Amherst College, DEPARTMENT OF MATHEMATICS

Math 11 Final Examination, December 22, 2010

Answer Key

• This examination booklet consists of **12** problems on **15** numbered pages. If you have received a defective copy, please notify your instructor immediately.

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, or $e^{3\ln 3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [20 Points] Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim_{x\to 2} \frac{x^2-4}{x^2+4} = \frac{0}{8} = \boxed{0}$ by DSP

(b) $\lim_{x\to 5^-} \frac{x^2+4x-5}{g(x^2)-49}$, where $g(x) = 2x-1$.

$$\lim_{x\to 5^-} \frac{x^2+4x-5}{g(x^2)-49} = \lim_{x\to 5^-} \frac{x^2+4x-5}{(2x^2-1)-49} = \lim_{x\to 5^-} \frac{x^2+4x-5}{2x^2-50} = \lim_{x\to 5^-} \frac{x^2+4x-5}{2(x^2-25)}$$

$$= \lim_{x\to 5^-} \frac{(x+5)(x-1)}{2(x-5)(x+5)} = \lim_{x\to 5^-} \frac{x-1}{2(x-5)} = \frac{4}{2\cdot 0^-} = \boxed{-\infty}$$

(c) $\lim_{x\to 5} \frac{5-x}{\sqrt{x+4}-3} = \lim_{x\to 5} \frac{5-x}{\sqrt{x+4}-3}\cdot\frac{\sqrt{x+4}+3}{\sqrt{x+4}+3} = \lim_{x\to 5} \frac{(5-x)(\sqrt{x+4}+3)}{(x+4)-9}$

$$= \lim_{x\to 5} \frac{(5-x)(\sqrt{x+4}+3)}{x-5} = \lim_{x\to 5} \frac{-(x-5)(\sqrt{x+4}+3)}{x-5} = \lim_{x\to 5} -(\sqrt{x+4}+3)$$

$$= -(\sqrt{5+4}+3) = -(\sqrt{9}+3) = -(3+3) = \boxed{-6}$$

(d) $\lim_{x\to 3^+} \frac{x^2-9}{|3-x|} = \lim_{x\to 3^+} \frac{(x-3)(x+3)}{-(3-x)} = \lim_{x\to 3^+} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3^+} x+3 = \boxed{6}$

2. [30 Points] Compute each of the following derivatives. Do not simplify your answers.

(a) $f'\left(\frac{\pi}{6}\right)$, where $f(x) = \tan^2 x + \cos(2x)$.

$f'(x) = 2\tan x\cdot\sec^2 x - 2\sin(2x)$

$$f'\left(\frac{\pi}{6}\right) = 2\tan\left(\frac{\pi}{6}\right)\cdot\sec^2\left(\frac{\pi}{6}\right) - 2\sin\left(2\left(\frac{\pi}{6}\right)\right) = 2\tan\left(\frac{\pi}{6}\right)\cdot\sec^2\left(\frac{\pi}{6}\right) - 2\sin\left(\frac{\pi}{3}\right)$$

$$= 2\left(\frac{1}{\sqrt{3}}\right)\cdot\left(\frac{2}{\sqrt{3}}\right)^2 - 2\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{8}{3\sqrt{3}} - \sqrt{3}} = \frac{8}{3\sqrt{3}} - \frac{9}{3\sqrt{3}} = \boxed{-\frac{1}{3\sqrt{3}}}$$

(b) $\dfrac{d}{dx}\ln\left(\dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{e^{\sec x}}\right) = \dfrac{d}{dx}\left[\ln\left(x^{\frac{3}{4}}\sqrt{x^2+1}\right) - \ln\left(e^{\sec x}\right)\right]$

$$= \frac{d}{dx}\left[\ln\left(x^{\frac{3}{4}}\right) + \ln\left(\sqrt{x^2+1}\right) - \ln\left(e^{\sec x}\right)\right]$$

$$= \frac{d}{dx}\left[\ln\left(x^{\frac{3}{4}}\right) + \ln\left((x^2+1)^{\frac{1}{2}}\right) - \ln\left(e^{\sec x}\right)\right]$$

$$= \frac{d}{dx}\left[\frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - \sec x\right]$$

$$= \frac{3}{4}\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{1}{x^2+1}\right)(2x) - \sec x\tan x = \boxed{\frac{3}{4x} + \frac{x}{x^2+1} - \sec x\tan x}$$

(c) $\dfrac{dy}{dx}$, if $e^{xy} = y^3\ln x + e^7$.

Implicitly differentiate both sides with respect to x:

$$\frac{d}{dx}\left(e^{xy}\right) = \frac{d}{dx}\left(y^3\ln x + e^7\right)$$

$$\frac{d}{dx}\left(e^{xy}\right) = \frac{d}{dx}\left(y^3\ln x + e^7\right)$$

$$e^{xy}\left(x\frac{dy}{dx} + y(1)\right) = y^3\left(\frac{1}{x}\right) + (\ln x)(3y^2)\frac{dy}{dx} + 0$$

$$xe^{xy}\frac{dy}{dx} + ye^{xy} = \frac{y^3}{x} + 3y^2\ln x\frac{dy}{dx}$$

$$xe^{xy}\frac{dy}{dx} - 3y^2\ln x\frac{dy}{dx} = \frac{y^3}{x} - ye^{xy}$$

$$\left(xe^{xy} - 3y^2\ln x\right)\frac{dy}{dx} = \frac{y^3}{x} - ye^{xy}$$

Finally, solve for

$$\frac{dy}{dx} = \boxed{\frac{\frac{y^3}{x} - ye^{xy}}{xe^{xy} - 3y^2\ln x}}$$

(d) $g'(0)$, where $g(x) = \dfrac{f(x)}{e^{3x}}$ with $f(0) = 2$ and $f'(0) = 7$.

$$g'(x) = \frac{e^{3x}f'(x) - f(x)e^{3x}(3)}{\left(e^{3x}\right)^2}$$

$$g'(0) = \frac{e^0 f'(0) - f(0)e^0(3)}{(e^0)^2} = \frac{(1)(7) - (2)(1)(3)}{(1)^2} = 7 - 6 = \boxed{1}$$

(e) $g''(x)$, where $g(x) = \displaystyle\int_x^9 \sqrt{1+\cos t}\ dt$.

First, $g'(x) = \dfrac{d}{dx}\displaystyle\int_x^9 \sqrt{1+\cos t}\ dt = -\dfrac{d}{dx}\displaystyle\int_9^x \sqrt{1+\cos t}\ dt = -\sqrt{1+\cos x}$ by FTC (Part I)

Second, $g''(x) = -\dfrac{1}{2\sqrt{1+\cos x}}(-\sin x) = \boxed{\dfrac{\sin x}{2\sqrt{1+\cos x}}}$ by chain rule.

(f) $\dfrac{d}{dx}(\sin x)^x$.

We have two options for this problem:

First, try logarithmic differentiation.

Set $y = (\sin x)^x$.

Then $\ln y = \ln\left((\sin x)^x\right) = x\ln(\sin x)$ by algebraic properties of the natural log.

Now implicitly differentiate both sides:

$\dfrac{d}{dx}(\ln y) = \dfrac{d}{dx}(x\ln(\sin x))$

As a result, using chain and product rules,

$\dfrac{1}{y}\dfrac{dy}{dx} = x\left(\dfrac{1}{\sin x}\right)(\cos x) + \ln(\sin x)(1)$

Finally,

$\dfrac{dy}{dx} = y\left(\dfrac{x\cos x}{\sin x} + \ln(\sin x)\right)$

and

$\dfrac{dy}{dx} = \boxed{(\sin x)^x\left(\dfrac{x\cos x}{\sin x} + \ln(\sin x)\right)}$

Secondly, you could try the following:

$\dfrac{d}{dx}(\sin x)^x = \dfrac{d}{dx}e^{\ln((\sin x)^x)} = \dfrac{d}{dx}e^{x\ln(\sin x)} = e^{x\ln(\sin x)}\left(x\left(\dfrac{1}{\sin x}\right)\cos x + \ln(\sin x)(1)\right)$

$= \boxed{(\sin x)^x\left(\dfrac{x\cos x}{\sin x} + \ln(\sin x)\right)}$

3. [25 Points] Compute each of the following integrals.

(a) $\displaystyle\int_0^{\ln 3} \frac{e^{2x}}{1+e^{2x}}\ dx = \frac{1}{2}\int_2^{10} \frac{1}{u}\ du = \frac{1}{2}\ln|u|\Big|_2^{10} = \frac{1}{2}\left(\ln|u|\Big|_2^{10}\right) = \frac{1}{2}\left(\ln(10) - \ln 2\right)$

$$= \frac{1}{2}\left(\ln\left(\frac{10}{2}\right)\right) = \boxed{\frac{\ln 5}{2}}$$

Here $\begin{cases} u &= 1+e^{2x} \\ du &= 2e^{2x}dx \\ \frac{1}{2}du &= e^{2x}dx \end{cases}$ and $\begin{cases} x=0 & \Longrightarrow u = 1+e^0 = 1+1=2 \\ x=\ln 3 & \Longrightarrow u = 1+e^{2\ln 3} = 1+e^{\ln(3^2)} = 1+e^{\ln(9)} = 1+9=10 \end{cases}$

(b) $\displaystyle\int_0^3 |4-x^2|\ dx = \int_0^2 4-x^2\ dx + \int_2^3 -(4-x^2)\ dx = \left(4x - \frac{x^3}{3}\right)\Big|_0^2 + \left(\frac{x^3}{3} - 4x\right)\Big|_2^3$

$$= \left(8-\frac{8}{3}\right) - 0 + (9-12) - \left(\frac{8}{3}-8\right) = 8 - \frac{8}{3} - 3 - \frac{8}{3} + 8 = 13 - \frac{16}{3} = \boxed{\frac{10}{3}}$$

(c) $\displaystyle\int \frac{(x+1)(x+2)}{x^3}\ dx = \int \frac{x^2+3x+2}{x^3}\ dx = \int \frac{1}{x} + \frac{3}{x^2} + \frac{2}{x^3}\ dx$

$$= \int x^{-1} + 3x^{-2} + 2x^{-3} = \ln|x| - 3x^{-1} - x^{-2} + C = \boxed{\ln|x| - \frac{3}{x} - \frac{1}{x^2} + C}$$

(d) $\displaystyle\int_0^1 \frac{d}{dx}\left(\frac{\sqrt{1+3x^2}}{x+e^x}\right)\ dx = \left(\frac{\sqrt{1+3x^2}}{x+e^x}\right)\Big|_0^1 = \frac{\sqrt{1+3}}{1+e^1} - \frac{\sqrt{1+0}}{0+e^0}$

$$= \frac{2}{1+e} - 1 = \frac{2}{1+e} - \frac{1+e}{1+e} = \frac{2-(1+e)}{1+e} = \boxed{\frac{1-e}{1+e}}$$

(e) $\displaystyle\int e^{x^2+\ln x}\ dx = \int e^{x^2}e^{\ln x}\ dx = \int e^{x^2}x\ dx = \frac{1}{2}\int e^u\ du = \frac{1}{2}e^u + C = \boxed{\frac{1}{2}e^{x^2} + C}$

Here $\begin{cases} u &= x^2 \\ du &= 2x\ dx \\ \frac{1}{2}du &= x\ dx \end{cases}$

4. [10 Points] Give an ε–δ proof that $\displaystyle\lim_{x\to -2} \frac{1}{2}x + 3 = 2$.

Scratchwork: we want $|f(x) - L| = \left|\left(\frac{1}{2}x+3\right) - 2\right| < \varepsilon$

$|f(x) - L| = \left|\left(\frac{1}{2}x+3\right) - 2\right| = \left|\frac{1}{2}x+1\right| = \left|\frac{1}{2}(x+2)\right| = \left|\frac{1}{2}\right| |x-(-2)| = \frac{1}{2}|x-(-2)|$ (want $< \varepsilon$)

$$\frac{1}{2}|x-(-2)| < \varepsilon \text{ means } |x-(-2)| < 2\varepsilon$$

So choose $\delta = 2\varepsilon$ to restrict $0 < |x-(-2)| < \delta$. That is $0 < |x-(-2)| < 2\varepsilon$.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = 2\varepsilon$. Given x such that $0 < |x-(-2)| < \delta$, then

$|f(x) - L| = \left|\left(\frac{1}{2}x+3\right) - 2\right| = \left|\frac{1}{2}x+1\right| = \left|\frac{1}{2}(x+2)\right| = \left|\frac{1}{2}\right| |x-(-2)| = \frac{1}{2}|x-(-2)| < \frac{1}{2}\cdot(2\varepsilon) = \varepsilon$

as desired.

□

5. [10 Points] Let $f(x) = \dfrac{x+1}{x+2}$. Calculate $f'(x)$, using the **limit definition** of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{(x+h)+1}{(x+h)+2} - \dfrac{x+1}{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(x+h+1)(x+2)-(x+1)(x+h+2)}{(x+h+2)(x+2)}}{h}$$

$$= \lim_{h\to 0} \frac{(x^2+xh+x+2x+2h+2)-(x^2+xh+2x+x+h+2)}{h(x+h+2)(x+2)}$$

$$= \lim_{h\to 0} \frac{(x^2+xh+3x+2h+2)-(x^2+xh+3x+h+2)}{h(x+h+2)(x+2)}$$

$$= \lim_{h\to 0} \frac{x^2+xh+3x+2h+2-x^2-xh-3x-h-2}{h(x+h+2)(x+2)}$$

$$= \lim_{h\to 0} \frac{h}{h(x+h+2)(x+2)}$$

$$= \lim_{h\to 0} \frac{1}{(x+h+2)(x+2)} = \boxed{\frac{1}{(x+2)^2}}$$ (Double check using quotient rule.)

6. [15 Points] Consider the function defined by

$$f(x) = \begin{cases} |x-1| & \text{if } x \le 7 \\ \dfrac{1}{x-7} & \text{if } x > 7 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$. SEE ME FOR A SKETCH

(b) State the domain of the function $f(x)$. Domain=$\{x|x \neq 0\}$

(c) Compute
$$\begin{cases} \lim_{x\to 7^+} f(x) = +\infty \\ \lim_{x\to 7^-} f(x) = 6 \\ \lim_{x\to 7} f(x) = \text{DNE since LHL} \neq \text{RHL} \end{cases}$$

(d) State the value(s) of x at which f is discontinuous. Justify your answer(s) using the definition of continuity.

Note that $f(7)$ is defined, but $f(x)$ is discontinuous at $x = 7$ since $\lim_{x\to 7} f(x)$ DNE.

(e) State the value(s) of x where $f(x)$ is *not* differentiable. Justify your answer(s).

$f(x)$ is not differentiable at $x = 1$ because of the sharp corner there, but also not differentiable at $x = 7$ since it's not continuous there.

7. [10 Points] Find the equation of the tangent line to $y = \sin(e^x)$ at the point where the x-coordinate is $\ln \pi$.

First, $y' = \cos(e^x) \cdot e^x$. Then $y'(\ln \pi) = \cos\left(e^{\ln \pi}\right) \cdot e^{\ln \pi} = \cos(\pi) \cdot (\pi) = -\pi$.

The point is $(\ln \pi, y(\ln \pi)) = (\ln \pi, \sin(e^{\ln \pi}) = (\ln \pi, \sin(\pi)) = (\ln \pi, 0)$.

Using *point-slope form*, the equation of the tangent line at the point $(\ln \pi, 0)$ is given by

$y - 0 = -\pi(x - \ln \pi)$ or $\boxed{y = -\pi x + \pi \ln \pi}$

8. [20 Points] Let $\boldsymbol{f(x) = \dfrac{x-3}{x^4}}$.

Sketch the graph of $y = f(x)$. State the domain for $f(x)$. Clearly indicate horizontal and vertical asymptotes, local minima/maxima, and inflection points on the graph, as well as where the graph is increasing, decreasing, concave up and concave down. Take my word that

$$\boldsymbol{f'(x) = \frac{3(4-x)}{x^5}} \quad \text{and} \quad \boldsymbol{f''(x) = \frac{12(x-5)}{x^6}}.$$

• Domain: $f(x)$ has domain $\{x | x \neq 0\}$

• VA: Vertical asymptotes $x = 0$. Note that,

$\lim_{x\to 0^-} \dfrac{x-3}{x^4} = \dfrac{-3}{0^+} = -\infty$ and $\lim_{x\to 0^+} \dfrac{x-3}{x^4} = \dfrac{-3}{0^+} = -\infty$

• HA: Horizontal asymptote is $y = 0$ for this f since

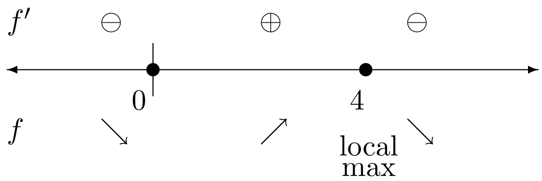
$\lim_{x\to\pm\infty} \dfrac{x-3}{x^4} = \lim_{x\to\pm\infty} \dfrac{x}{x^4} - \dfrac{3}{x^4} = \lim_{x\to\pm\infty} \dfrac{1}{x^3} - \dfrac{3}{x^4} = 0 - 0 = 0$

or because

$\lim_{x\to\pm\infty} \dfrac{x-3}{x^4} \cdot \dfrac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} = \lim_{x\to\pm\infty} \dfrac{\frac{1}{x^3} - \frac{3}{x^4}}{1} = \lim_{x\to\pm\infty} \dfrac{0}{1} = 0$

• First Derivative Information:

We know $f'(x) = \dfrac{3(4-x)}{x^5}$. Set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined or zero. The former happens when $x = 0$, but $x = 0$ was not in the domain of the original function, so it isn't technically a critical number. (We will still sign test around it.) The latter happens when $x = 4$. As a result, $x = 4$ is the critical number. Using sign testing/analysis for f',



or our f' chart is

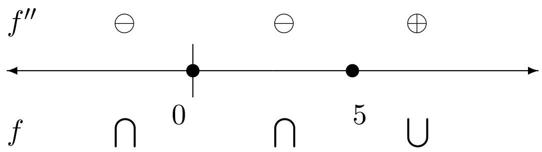
x	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$
$f'(x)$	$\ominus$	$\oplus$	$\ominus$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

So f is decreasing on $(-\infty, 0)$ and $(4, \infty)$ and increasing on $(0, 4)$. Moreover, f has a local max at $(4, f(4)) = \left(4, \dfrac{1}{128}\right)$.

• Second Derivative Information:

Meanwhile, $f'' = \dfrac{12(x-5)}{x^6}$.

We have a possible inflection point at $x = 5$. We note again that $x = 0$ here since it's not in the domain for f. (We will still sign test around it.) Using sign testing/analysis for f'',

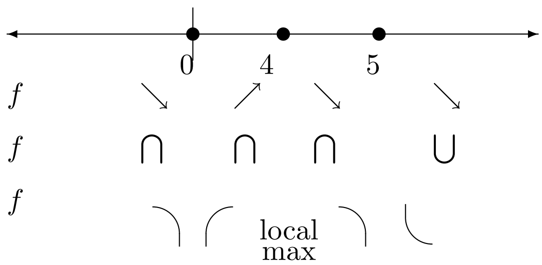


or our f'' chart is

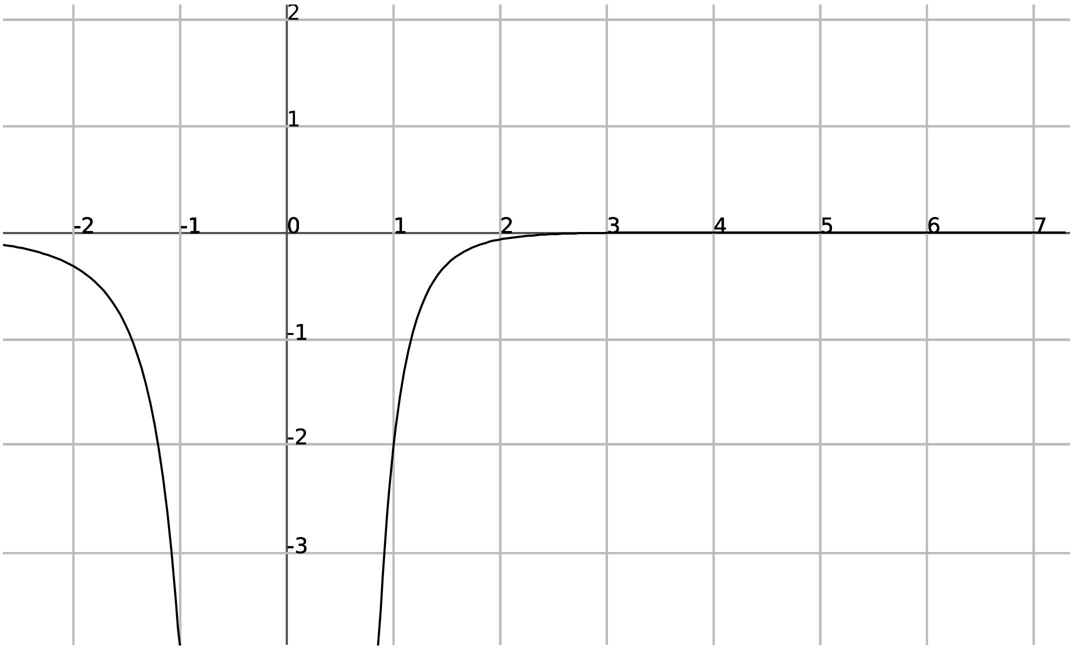
x	$(-\infty, 0)$	$(0, 5)$	$(5, \infty)$
$f''(x)$	$\ominus$	$\ominus$	$\oplus$
$f(x)$	$\cap$	$\cap$	$\cup$

So f is concave down on $(-\infty, 0)$ and $(0, 5)$, and concave up on $(5, \infty)$.

• Piece the first and second derivative information together:

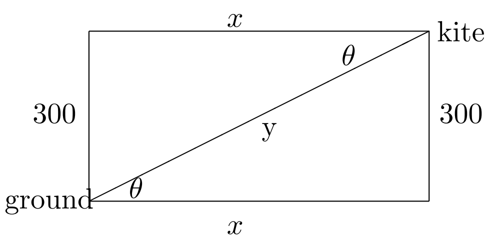


• Sketch:



9. [15 Points] A kite starts flying 300 feet directly above the ground. The kite is being blown horizontally at 10 feet per second. When the kite has blown horizontally for 40 seconds, how fast is the angle between the string and the ground changing?

• Diagram



The picture at arbitrary time t is:

• Variables

Let $x =$ distance kite has travelled horizontally at time t

Let $y =$ length of string from ground to kite at time t

Let $\theta =$ the angle between the string/horizontal

Given $\dfrac{dx}{dt} = 10\dfrac{\text{ft}}{\text{sec}}$,

find $\dfrac{d\theta}{dt} =?$ when $x = 10\frac{\text{ft}}{\text{sec}} \cdot (40)\text{sec} = 400$ feet

• Equation relating the variables:

The trigonometry of the triangle yields $\tan\theta = \dfrac{300}{x}$.

Note, you can also use $\cot\theta = \dfrac{x}{300}$

• Differentiate both sides w.r.t. time t.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{300}{x}\right) \implies \sec^2\theta\frac{d\theta}{dt} = -\frac{300}{x^2}\frac{dx}{dt}. \text{ (Related Rates!)}$$

• Substitute Key Moment Information (now and not before now!!!):

At the key instant when $x = 400$, we have $y = 500$ by the Pythagorean Theorem.

Therefore, $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{500}{400} = \frac{5}{4}$ and

$$\left(\frac{5}{4}\right)^2\frac{d\theta}{dt} = -\frac{300}{(400)^2}\cdot(10).$$

• Solve for the desired quantity:

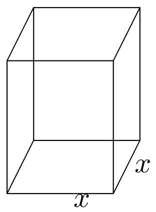
$$\frac{d\theta}{dt} = \frac{-300(10)(4^2)}{(400)^2(5)^2} = -\frac{3}{250}\frac{\text{rad}}{\text{sec}}$$

• Answer: The angle is decreasing at a rate of $\boxed{\frac{3}{250}}$ radians every second.

10. [15 Points] A large box with a square base and top is to be made to hold a fixed volume of 54 cubic feet. The sides cost \$1 per square foot. The top and bottom cost \$2 per square foot. Determine the dimensions that minimize the cost of materials.

(Remember to state the domain of the function you are computing extreme values for.)

• Diagram:



• Variables:

Let x =length of base of box.
Let y =height of the box.
Let V =volume of box.
Let C =cost of amount of material to make box.

• Equations:

We know the volume of the box given by $V = x^2y = 54$ is fixed, so that $y = \frac{54}{x^2}$.

Then the Cost of materials, which must be minimized, is given as

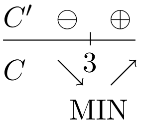
$$\begin{aligned} C &= \text{cost of bottom} + \text{cost of top} + \text{cost of 4 sides} \\ &= x^2(\$2) + x^2(\$2) + 4xy(\$1) \\ &= 4x^2 + 4xy \\ &= 4x^2 + 4x\left(\frac{54}{x^2}\right) \\ &= 4x^2 + \left(\frac{216}{x}\right) \end{aligned}$$

The (common-sense-bounds)domain of Cost is $\{x : x > 0\}$.

• Minimize:

Next $C' = 8x - \dfrac{216}{x^2}$. Setting $C' = 0$ we solve $x^3 = \dfrac{216}{8} = 27 \Longrightarrow x = 3$.

Sign-testing the critical number does indeed yield a minimum for the cost function.

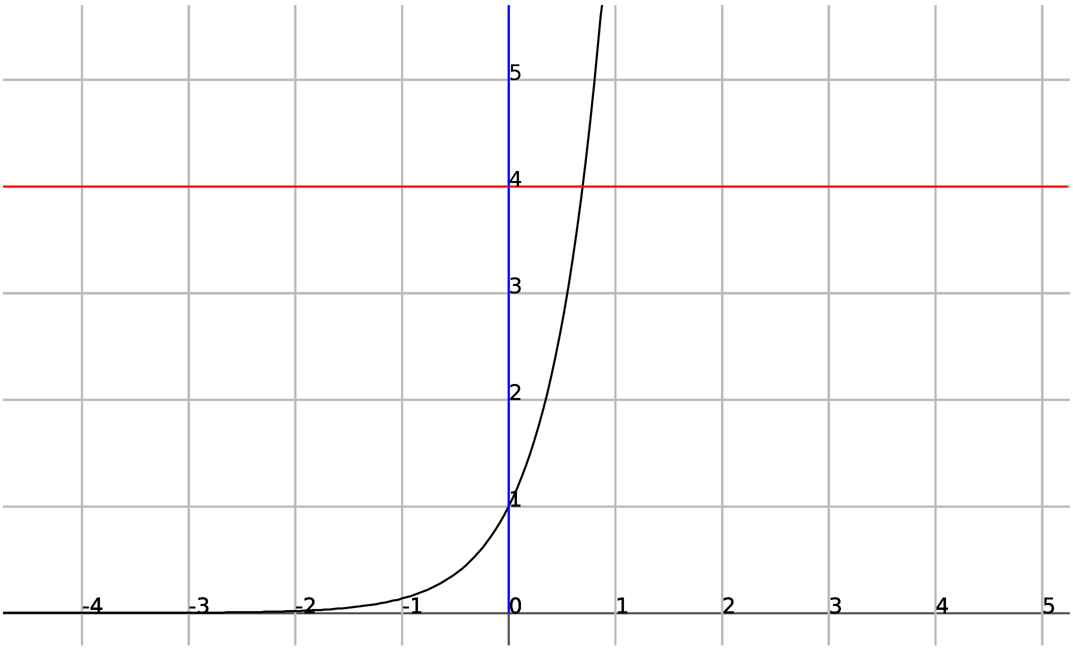


• Answer:

When $x = 3$ then $y = \dfrac{54}{(3)^2} = 6$. As a result, the most economical box has dimensions $\boxed{\text{3x3x6}}$ each in feet.

11. [20 Points] Consider the region in the first quadrant bounded by $y = e^{2x}$, $y = 4$, and the y–axis.

(a) Draw a picture of the region.



Note the curves intersect when $e^{2x} = 4$ or when $2x = \ln 4$. That is, $x = \dfrac{1}{2}\ln 4 = \ln\left(4^{\frac{1}{2}}\right) = \ln(\sqrt{4}) = \ln 2$.

(b) Compute the area of the region.

Area$= \displaystyle\int_0^{\ln 2} \text{top}-\text{bottom}\, dx = \int_0^{\ln 2} 4 - e^{2x}\, dx = 4x - \frac{1}{2}e^{2x}\Big|_0^{\ln 2} = \left(4\ln 2 - \frac{1}{2}e^{2\ln 2}\right) - \left(0 - \frac{1}{2}e^0\right) = \left(4\ln 2 - \frac{1}{2}e^{\ln(2^2)}\right) - \left(0 - \frac{1}{2}\right) = \ln\left(2^4\right) - \frac{1}{2}(4) + \frac{1}{2} = \ln(16) - 2 + \frac{1}{2} = \boxed{\ln(16) - \frac{3}{2}}$

(c) Compute the volume of the three-dimensional object obtained by rotating the region about the x–axis.

SEE ME FOR A SKETCH OF THE 3-D SOLID.

Volume$= \int_0^{\ln 2} \pi[(\text{outer radius})^2 - (\text{inner radius})^2]\ dx = \pi \int_0^{\ln 2} (4)^2 - (e^{2x})^2\ dx$

$= \pi \int_0^{\ln 2} 16 - e^{4x}\ dx = \pi \left(16x - \frac{1}{4}e^{4x}\right)\Big|_0^{\ln 2} = \pi\left(\left(16\ln 2 - \frac{1}{4}e^{4(\ln 2)}\right) - \left(0 - \frac{1}{4}e^0\right)\right)$

$= \pi\left(16\ln 2 - \frac{1}{4}e^{\ln(2^4)} + \frac{1}{4}\right) = \pi\left(16\ln 2 - \frac{1}{4}(16) + \frac{1}{4}\right) = \pi\left(16\ln 2 - 4 + \frac{1}{4}\right) = \boxed{\pi\left(16\ln 4 - \frac{15}{4}\right)}$

12. [15 Points] Consider an object moving on the number line such that its velocity at time t is $\boldsymbol{v(t) = \sin t}$ feet per second. Also assume that $s(0) = 2$ feet, where as usual $s(t)$ is the position of the object at time t.

(a) Compute the acceleration function $a(t)$ and position function $s(t)$.

$a(t) = \boxed{v'(t) = \cos t}$

and

$s(t) = \int v(t)\ dt = \int \sin t\ dt = -\cos t + C$

Use initial condition $s(0) = 2$ to solve for C here.

$s(0) = -\cos(0) + C = -1 + C \overset{\text{set}}{=} 2$, so $C = 3$. Finally, $\boxed{s(t) = -\cos t + 3}$

(b) Draw the graph of $v(t)$ for $0 \le t \le 2\pi$, and explain why the object is *not* always moving to the right.

This graph is just the plot of $y = \sin t$. Recall that the sine graph is positive between 0 and π, and negative between π and 2π. As a result the position is increasing along the number line for t between 0 and π, that is *moving to the right*, but the position is decreasing on the number line between π and 2π, that is *moving to the left.*

(c) Compute the **displacement** and **total distance** travelled for $0 \le t \le 2\pi$.

First note that $\sin t$ is an odd function, so that by symmetry of the graph, the displacement will automatically be 0. But you can also blast through the computation as follows:

Displacement$= \int_0^{2\pi} v(t)\ dt = \int_0^{2\pi} \sin t\ dt = -\cos t\Big|_0^{2\pi} = -\cos(2\pi) - (-\cos 0) = -1 - (-1) = -1 + 1 = \boxed{0}$.

Total Distance$= \int_0^{2\pi} |v(t)|\ dt = \int_0^{2\pi} |\sin t|\ dt = \int_0^{\pi} \sin t\ dt + \int_\pi^{2\pi} -\sin t\ dt = -\cos t\Big|_0^{\pi} + \cos t\Big|_\pi^{2\pi} = -\cos(\pi) - (-\cos 0) + \cos(2\pi) - \cos(\pi) = -(-1) + 1 + 1 - (-1) = 2 + 2 = \boxed{4}$