Math 111 Final Examination December 19, 2012

Answer Key

1. [20 Points] Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $$\lim_{x\to -2} \frac{x^2+3x+2}{x^2+x-2} = \lim_{x\to -2} \frac{(x+2)(x+1)}{(x+2)(x-1)} = \lim_{x\to -2} \frac{x+1}{x-1} = \frac{-1}{-3} = \boxed{\frac{1}{3}}$$

(b) $$\lim_{x\to 1^-} \frac{g(x+1)-2x-5}{(x-1)^2}, \quad \text{where} \quad g(x) = x^2+3.$$

$$\lim_{x\to 1^-} \frac{g(x+1)-2x-5}{(x-1)^2} = \lim_{x\to 1^-} \frac{(x+1)^2+3-2x-5}{(x-1)^2} = \lim_{x\to 1^-} \frac{x^2+2x+1+3-2x-5}{(x-1)^2}$$

$$= \lim_{x\to 1^-} \frac{x^2-1}{(x-1)^2} = \lim_{x\to 1^-} \frac{(x-1)(x+1)}{(x-1)^2} = \lim_{x\to 1^-} \frac{x+1}{x-1} = \frac{2}{0^-} = \boxed{-\infty}$$

(c) $$\lim_{x\to\infty} \frac{3-2x^2}{3x^2+5x} = \lim_{x\to\infty} \frac{3-2x^2}{3x^2+5x} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{\frac{3}{x^2} - \frac{2x^2}{x^2}}{\frac{3x^2}{x^2} + \frac{5x}{x^2}} = \lim_{x\to\infty} \frac{\frac{3}{x^2} - 2}{3 + \frac{5}{x}} = \boxed{-\frac{2}{3}}$$

(d) $$\lim_{x\to 5} \frac{25-x^2}{\sqrt{x+4}-3} = \lim_{x\to 5} \frac{25-x^2}{\sqrt{x+4}-3} \cdot \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3} = \lim_{x\to 5} \frac{(25-x^2)(\sqrt{x+4}+3)}{(x+4)-9}$$

$$= \lim_{x\to 5} \frac{(5-x)(5+x)(\sqrt{x+4}+3)}{x-5} = \lim_{x\to 5} \frac{-(x-5)(5+x)(\sqrt{x+4}+3)}{x-5}$$

$$= \lim_{x\to 5} -(5+x)(\sqrt{x+4}+3) = -10(\sqrt{9}+3) = -10(3+3) = -10(6) = \boxed{-60}$$

(e) $$\lim_{x\to 7} \frac{|7-x|}{x^2-x-42}$$ $\boxed{\text{DNE}}$ since RHL$\neq$LHL. see below.

Note: $$|7-x| = \begin{cases} 7-x & \text{if } 7-x \geq 0, \text{ that is } x \leq 7 \\ -(7-x) & \text{if } 7-x < 0, \text{ that is } x > 7 \end{cases}$$

RHL: $$\lim_{x\to 7+} \frac{|7-x|}{x^2-x-42} = \lim_{x\to 7+} \frac{-(7-x)}{x^2-x-42} = \lim_{x\to 7+} \frac{x-7}{(x-7)(x+6)} = \lim_{x\to 7+} \frac{1}{x+6} = \frac{1}{13}$$

LHL: $$\lim_{x\to 7-} \frac{|7-x|}{x^2-x-42} = \lim_{x\to 7+} \frac{7-x}{x^2-x-42} = \lim_{x\to 7+} \frac{-(x-7)}{(x-7)(x+6)} = \lim_{x\to 7+} \frac{-1}{x+6} = -\frac{1}{13}$$

2. [30 Points] Compute each of the following derivatives.

(a) $f'\left(\frac{\pi}{6}\right)$, where $f(x) = \cos^2 x + \tan(2x) + \sin x$. Simplify.

$$f'(x) = 2\cos x(-\sin x) + \sec^2(2x)(2) + \cos x$$

$$f'\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{6}\right)\left(-\sin\left(\frac{\pi}{6}\right)\right) + 2\sec^2\left(2\left(\frac{\pi}{6}\right)\right) + \cos\left(\frac{\pi}{6}\right)$$

$$= 2\cos\left(\frac{\pi}{6}\right)\left(-\sin\left(\frac{\pi}{6}\right)\right) + 2\sec^2\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) + 2(2)^2 + \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} + 8 + \frac{\sqrt{3}}{2} = \boxed{8}$$

(b) $\frac{d}{dx} \ln\left(\frac{(x^2+1)^{\frac{4}{7}}\ e^{\tan x}}{\sqrt{1+\sqrt{x}}}\right)$ Hint: you might want to simplify before differentiating.

$$\frac{d}{dx} \ln\left(\frac{(x^2+1)^{\frac{4}{7}}\ e^{\tan x}}{\sqrt{1+\sqrt{x}}}\right) = \frac{d}{dx}\left[\ln\left((x^2+1)^{\frac{4}{7}}\right) + \ln e^{\tan x} - \ln\sqrt{1+\sqrt{x}}\right]$$

$$= \frac{d}{dx}\left[\frac{4}{7}\ln(x^2+1) + \tan x - \frac{1}{2}\ln(1+\sqrt{x})\right]$$

$$= \boxed{\frac{4}{7}\left(\frac{1}{x^2+1}\right)\cdot 2x + \sec^2 x - \frac{1}{2}\left(\frac{1}{1+\sqrt{x}}\right)\cdot\left(\frac{1}{2\sqrt{x}}\right)}$$

$$\overset{\text{OR}}{=} \boxed{\frac{8x}{7(x^2+1)} + \sec^2 x - \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}}$$

(c) $g'(x)$, where $g(x) = \sqrt{\cos(x^2+e^x)} + \cos\sqrt{x^2+e^x} + e^{\sqrt{x^2+\cos x}}$. Do not simplify here.

$$\boxed{g'(x) = \frac{1}{2\sqrt{\cos(x^2+e^x)}}(-\sin(x^2+e^x))(2x+e^x) - \sin\sqrt{x^2+e^x}\,\frac{1}{2\sqrt{x^2+e^x}}(2x+e^x)}$$

continued $\boxed{+e^{\sqrt{x^2+\cos x}}\frac{1}{2\sqrt{x^2+\cos x)}}(2x-\sin x)}$

(d) $\frac{dy}{dx}$, if $\sin(xy) = \sec x + \cos(e^9) - y$.

$$\frac{d}{dx}(\sin(xy)) = \frac{d}{dx}\left(\sec x + \cos(e^9) - y\right)$$

$$\cos(xy)\left(x\frac{dy}{dx} + y\right) = \sec x\tan x - \frac{dy}{dx}$$

$$x\cos(xy)\frac{dy}{dx} + y\cos(xy) = \sec x\tan x - \frac{dy}{dx}$$

$$x\cos(xy)\frac{dy}{dx} + \frac{dy}{dx} = \sec x\tan x - y\cos(xy)$$

$$(x\cos(xy)+1)\frac{dy}{dx} = \sec x\tan x - y\cos(xy)$$

$$\frac{dy}{dx} = \boxed{\frac{\sec x \tan x - y\cos(xy)}{x\cos(xy)+1}}$$

(e) $g''(x)$, where $g(x) = \displaystyle\int_x^{2012} \sqrt{\ln t} + \ln\sqrt{t}\ dt.$

$$g'(x) = \frac{d}{dx}\int_x^{2011} \sqrt{\ln t}+\ln\sqrt{t}\ dt = -\frac{d}{dx}\int_{2011}^{x} \sqrt{\ln t}+\ln\sqrt{t}\ dt = -\left(\sqrt{\ln x}+\ln\sqrt{x}\right) \text{ (FTC Part I)}$$

$$g''(x) = -\left(\frac{1}{2\sqrt{\ln x}}\left(\frac{1}{x}\right) + \frac{1}{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)\right) = \boxed{-\left(\frac{1}{2x\sqrt{\ln x}} + \frac{1}{2x}\right)}$$

(f) $f''(x)$, where $f(x) = \dfrac{x^4}{e^x}$. Simplify here.

use Quotient Rule or Product Rule+Chain Rule:

$$f'(x) = x^4e^{-x}(-1) + e^{-x}(4x^3) = e^{-x}(4x^3 - x^4) = e^{-x}x^3(4-x) = \frac{x^3(4-x)}{e^x} = \boxed{\frac{4x^3 - x^4}{e^x}}$$

$$f'' = e^{-x}(12x^2 - 4x^3) - e^{-x}(4x^3 - x^4) = e^{-x}(12x^2 - 4x^3 - 4x^3 + x^4) = e^{-x}x^2(12 - 8x + x^2)$$

$$= e^{-x}x^2(x-2)(x-6) = \frac{x^2(x-2)(x-6)}{e^x} = \boxed{\frac{x^4 - 8x^3 + 12x^2}{e^x}}$$

3. [25 Points] Compute each of the following integrals.

(a) $$\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \tan(3x)\,dx = \int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \frac{\sin(3x)}{\cos(3x)}\,dx = -\frac{1}{3}\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{u}\,du = -\frac{1}{3}\ln|u|\Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} = -\frac{1}{3}\left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= -\frac{1}{3}\left(\ln\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right) = -\frac{1}{3}\left(\ln\left(\frac{1}{\sqrt{3}}\right)\right) = -\frac{1}{3}\left(\ln 1 - \ln\sqrt{3}\right) = -\frac{1}{3}\left(0 - \ln\sqrt{3}\right) = \boxed{\frac{\ln\sqrt{3}}{3}} \overset{\text{or}}{=} \boxed{\frac{\ln 3}{6}}$$

Here $\boxed{\begin{aligned} u &= \cos(3x) \\ du &= -3\sin(3x)dx \\ -\tfrac{1}{3}du &= \sin(3x)dx \end{aligned}}$ and $\boxed{\begin{aligned} x = \tfrac{\pi}{18} &\implies u = \cos\left(\tfrac{3\pi}{18}\right) = \cos\left(\tfrac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\ x = \tfrac{\pi}{9} &\implies u = \cos\left(\tfrac{3\pi}{9}\right) = \cos\left(\tfrac{\pi}{3}\right) = \frac{1}{2} \end{aligned}}$

(b) $$\int \frac{(3-\sqrt{x})(1+2\sqrt{x})}{x^2}\,dx = \int \frac{3 + 6\sqrt{x} - \sqrt{x} - 2x}{x^2}\,dx = \int \frac{3 + 5\sqrt{x} - 2x}{x^2}\,dx$$

$$= \int \frac{3}{x^2} + \frac{5\sqrt{x}}{x^2} - \frac{2x}{x^2}\,dx = \int \frac{3}{x^2} + \frac{5}{x^{\frac{3}{2}}} - \frac{2}{x}\,dx = \int \frac{3}{x^2} + \frac{5}{x^{\frac{3}{2}}} - \frac{2}{x}\,dx$$

$$= \int 3x^{-2} + 5x^{-\frac{3}{2}} - \frac{2}{x}\,dx = -3x^{-1} + 5(-2)x^{-\frac{1}{2}} - 2\ln|x| + C = \boxed{-\frac{3}{x} - \frac{10}{\sqrt{x}} - 2\ln|x| + C}$$

(c) $\displaystyle\int x\sqrt{x+1}\ dx = \int (u-1)\sqrt{u}\ du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$

$= \boxed{\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C}$

Here $\boxed{\begin{aligned} u &= x+1 \to x = u-1 \\ du &= dx \end{aligned}}$

(d) $\displaystyle\int_e^{e^3} \frac{4}{x(\ln x)^2}\ dx = 4\int_1^3 \frac{1}{u^2}\ du = 4\int_1^3 u^{-2}\ du = -4u^{-1}\Big|_1^3 = -\frac{4}{u}\Big|_1^3 = -\frac{4}{3} - (-4)$

$= -\frac{4}{3} + 4 = -\frac{4}{3} + \frac{12}{3} = \boxed{\frac{8}{3}}$

Here $\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\ dx \end{aligned}}$ and $\boxed{\begin{aligned} x = e &\implies u = \ln e = 1 \\ x = e^3 &\implies u = \ln e^3 = 3 \end{aligned}}$

(e) $\displaystyle\int_{\ln 3}^{\ln 8} \frac{e^x}{\sqrt{1+e^x}}\ dx = \int_4^9 \frac{1}{\sqrt{u}}\ du = \int_4^9 u^{-\frac{1}{2}}\ du = 2\sqrt{u}\Big|_4^9 = 2\sqrt{9} - 2\sqrt{4} = 2(3) - 2(2) = 6 - 4 = \boxed{2}$

Here $\boxed{\begin{aligned} u &= 1+e^x \\ du &= e^x\ dx \end{aligned}}$ and $\boxed{\begin{aligned} x = \ln 3 &\implies u = 1 + e^{\ln 3} = 1 + 3 = 4 \\ x = \ln 8 &\implies u = 1 + e^{\ln 8} = 1 + 8 = 9 \end{aligned}}$

4. [10 Points] Give an ε–δ proof that $\displaystyle\lim_{x\to 3} 7 - 5x = -8$.

Scratchwork: we want $|f(x) - L| = |(7-5x) - (-8)| < \varepsilon$

$$|f(x) - L| = |(7-5x) - (-8)| = |7 - 5x + 8| = |15 - 5x| = |-5(x-3)| = |-5||x-3| = 5|x-3|$$

(want $< \varepsilon$)

$$5|x-3| < \varepsilon \text{ means } |x-3| < \frac{\varepsilon}{5}$$

$$\text{So choose } \delta = \frac{\varepsilon}{5}.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{5}$. Given x such that $0 < |x-3| < \delta$, then as desired

$$|f(x) - L| = |(7-5x) - (-8)| = |-5x + 15| = |-5(x-3)| = |-5||x-3| = 5|x-3| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon.$$

$\square$

5. [10 Points] Let $f(x) = \dfrac{1}{x^2}$. Calculate $f'(x)$, using the **limit definition** of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{x^2-(x+h)^2}{(x+h)^2x^2}\right)}{h} = \lim_{h\to 0} \frac{\left(\dfrac{x^2-(x^2+2xh+h^2)}{(x+h)^2x^2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{x^2-x^2-2xh-h^2}{(x+h)^2x^2}\right)}{h} = \lim_{h\to 0} \frac{\left(\dfrac{-2xh-h^2}{(x+h)^2x^2}\right)}{h} = \lim_{h\to 0} \frac{-2xh-h^2}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{h(-2x-h)}{h(x+h)^2x^2}$$

$$= \lim_{h\to 0} \frac{-2x-h}{(x+h)^2x^2} = \frac{-2x}{x^4} = \boxed{-\frac{2}{x^3}}$$

Free double check for yourself using the Power Rule:

$f'(x) = -2x^{-3} = -\dfrac{2}{x^3}$ Match!!

6. [15 Points] Compute $\displaystyle\int_1^3 4-x^2\,dx$ using each of the following **two** different methods:

(a) Fundamental Theorem of Calculus,
(b) Riemann Sums and the limit definition of the definite integral $***$.

***Recall

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} 1 = n$$

(a) Fundamental Theorem of Calculus.

$$\int_1^3 4-x^2\,dx = 4x - \frac{x^3}{3}\Big|_1^3 = (12-9) - \left(4-\frac{1}{3}\right) = 3-4+\frac{1}{3} = -1+\frac{1}{3} = \boxed{-\frac{2}{3}}$$

(b) Riemann Sums and the limit definition of the definite integral ***.

Here $a=1, b=3, \Delta x = \dfrac{b-a}{n} = \dfrac{2}{n}$, and $x_i = a + i\Delta x = 1 + \dfrac{2i}{n}$

$$
\begin{aligned}
\int_1^3 4 - x^2\, dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \quad &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\frac{2}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(4-\left(1+\left(\frac{2i}{n}\right)^2\right)\right)\frac{2}{n}\\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} 4-\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)\\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} 3-\frac{4i}{n}-\frac{4i^2}{n^2}\\
&= \lim_{n\to\infty}\left(\frac{2}{n}\sum_{i=1}^{n}3-\frac{2}{n}\sum_{i=1}^{n}\frac{4i}{n}-\frac{2}{n}\sum_{i=1}^{n}\frac{4i^2}{n^2}\right)\\
&= \lim_{n\to\infty}\left(\frac{6}{n}\sum_{i=1}^{n}1-\frac{8}{n^2}\sum_{i=1}^{n}i-\frac{8}{n^3}\sum_{i=1}^{n}i^2\right)\\
&= \lim_{n\to\infty}\left(\frac{6}{n}(n)-\frac{8}{n^2}\frac{n(n+1)}{2}-\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)\\
&= \lim_{n\to\infty}\left(6-\frac{8}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)-\frac{8}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right)\\
&= \lim_{n\to\infty}\left(6--\frac{8}{2}(1)\left(1+\frac{1}{n}\right)-\frac{8}{6}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right)\\
&= 6-4-\frac{8}{3}\\
&= 2-\frac{8}{3} = \boxed{-\frac{2}{3}} \qquad \text{Match}
\end{aligned}
$$

7. [10 Points] Find the equation of the tangent line to

$$y = \cos(\ln(x+1)) \;+\; \ln(\cos x) \;+\; e^{\sin x} \;+\; \sin(e^x - 1)$$

at the point where $x = 0$.

$y' = -\sin(\ln(x+1))\left(\frac{1}{x+1}\right) \;+\; \frac{1}{\cos x}(-\sin x) \;+\; e^{\sin x}\cos x \;+\; \cos(e^x-1)e^x$

$y'(0) = -\sin(\ln(0+1))\left(\frac{1}{0+1}\right) \;+\; \frac{1}{\cos 0}(-\sin 0) \;+\; e^{\sin 0}\cos 0 \;+\; \cos(e^0-1)e^0$

$= 0+0+1+1 = 2 \longleftarrow$ Slope

Point $(0, y(0)) = (0, 2)$

because $y(0) = \cos(\ln(0+1)) + \ln(\cos 0) + e^{\sin 0} + \sin(e^0 - 1) = \cos 0 + \ln 1 + e^0 + \sin 0$

$= 1 + 0 + 1 + 0 = 2$

Point-Slope Form

$y - 2 = 2(x - 0)$

$\boxed{y = 2x + 2}$

8. [20 Points] Let $\boldsymbol{f(x) = \frac{x^4}{e^x} = x^4 e^{-x}}$.

For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

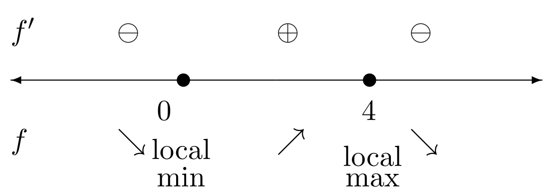
Take my word that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = +\infty$.

Also take my word that $f'(x) = \dfrac{x^3(4-x)}{e^x}$ and $f''(x) = \dfrac{x^2(x-2)(x-6)}{e^x}$.

- Domain$= \mathbb{R}$. Note that $f(x) = \dfrac{x^4}{e^x}$ and e^x is never zero in the denominator.
- It has no vertical asymptotes.
- There is a horizontal asymptote for this f at $y = 0$ because $\lim_{x\to\infty} f(x) = 0$.
- First Derivative Information

We use the given derivative $f'(x) = \dfrac{x^3(4-x)}{e^x}$ to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when $x = 0$ or $x = 4$.

Using sign testing/analysis for f',

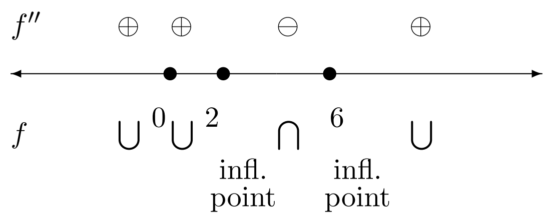


So f is increasing on the interval $(0, 4)$; and f is decreasing on $(-\infty, 0)$ and $(4, \infty)$. Moreover, f has a local max at $x = 4$ with $f(4) = 256e^{-4}$, and a local min at $x = 0$ with $f(0) = 0$.

- Second Derivative Information

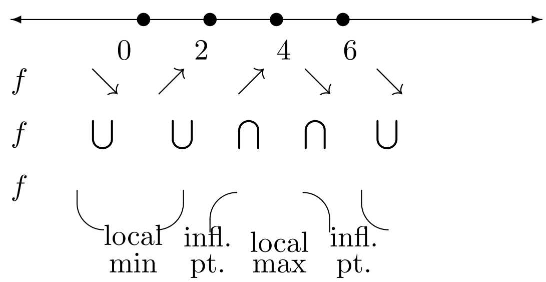
Setting $f'' = 0$ we solve for our possible inflection points $x = 0, x = 2$, or $x = 6$.

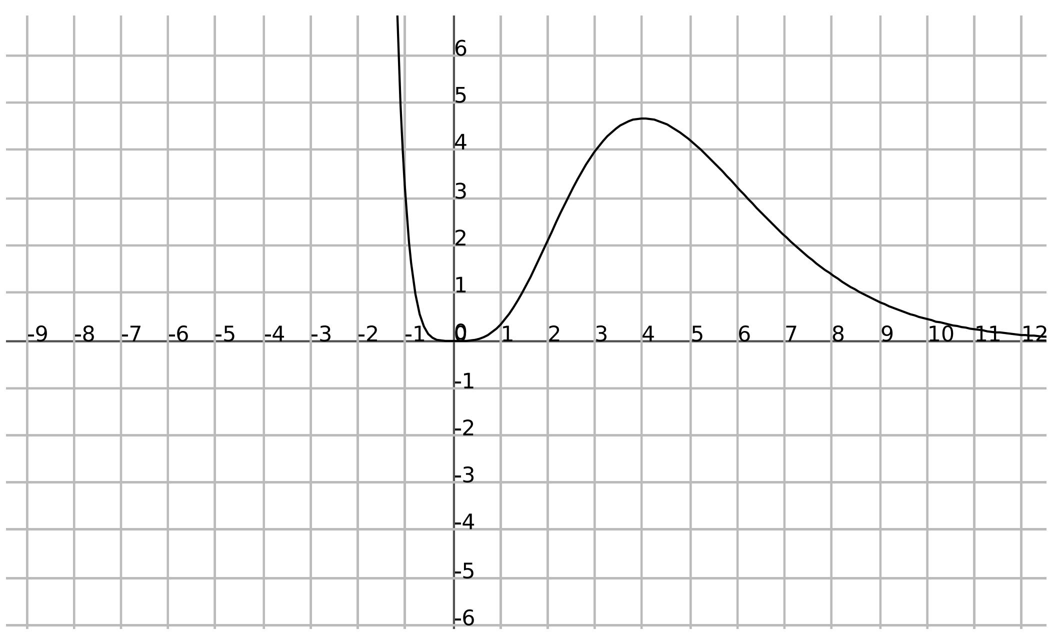
Using sign testing/analysis for f'',



So f is concave down on the interval $(2, 6)$ and concave up on $(-\infty, 2)$ and $(6, \infty)$, with inflection points at $x = 2$ and $x = 6$.

- Piece the first and second derivative information together



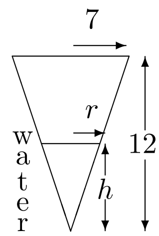


9. [15 Points] A conical tank, 14 feet across the entire top and 12 feet deep, is leaking water.

The water is leaking at the rate of 2 cubic feet per minute. How fast is the radius of the water level changing when the radius of the water level is 3 feet?

**Recall the volume of the cone is given by $V = \frac{1}{3}\pi r^2 h$

The cross section (with water level drawn in) looks like:



• Diagram

• Variables

Let $r =$ radius of the water level at time t

Let $h =$ height of the water level at time t

Let $V =$ volume of the water in the tank at time t

Find $\frac{dr}{dt}$ =? when $r = 3$ feet

and $\frac{dV}{dt} = -2\frac{\text{ft}^3}{\text{min}}$

• Equation relating the variables:

Volume= $V = \frac{1}{3}\pi r^2 h$

• Extra solvable information: Note that h is not mentioned in the problem's info. But there is a relationship, via similar triangles, between r and h. We must have

$$\frac{r}{7} = \frac{h}{12} \implies h = \frac{12r}{7}$$

After substituting into our previous equation, we get:

$V = \frac{1}{3}\pi r^2 \left(\frac{12r}{7}\right) = \frac{4}{7}\pi r^3$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{7}\pi r^3\right) \implies \frac{dV}{dt} = \frac{4}{7}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \implies \frac{dV}{dt} = \frac{12}{7}\pi r^2 \frac{dr}{dt}$

• Substitute Key Moment Information (now and not before now!!!):

$-2 = \frac{12}{7}\pi(3)^2\frac{dr}{dt}$

• Solve for the desired quantity:

$\boxed{\frac{dr}{dt} = -\frac{7}{54\pi}\frac{\text{ft}}{\text{min}}}$

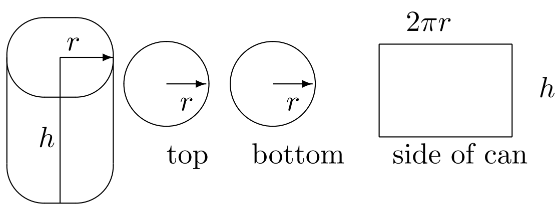
• Answer the question that was asked: The radius of the water level is decreasing out of the tank at a rate of $\frac{7}{54\pi}$ feet every minute.

10. [15 Points] A cylindrical can, with a bottom and a top, has a fixed volume of 2000 π cubic units. Determine the height and radius of the can that has the least surface area.

(Recall, the volume of a cylinder with radius r and height h, is given by $V = \pi r^2 h$.)

(Remember to state the domain of the function you are computing extreme values for.)

- Diagram:



- Variables:

Let r =radius of can.
Let h =height of can.
Let M =amount of material (surface area).
Let V =volume of can.

- Equations:

We know that the can's volume $V = \pi r^2 h = 2000\pi$ is fixed so that $h = \dfrac{2000\pi}{\pi r^2} = \dfrac{2000}{r^2}$

Then the amount of material used $M = 2\pi r^2 + 2\pi r h$ is must be minimized.

Subsitute for h: $M = 2\pi r^2 + 2\pi r\left(\dfrac{2000}{r^2}\right) = 2\pi r^2 + \left(\dfrac{4000\pi}{r}\right)$ Minimize!

The (common-sense-bounds)domain of M is $\{r : r > 0\}$.

- Maximize:

Next $M' = 4\pi r - \dfrac{4000\pi}{r^2}$. Setting $M' = 0$ yields $r^3 = 1000$ or $r = 10$ as the critical number.

Sign-testing the critical number does indeed yield a minimum for the volume function.

M' $\ominus$ $\oplus$
M $\searrow$ 10 $\nearrow$
MIN

- Answer:

Since $r = 10$ then $h = \dfrac{2000\pi}{\pi(10)^2} = \dfrac{2000\pi}{100\pi} = 20$.

The dimensions of the can with the smallest surface area are $\boxed{r = 10}$ units and $\boxed{h = 20}$ units.

11. [15 Points] Consider the region in the first quadrant bounded by $y = e^x + 1$, $y = 4$, and the y–axis.

(a) Draw a picture of the region.

See me for a sketch.

Note that the curves intersect when $1 + e^x = 4$ which is when $e^x = 3$ which implies $x = \ln 3$.

(b) Compute the area of the region.

Area= $\displaystyle\int_0^{\ln 3} \text{top} - \text{bottom}\ dx = \int_0^{\ln 3} 4 - (e^x + 1)\ dx = \int_0^{\ln 3} 3 - e^x\ dx = 3x - e^x \Big|_0^{\ln 3}$

$= (3\ln 3 - e^{\ln 3}) - (0 - e^0) = 3\ln 3 - 3 + 1 = \boxed{\ln 27 - 2}$

(c) Compute the volume of the three-dimensional solid obtained by rotating the region about the horizontal line $\boldsymbol{y = -2}$. Sketch the solid, along with one of the approximating washers.

See me for a sketch.

Volume= $\displaystyle\int_0^{\ln 3} \pi[(\text{outer radius})^2 - (\text{inner radius})^2]\ dx = \int_0^{\ln 3} \pi[6^2 - (3 + e^x)^2]\ dx$

$\displaystyle= \int_0^{\ln 3} \pi[36 - (9 + 6e^x + e^{2x})]\ dx = \int_0^{\ln 3} \pi[27 - 6e^x - e^{2x}]\ dx = \pi[27x - 6e^x - \frac{1}{2}e^{2x}]\Big|_0^{\ln 3}$

$= \pi[(27\ln 3 - 6e^{\ln 3} - \frac{1}{2}e^{2\ln 3}) - (0 - 6e^0 - \frac{1}{2}e^0)] = \pi[27\ln 3 - 6(3) - \frac{1}{2}e^{\ln(3^2)} + 6 + \frac{1}{2}]$

$= \pi[27\ln 3 - 18 - \frac{9}{2} + 6 + \frac{1}{2}] = \pi[27\ln 3 - 12 - \frac{8}{2}] = \pi[27\ln 3 - 12 - 4] = \boxed{\pi[27\ln 3 - 16]}$

12. [15 Points] Consider an object moving on the number line such that its velocity at time t seconds is $\boldsymbol{v(t) = 4 - t^2}$ feet per second. Also assume that the position of the object at one second is $\frac{5}{3}$.

(a) Compute the acceleration function $a(t)$ and the position function $s(t)$.

$a(t) = \boxed{-2t}$

$\displaystyle s(t) = \int 4 - t^2\ dt = 4t - \frac{t^3}{3} + C$

Use the initial condition $s(1) = \dfrac{5}{3}$

$\displaystyle s(1) = 4 - \frac{1}{3} + C \overset{\text{set}}{=} \frac{5}{3} \Rightarrow C = -2$

Finally, $s(t) = \boxed{4t - \dfrac{t^3}{3} - 2}$

(b) Compute the **total distance** travelled for $0 \le t \le 3$.

$$
\begin{aligned}
\text{Total Distance} = \int_0^3 |4 - t^2| \; dt &= \int_0^2 4 - t^2 \; dt + \int_2^3 -(4 - t^2) \; dt \\
&= 4t - \frac{t^3}{3} \bigg|_0^2 + \left(-4t + \frac{t^3}{3} \right) \bigg|_2^3 \\
&= \left(8 - \frac{8}{3} \right) - (0 - 0) + (-12 + 9) - \left(-8 + \frac{8}{3} \right) \\
&= 8 - \frac{8}{3} - 3 + 8 - \frac{8}{3} \\
&= 13 - \frac{16}{3} \\
&= \frac{39}{3} - \frac{16}{3} \\
&= \boxed{\frac{23}{3}}
\end{aligned}
$$