Extra Examples of ε and δ precise proofs for Limits

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Math 111

1. Prove : $\lim_{x \to 2} 7x - 6 = 8$ using the $\varepsilon - \delta$ definition of the limit.

Scratchwork: we want $|f(x) - L| = |(7x-6) - 8| < \varepsilon$; what restrictions on $|x-2|$ make that possible?

$$|f(x) - L| = |(7x-6) - 8| = |7x - 14| = |7(x-2)| = |7||x-2| = 7|x-2| \text{ (want } < \varepsilon)$$
$$7|x-2| < \varepsilon \text{ means } |x-2| < \frac{\varepsilon}{7}, \qquad \text{so choose } \delta = \frac{\varepsilon}{7}.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{7}$. Given x such that $0 < |x-2| < \delta$, then

$$|f(x) - L| = |(7x-6) - 8| = |7x - 14| = |7(x-2)| = |7||x-2| = 7|x-2| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon.$$

$\square$

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2. Prove : $\lim_{x \to 1} 4 - 5x = -1$ using the $\varepsilon - \delta$ definition of the limit.

Scratchwork: we want $|f(x) - L| = |(4-5x) - (-1)| < \varepsilon$

$$|f(x) - L| = |(4-5x) - (-1)| = |4 - 5x + 1| = |5 - 5x| = |-5x + 5| = |-5(x-1)| =$$
$$|-5||x-1| = 5|x-1| \text{ (want } < \varepsilon)$$
$$5|x-1| < \varepsilon \text{ means } |x-1| < \frac{\varepsilon}{5}, \qquad \text{so choose } \delta = \frac{\varepsilon}{5}$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{5}$. Given x such that $0 < |x-1| < \delta$, then

$$|f(x) - L| = |(4-5x) - (-1)| = |4 - 5x + 1| = |5 - 5x| = |-5x + 5| = |-5(x-1)|$$
$$= |-5||x-1| = 5|x-1| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon.$$

$\square$

3. Prove : $\lim_{x\to 12} \frac{x}{6} - 4 = -2$ using the $\varepsilon - \delta$ definition of the limit.

Scratchwork: we want $|f(x) - L| = \left|(\frac{x}{6} - 4) - (-2)\right| < \varepsilon$

$$|f(x)-L| = \left|(\frac{x}{6} - 4) - (-2)\right| = \left|\frac{x}{6} - 4 + 2\right| = \left|\frac{x}{6} - 2\right| = \left|\frac{1}{6}(x - 12)\right| = \left|\frac{1}{6}\right| |x-12| = \frac{1}{6}|x-12|$$

(want $< \varepsilon$)

$$\frac{1}{6}|x - 12| < \varepsilon \text{ means } |x - 12| < 6 \cdot \varepsilon, \qquad \text{so choose } \delta = 6 \cdot \varepsilon$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = 6 \cdot \varepsilon$. Given x such that $0 < |x - 12| < \delta$, then

$$|f(x) - L| = \left|(\frac{x}{6} - 4) - (-2)\right| = \left|\frac{x}{6} - 4 + 2\right| = \left|\frac{x}{6} - 2\right| = \left|\frac{1}{6}(x - 12)\right|$$
$$= \left|\frac{1}{6}\right| |x - 12| = \frac{1}{6}|x - 12| < \frac{1}{6} \cdot 6 \cdot \varepsilon = \varepsilon.$$

□

- Please try and pay attention to the format of your proof.
- Be explicit about where your proof starts and ends.
- Be careful to follow the $\varepsilon - \delta$ definition of the limit. The idea being: *for every $\varepsilon > 0$ there exists a δ such that . . .* So after fixing an epsilon, FIND the delta (which usually depends on epsilon). Be clear on what your choice of δ is.
- When examining $|f(x) - L|$, you are on an algebraic mission to find $|x - a|$ pop out.
- Show all of your work in order to justify your statements. Eventually you will practice enough that the scratchwork piece will be more obvious.