 0} \frac{x+h-7-x+7}{h(\sqrt{(x+h)-7} + \sqrt{x-7})} = \lim_{h\to 0} \frac{h}{h(\sqrt{(x+h)-7} + \sqrt{x-7})} = \lim_{h\to 0} \frac{1}{\sqrt{(x+h)-7} + \sqrt{x-7}}$$

$$= \boxed{\frac{1}{2\sqrt{x-7}}}$$

101. $f(x) = \sqrt{3x-7}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3(x+h)-7} - \sqrt{3x-7}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)-7} - \sqrt{3x-7}}{h} \cdot \frac{\sqrt{3(x+h)-7} + \sqrt{3x-7}}{\sqrt{3(x+h)-7} + \sqrt{3x-7}} = \lim_{h\to 0} \frac{(3(x+h)-7)-(3x-7)}{h(\sqrt{3(x+h)-7} + \sqrt{3x-7})} =$$

$$\lim_{h\to 0} \frac{3x+3h-7-3x+7}{h(\sqrt{3(x+h)-7} + \sqrt{3x-7})} = \lim_{h\to 0} \frac{3h}{h(\sqrt{3(x+h)-7} + \sqrt{3x-7})}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)-7} + \sqrt{3x-7}} = \boxed{\frac{3}{2\sqrt{3x-7}}}$$

102. $f(x) = \dfrac{1}{x^3}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^3} - \dfrac{1}{x^3}}{h} = \lim_{h\to 0} \frac{\left(\dfrac{x^3-(x+h)^3}{(x+h)^3x^3}\right)}{h}$$

$$= \lim_{h\to 0} \frac{x^3 - x^3 - 3x^2h - 3xh^2 - h^3}{h(x+h)^3x^3} = \lim_{h\to 0} \frac{-3x^2h - 3xh^2 - h^3}{h(x+h)^3x^3} = \lim_{h\to 0} \frac{h(-3x^2-3xh-h^2)}{h(x+h)^3x^3}$$

$$= \lim_{h\to 0} \frac{-3x^2-3xh-h^2}{(x+h)^3x^3} = \frac{-3x^2}{x^6} = \boxed{\frac{-3}{x^4}}$$

103. $f(x) = \dfrac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{x+h}} - \dfrac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\left(\dfrac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} = \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \cdot \left(\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right) = \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{(\sqrt{x})^2 2\sqrt{x}} = \boxed{\frac{-1}{2x^{\frac{3}{2}}}}$$

Functions and Limit Practice Problems Evaluate the following limits:

104. Let $g(x) = 2x+1$. Compute $\displaystyle\lim_{x\to 1} \frac{x-1}{g(x^2)-3} =$

$$\lim_{x\to 1} \frac{x-1}{(2x^2+1)-3} = \lim_{x\to 1} \frac{x-1}{2x^2-2} = \lim_{x\to 1} \frac{x-1}{2(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{2(x+1)} = \boxed{\frac{1}{4}}$$

105. Let $G(u) = u^2+u$. Compute $\displaystyle\lim_{u\to 2} \frac{u^2-2u}{G(u-3)} =$

$$\lim_{u\to 2} \frac{u^2-2u}{(u-3)^2+(u-3)} = \lim_{u\to 2} \frac{u(u-2)}{u^2-5u+6} = \lim_{u\to 2} \frac{u(u-2)}{(u-3)(u-2)} = \lim_{u\to 2} \frac{u}{u-3} = \frac{2}{-1} = \boxed{-2}$$

106. Let $F(x) = |x|+1$. Compute $\displaystyle\lim_{x\to 4} \frac{F(x-1)}{F(x-5)} =$

$$\lim_{x\to 4} \frac{|x-1|+1}{|x-5|+1} = \lim_{x\to 4} \frac{(x-1)+1}{-(x-5)+1} == \frac{3+1}{1+1} = \frac{4}{2} = \boxed{2}$$

107. Let $h(y) = y^2-3$. Compute $\displaystyle\lim_{x\to -2} \frac{x+2}{h(2x)-h(x+6)} =$

$$\lim_{x\to -2} \frac{x+2}{((2x)^2-3)-((x+6)^2-3)} = \lim_{x\to -2} \frac{x+2}{(4x^2-3)-(x^2+12x+36-3)}$$

$$= \lim_{x\to -2} \frac{x+2}{4x^2-3-x^2-12x-33} = \lim_{x\to -2} \frac{x+2}{3x^2-12x-36} = \lim_{x\to -2} \frac{x+2}{3(x^2-4x-12)}$$

$$= \lim_{x\to -2} \frac{x+2}{3(x-6)(x+2)} = \lim_{x\to -2} \frac{1}{3(x-6)} = \boxed{-\frac{1}{24}}$$

108. Let $g(x) = \sqrt{x}$. Compute $\displaystyle\lim_{s\to 1} \frac{g(s^2+8)-3}{s-1} =$

$$\lim_{s\to 1} \frac{\sqrt{s^2+8}-3}{s-1} = \lim_{s\to 1} \frac{\sqrt{s^2+8}-3}{s-1} \cdot \frac{\sqrt{s^2+8}+3}{\sqrt{s^2+8}+3} = \lim_{s\to 1} \frac{s^2+8-9}{(s-1)(\sqrt{s^2+8}+3)}$$

$$= \lim_{s\to 1} \frac{s^2-1}{(s-1)(\sqrt{s^2+8}+3)} = \lim_{s\to 1} \frac{(s-1)(s+1)}{(s-1)(\sqrt{s^2+8}+3)} = \lim_{s\to 1} \frac{s+1}{\sqrt{s^2+8}+3} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

109. Let $f(t) = \dfrac{1}{t}$. Compute $\displaystyle\lim_{t\to 2} \frac{f(t-1)-2f(t)}{t^2-4} =$

$$\lim_{t\to 2}\frac{\left(\dfrac{1}{t-1}-\dfrac{2}{t}\right)}{t^2-4}=\lim_{t\to 2}\frac{\left(\dfrac{t-2(t-1)}{(t-1)t}\right)}{t^2-4}=\lim_{t\to 2}\frac{\left(\dfrac{-t+2}{(t-1)t}\right)}{t^2-4}=\lim_{t\to 2}\frac{-(t-2)}{(t-1)t(t-2)(t+2)}$$

$$=\lim_{t\to 2}\frac{-1}{(t-1)t(t+2)}=\frac{-1}{1\cdot 2\cdot 4}=\boxed{-\frac{1}{8}}$$

More Tangent Lines Please use the limit definition for the derivative when computing derivatives in this section.

110. Find an equation for the tangent line to the graph of $f(x)=\dfrac{1}{x-1}$ at the point $(0,-1)$.

First we compute the slope $f'(x)$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left(\dfrac{1}{(x+h)-1}-\dfrac{1}{x-1}\right)}{h}=\lim_{h\to 0}\frac{\left(\dfrac{(x-1)-(x+h-1)}{((x+h)-1)(x-1)}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\dfrac{x-1-x-h+1)}{((x+h)-1)(x-1)}\right)}{h}=\lim_{h\to 0}\frac{-h}{h(x+h-1)(x-1)}=\lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)}=-\frac{1}{(x-1)^2}$$

Note: $f'(0)=-1$. Therefore, using *point slope form*, the equation of the tangent line through the point $(0,-1)$ with slope equal to -1 is given by $y-(-1)=-1(x-0)$ or $\boxed{y=-x-1}$.

111. Find an equation for the tangent line to the graph of $f(x)=\dfrac{1}{x+1}$ at the point $\left(1,\dfrac{1}{2}\right)$.

First we compute the slope $f'(x)$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left(\dfrac{1}{(x+h)+1}-\dfrac{1}{x+1}\right)}{h}=\lim_{h\to 0}\frac{\left(\dfrac{(x+1)-(x+h+1)}{((x+h)+1)(x+1)}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\dfrac{x+1-x-h-1)}{((x+h)+1)(x+1)}\right)}{h}=\lim_{h\to 0}\frac{-h}{h(x+h+1)(x+1)}=\lim_{h\to 0}\frac{-1}{(x+h+1)(x+1)}=-\frac{1}{(x+1)^2}$$

Note: $f'(1)=-\dfrac{1}{4}$. Therefore, using *point slope form*, the equation of the tangent line through the point $\left(1,\dfrac{1}{2}\right)$ with slope equal to $-\dfrac{1}{4}$ is given by $y-\dfrac{1}{2}=-\dfrac{1}{4}(x-1)$ or $\boxed{y=-\dfrac{1}{4}x+\dfrac{3}{4}}$.

112. Find an equation for the tangent line to the graph of $y=\dfrac{3}{x}+1$ when $x=1$.

First we compute the slope $f'(x)$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left(\dfrac{3}{x+h}+1\right)-\left(\dfrac{3}{x}+1\right)}{h}=\lim_{h\to 0}\frac{\dfrac{3}{x+h}-\dfrac{3}{x}}{h}$$

$$=\lim_{h\to 0}\frac{\left(\dfrac{3x-3(x+h)}{(x+h)x}\right)}{h}=\lim_{h\to 0}\frac{\left(\dfrac{3x-3x-3h}{(x+h)x}\right)}{h}=\lim_{h\to 0}\frac{-3h}{h(x+h)x}=\lim_{h\to 0}\frac{-3}{(x+h)x}=-\frac{3}{x^2}$$

Note: $f'(1) = -\frac{3}{1} = -3$. Therefore, using *point slope form*, the equation of the tangent line through the point $(1, f(1)) = \left(1, \frac{3}{1}+1\right) = (1,4)$ with slope equal to -3 is given by $y - 4 = -3(x-1)$ or $\boxed{y = -3x+7}$.

113. Find an equation for the tangent line to the graph of $y = x^2 - 4x + 2$ when $x = 1$. At what point is the tangent line to this curve horizontal?

First compute the derivative $f'(x)$.

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^2 - 4(x+h)+2) - (x^2-4x+2)}{h}$

$= \lim_{h\to 0} \frac{x^2+2xh+h^2+-4x-4h+2-x^2+4x-2}{h} = \lim_{h\to 0} \frac{2xh+h^2-4h}{h} = \lim_{h\to 0} \frac{h(2x+h-4)}{h}$

$= \lim_{h\to 0} 2x+h-4 = 2x-4$

Note: $f'(1) = -2$. Therefore, using *point slope form*, the equation of the tangent line through the point $(1, f(1)) = (1,-1))$ with slope equal to -2 is given by $y-(-1) = -2(x-1)$ or $\boxed{y = -2x+1}$.

Functions Please state what the domain is for each of the following functions.

114. $f(x) = \frac{x+2}{x-1}$ Domain= $\{x|x \neq 1\}$ or $(-\infty, 1) \cup (1, \infty)$.

115. $g(x) = \sqrt{x-2}$ Domain= $\{x|x-2 \geq 0\}$ or $\{x|x \geq 2\}$ or $[2, \infty)$.

116. $g(x) = \sqrt{2-x}$ Domain= $\{x|2-x \geq 0\}$ or $\{x|x \leq 2\}$ or $(-\infty, 2]$.

117. $g(x) = \frac{1}{\sqrt{2-x}}$ Domain= $\{x|2-x > 0\}$ or $\{x|x < 2\}$ or $(-\infty, 2)$.

118. $f(x) = \frac{x-3}{x^2+3}$ Domain= $\mathbb{R}$ or $(-\infty, \infty)$.

119. $w(x) = \frac{1}{x-4}$ Domain= $\{x|x \neq 4\}$ or $(-\infty, 4) \cup (4, \infty)$.

120. $f(x) = \frac{x^2+6x+8}{x+2}$ Domain= $\{x|x \neq -2\}$ or $(-\infty, -2) \cup (-2, \infty)$.

More Functions

121. Let $g(x) = \frac{x+1}{x}$. Compute (and simplify, if possible) the following:

(a) $g(2) = \boxed{\frac{3}{2}}$

(b) $g(0) = \boxed{\text{undefined}}$

(c) $g\left(\frac{1}{2}\right) = \frac{\frac{1}{2}+1}{\frac{1}{2}} = \boxed{3}$

(d) $g\left(\frac{1}{10}\right) = \frac{\frac{1}{10}+1}{\frac{1}{10}} = \boxed{11}$

(e) $g(t-2) = \frac{(t-2)+1}{t-2} = \boxed{\frac{t-1}{t-2}}$

(f) $\frac{g(2+h)-g(2)}{h} = \frac{\left(\frac{(2+h)+1}{2+h}\right)-\frac{3}{2}}{h} = \frac{\left(\frac{(3+h)2-3(2+h)}{(2+h)2}\right)}{h} = \frac{6+2h-6-3h}{h(2+h)2} =$
$\frac{-h}{h(2+h)2} = \boxed{\frac{-1}{(2+h)2}}$

122. Let $f(x) = \frac{1}{x+1} - \frac{1}{x}$. Compute (and simplify, if possible) the following:

(a) $f(1) = \frac{1}{1+1} - \frac{1}{1} = \frac{1}{2} - 1 = \boxed{-\frac{1}{2}}$

(b) $f(-1) = \boxed{\text{undefined}}$

(c) $f\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}+1} - \frac{1}{-\frac{1}{2}} = 2+2 = \boxed{4}$

(d) $f(t-1) = \frac{1}{(t-1)+1} - \frac{1}{t-1} = \frac{1}{t} - \frac{1}{t-1} = \frac{(t-1)-t}{t(t-1)} = \boxed{\frac{-1}{t(t-1)}}$

(e) $f\left(\frac{1}{t}\right) = \frac{1}{\left(\frac{1}{t}+1\right)} - \frac{1}{\left(\frac{1}{t}\right)} = \frac{1}{\left(\frac{1+t}{t}\right)} - \frac{1}{\left(\frac{1}{t}\right)} = \frac{t-t(t+1)}{t+1} = \frac{t-t^2-t}{t+1} = \boxed{\frac{-t^2}{t+1}}$

More Functions

123. Let $f(x) = \sqrt{x}$, $g(x) = x^2+4$, and $h(x) = \frac{1}{x}$. Compute (and simplify, if possible) the following:

(a) $f \circ g(x) = f(g(x)) = f(x^2+4) = \boxed{\sqrt{x^2+4}}$

(b) $g \circ f(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + 4 = \boxed{x+4}$

(c) $g \circ h(x) - h \circ g(x) = g(h(x)) - h(g(x)) = g\left(\frac{1}{x}\right) - h(x^2+4) = \boxed{\left(\left(\frac{1}{x}\right)^2 + 4\right) - \frac{1}{x^2+4}}$

(d) $f \circ h(x) - h \circ f(x) = f(h(x)) - h(f(x)) = f\left(\frac{1}{x}\right) - h(\sqrt{x}) = \sqrt{\frac{1}{x}} - \frac{1}{\sqrt{x}} = \boxed{0}$

(e) $h \circ g \circ f(x) = h(g(f(x))) = h(g(\sqrt{x})) = h((\sqrt{x})^2+4) = h(x+4) = \boxed{\frac{1}{x+4}}$

(f) $g \circ f \circ f(x) = g(f(f(x))) = g(f(\sqrt{x})) = g\left(x^{\frac{1}{4}}\right) = \left(x^{\frac{1}{4}}\right)^2 + 4 = \boxed{\sqrt{x}+4}$

(g) $g \circ g(x) = g(x^2+4) = (x^2+4)^2 + 4 = \boxed{x^4+8x^2+20}$

124. Suppose $\lim_{x\to 3} f(x) = 3$ and $\lim_{x\to 3} g(x) = -2$. Assume g is continuous at $x = 3$. Find each of the following values:

(a) $\lim_{x\to 3} 2f(x) - 4g(x) = \lim_{x\to 3} 2f(x) - \lim_{x\to 3} 4g(x) = 2\lim_{x\to 3} f(x) - 4\lim_{x\to 3} g(x) = 2(3) - 4(-2) = 6 + 8 = \boxed{14}$

All of these limits are split up appropriately because of the Limit Laws.

(b) $\lim_{x\to 3} g(x) \cdot \dfrac{x^2-9}{x-3} = \lim_{x\to 3} g(x) \cdot \lim_{x\to 3} \dfrac{x^2-9}{x-3} = \lim_{x\to 3} g(x) \cdot \lim_{x\to 3} \dfrac{(x-3)(x+3)}{x-3}$
$= \lim_{x\to 3} g(x) \cdot \lim_{x\to 3} x + 3 = (-2)\cdot 6 = \boxed{-12}$

(c) $g(3) = \lim_{x\to 3} g(x) = \boxed{-2}$ because g was assumed to be continuous at $x = 3$.

(d) $\lim_{x\to 3} g(f(x)) = g\left(\lim_{x\to 3} f(x)\right) = g(3) = -2$

(e) $\lim_{x\to 3} \sqrt{(f(x))^2 - 8g(x)} = \sqrt{\lim_{x\to 3}\left((f(x))^2 - 8g(x)\right)} = \sqrt{\lim_{x\to 3}(f(x))^2 - \lim_{x\to 3} 8g(x)}$
$= \sqrt{\left(\lim_{x\to 3} f(x)\right)^2 - 8\lim_{x\to 3} g(x)} = \sqrt{(3)^2 - 8(-2)} = \sqrt{9+16} = \sqrt{25} = \boxed{5}$

Consider each of the following piecewise defined functions. Answer the related questions. *Justify* your answers please.

SEE THE NEXT HANDOUT LINK FOR THE SKETCHES!

125. Let $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 2 \\ 8 - x & \text{if } x > 2 \end{cases}$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$\lim_{x\to 2} f(x) =$ DOES NOT EXIST since RHL $\neq$ LHL

$$\begin{cases} \text{LHL}: \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 4 \\ \text{RHL}: \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 8 - x = 6 \end{cases}$$

$\lim_{x\to 0} f(x) = 0$ since RHL= LHL

$$\begin{cases} \text{LHL}: \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0 \\ \text{RHL}: \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0 \end{cases}$$

Despite the fact that $f(2) = 4$ is defined, f is discontinuous at $x = 2$ since $\lim_{x\to 2} f(x)$ DOES NOT EXIST.

126. Let $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \le x \le 1 \\ 3 - x & \text{if } x > 1 \end{cases}$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$\lim_{x\to 2} f(x) = \lim_{x\to 2} 3 - x = 1$

$\lim_{x\to 1} f(x) = 2$ since RHL= LHL

$$\begin{cases} \text{LHL}: \lim\limits_{x \to^-} f(x) = \lim\limits_{x \to 1^-} 2x^2 = 2 \\ \text{RHL}: \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 3 - x = 2 \end{cases}$$

$\lim\limits_{x \to 0} f(x) =$ DOES NOT EXIST since RHL $\neq$ LHL

$$\begin{cases} \text{LHL}: \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x + 2 = 2 \\ \text{RHL}: \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 2x^2 = 0 \end{cases}$$

Despite the fact that $f(0) = 0$ is defined, f is discontinuous at $x = 0$ since $\lim\limits_{x \to 0} f(x)$ DOES NOT EXIST.

127. Let $f(x) = \begin{cases} \dfrac{1}{x - 4} & \text{if } x < 2 \\ \dfrac{1}{x} & \text{if } x \geq 2 \end{cases}$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{1}{x - 4} = -\dfrac{1}{3}$

$\lim\limits_{x \to 2} f(x) =$ DOES NOT EXIST since RHL $\neq$ LHL

$$\begin{cases} \text{LHL}: \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \dfrac{1}{x - 4} = -\dfrac{1}{2} \\ \text{RHL}: \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} = \dfrac{1}{x} = \dfrac{1}{2} \end{cases}$$

Despite the fact that $f(2) = \dfrac{1}{2}$ is defined, f is discontinuous at $x = 2$ since $\lim\limits_{x \to 2} f(x)$ DOES NOT EXIST.

128. Let $f(x) = \begin{cases} -3x + 4 & \text{if } x \leq 3 \\ -2 & \text{if } x > 3 \end{cases}$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$\lim\limits_{x \to 3} f(x) =$ DOES NOT EXIST since RHL $\neq$ LHL.

$$\begin{cases} \text{LHL}: \lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} -3x + 4 = -5 \\ \text{RHL}: \lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} -2 = -2 \end{cases}$$

$\lim\limits_{x \to -2} f(x) = \lim\limits_{x \to -2} -3x + 4 = 10$

Despite the fact that $f(3) = -5$ is defined, f is discontinuous at $x = 3$ since $\lim\limits_{x \to 3} f(x)$ DOES NOT EXIST.

129. Let $f(t) = \begin{cases} t - 3 & \text{if } t \leq 3 \\ 3 - t & \text{if } 3 < t < 5 \\ 1 & \text{if } t = 5 \\ 3 - t & \text{if } t > 5 \end{cases}$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$\lim\limits_{t \to 3} f(t) = 0$ since RHL $=$ LHL

$$\begin{cases} \text{LHL}: \lim_{t\to 3^-} f(t) = \lim_{t\to 3^-} t-3 = 0 \\ \text{RHL}: \lim_{t\to 3^+} f(t) = \lim_{t\to 3^+} t-3 = 0 \end{cases}$$

$\lim_{t\to 0} f(t) = \lim_{t\to 0} t - 3 = -3$

$\lim_{t\to 5} f(t) = -2$ since RHL=LHL

$$\begin{cases} \text{LHL}: \lim_{t\to 5^-} f(t) = \lim_{t\to 5^-} 3-t = -2 \\ \text{RHL}: \lim_{t\to 5^+} f(t) = \lim_{t\to 5^+} 3-t = -2 \end{cases}$$

Despite the fact that $f(5) = 1$ is defined, and $\lim_{x\to 5} f(x)$ exists and is equal to -2, f is discontinuous at $x = 5$ since those numbers are not equal. That is, $\lim_{x\to 5} f(x) \neq f(5)$.

130. Let $f(x) = \begin{cases} -2x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 2 \\ 6-x & \text{if } x > 2 \end{cases}$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$\lim_{x\to -2} f(x) = \lim_{x\to -2} -2x = 4$

$\lim_{x\to 0} f(x) = 0$ since RHL=LHL

$$\begin{cases} \text{LHL}: \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -2x = 0 \\ \text{RHL}: \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0 \end{cases}$$

$\lim_{x\to 2} f(x) = 4$ since RHL= LHL

$$\begin{cases} \text{LHL}: \lim_{x\to 2-} f(x) = \lim_{x\to 2^-} x^2 = 4 \\ \text{RHL}: \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 6-x = 4 \end{cases}$$

$\lim_{x\to 6} f(x) = \lim_{x\to 6} 6 - x = 0$

Notice that f is continuous at all real numbers since the three pieces of the graph of the curve match up at the break points. Specifically $\lim_{x\to a} f(x) = f(a)$ for every number $x = a$.

131. Let $f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ x & \text{if } -1 < x < 1 \\ 1-x & \text{if } x \ge 1 \end{cases}$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$\lim_{x\to -1} f(x) = -1$ since RHL= LHL

$$\begin{cases} \text{LHL}: \lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^3 = -1 \\ \text{RHL}: \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x = -1 \end{cases}$$

$\lim_{x\to 1} f(x) =$ DOES NOT EXIST since RHL $\neq$ LHL

$$\begin{cases} \text{LHL}: \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1 \\ \text{RHL}: \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 1-x = 0 \end{cases}$$

Despite the fact that $f(1) = 0$ is defined, f is discontinuous at $x = 1$ since $\lim_{x\to 1} f(x)$ DOES NOT EXIST. Also, f is discontinuous at $x = -1$ since $f(-1)$ is undefined.

132. Let $f(x) = \begin{cases} x-1 & \text{if } x < 2 \\ 1 & \text{if } 2 < x < 4 \\ 3 & \text{if } x = 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x - 1 = -1$

$\lim_{x\to 2} f(x) = 1$ since RHL= LHL

$$\begin{cases} \text{LHL}: \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x - 1 = 1 \\ \text{RHL}: \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 1 = 1 \end{cases}$$

$\lim_{x\to 4} f(x) =$ DOES NOT EXIST since RHL$\neq$LHL

$$\begin{cases} \text{LHL}: \lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 1 = 1 \\ \text{RHL}: \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x} = 2 \end{cases}$$

$f(4) = 3$

Despite the fact that $\lim_{x\to 2} f(x)$ exists and is equal to 1, $f(2)$ is undefined. Thus, f is discontinuous at $x = 2$. Also, despite the fact that $f(4) = 3$ is defined, f is discontinuous at $x = 4$ since $\lim_{x\to 4} f(x)$ DOES NOT EXIST.

133. Let $h(x) = \begin{cases} \dfrac{8}{x+2} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ \dfrac{1}{2}x - 4 & \text{if } 0 < x \leq 16 \\ \sqrt{x} & \text{if } x > 16 \end{cases}$

Sketch the graph. Find the numbers at which h is discontinuous. Evaluate:

$\lim_{x\to -2} h(x) =$ DOES NOT EXIST since RHL$\neq$LHL

$$\begin{cases} \text{LHL}: \lim_{x\to -2^-} h(x) = \lim_{x\to -2^-} \dfrac{8}{x+2} = -\infty \\ \text{RHL}: \lim_{x\to -2^+} h(x) = \lim_{x\to -2^+} \dfrac{8}{x+2} = +\infty \end{cases}$$

$\lim_{x\to 0} h(x) =$ DOES NOT EXIST since RHL$\neq$LHL

$$\begin{cases} \text{LHL}: \lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} \dfrac{8}{x+2} = 4 \\ \text{RHL}: \lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} \dfrac{1}{2}x - 4 = -4 \end{cases}$$

$\lim_{x\to 16} h(x) = 4$ since RHL=LHL

$$\begin{cases} \text{LHL}: \lim_{x\to 16^-} h(x) = \lim_{x\to 16^-} \dfrac{1}{2}x - 4 = 4 \\ \text{RHL}: \lim_{x\to 16^+} h(x) = \lim_{x\to 16^+} \sqrt{x} = 4 \end{cases}$$

Note that h is discontinuous at $x = -2$ since h is undefined there, as well as the fact that $\lim_{x \to -2} h(x)$ DOES NOT EXIST. Also, despite the fact that $h(0) = 2$ is defined, h is discontinuous at $x = 0$ since $\lim_{x \to 0} h(x)$ DOES NOT EXIST.

134. Let $h(x) = \begin{cases} \dfrac{8}{x-2} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ \dfrac{1}{2}x - 4 & \text{if } x > 0 \end{cases}$

Sketch the graph. Find the numbers at which h is discontinuous. Evaluate:

$\lim_{x \to 0} h(x) = -4$ since RHL=LHL

$$\begin{cases} \text{LHL}: \lim_{x \to 0^-} h(x) = \lim_{x \to 0^-} \dfrac{8}{x-2} = -4 \\ \text{RHL}: \lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} \dfrac{1}{2}x - 4 = -4 \end{cases}$$

$\lim_{x \to 2} h(x) = \lim_{x \to 2} \dfrac{1}{2}x - 4 = -3$

Despite the fact that $h(0) = 2$ is defined, and $\lim_{x \to 0} h(x)$ exists and is equal to -4, f is discontinuous at $x = 0$ since those numbers are not equal. That is, $\lim_{x \to 0} h(x) \neq h(0)$.

135. Let $h(x) = \begin{cases} \dfrac{8}{x-2} & \text{if } x \leq 0 \\ \\ \dfrac{1}{2}x - 4 & \text{if } x > 0 \end{cases}$

Sketch the graph. Find the numbers at which h is discontinuous. Evaluate:

$\lim_{x \to 0} h(x) = -4$ since RHL=LHL

$$\begin{cases} \text{LHL}: \lim_{x \to 0^-} h(x) = \lim_{x \to 0^-} \dfrac{8}{x-2} = -4 \\ \text{RHL}: \lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} \dfrac{1}{2}x - 4 = -4 \end{cases}$$

$\lim_{x \to 1} h(x) = \lim_{x \to 1} \dfrac{1}{2}x - 4 = -\dfrac{7}{2}$

$h(0) = -4$

Unlike the previous (similar) example, h is NO LONGER discontinuous at $x = 0$ since $h(0) = -4$ is defined, and $\lim_{x \to 0} h(x)$ exists and is equal to -4. Since those numbers are now equal, that is, $\lim_{x \to 0} h(x) = h(0)$, then h is now continuous at $x = 0$.