Math 111, Section 01, Fall 2014

Worksheet 5, Tuesday, October 7, 2014
Answer Key

1. Compute the derivative of each of the following functions. For these problems, you do **not** need to simplify your derivative.

(a) $y = \dfrac{1}{\sqrt{x^2 - 5x + 3}} = \left(x^2 - 5x + 3\right)^{-\frac{1}{2}}$

$y' = \boxed{-\dfrac{1}{2}\left(x^2 - 5x + 3\right)^{-\frac{3}{2}}(2x - 5)}$

(b) $y = \left(\dfrac{1}{x^3} + 7x\right)^{\frac{5}{7}}\left(x^4 - \dfrac{1}{x^7}\right)^{-5}$

$y' = \boxed{\left(\dfrac{1}{x^3} + 7x\right)^{\frac{5}{7}}(-5)\left(x^4 - \dfrac{1}{x^7}\right)^{-6}(4x^3 + 7x^{-8})}$ continued...

$\boxed{+\left(x^4 - \dfrac{1}{x^7}\right)^{-5}\left(\dfrac{5}{7}\right)\left(\dfrac{1}{x^3} + 7x\right)^{-\frac{2}{7}}\left(-\dfrac{3}{x^4} + 7\right)}$

(c) $y = \sqrt{\dfrac{x^2 + 5}{5 - 3x}}$

$y' = \boxed{\dfrac{1}{2\sqrt{\frac{x^2+5}{5-3x}}}\left(\dfrac{(5 - 3x)(2x) - (x^2 + 5)(-3)}{(5 - 3x)^2}\right)}$

(d) $H(\theta) = \sec^2(\cos\theta)$

$H'(\theta) = \boxed{2\sec(\cos\theta)(\sec(\cos\theta)\tan(\cos\theta))(-\sin\theta)}$

2. Compute each of the following derivatives. Simplify.

(a) $f'\left(\sqrt{\dfrac{\pi}{3}}\right)$, where $f(x) = \dfrac{1}{\tan(x^2)}$. Be careful with all the squares.

Rewrite $f(x) = \dfrac{1}{\tan(x^2)} = \left(\tan(x^2)\right)^{-1}$

Then $f'(x) = (-1)\left(\tan(x^2)\right)^{-2}\sec^2(x^2)(2x) = -\dfrac{1}{(\tan(x^2))^2}\sec^2(x^2)(2x)$

Finally $f'\left(\sqrt{\frac{\pi}{3}}\right) = -\frac{1}{\left(\tan\left(\left(\sqrt{\frac{\pi}{3}}\right)^2\right)\right)^2} \sec^2\left(\left(\sqrt{\frac{\pi}{3}}\right)^2\right)\left(2\left(\sqrt{\frac{\pi}{3}}\right)\right)$

$= -\frac{1}{\left(\tan\left(\frac{\pi}{3}\right)\right)^2} \sec^2\left(\frac{\pi}{3}\right)\left(2\left(\sqrt{\frac{\pi}{3}}\right)\right)$

$= -\frac{1}{(\sqrt{3})^2} (2)^2 \left(2\left(\sqrt{\frac{\pi}{3}}\right)\right) = \boxed{-\frac{8\sqrt{\pi}}{3\sqrt{3}}}$

(b) $g''\left(\frac{\pi}{6}\right)$, where $g(x) = \frac{\cos x}{1+\sin x}$. Hint: Simplify $g'(x)$ before computing $g''(x)$. Then compute the second derivative and evaluate $g''(x)$ at $x = \frac{\pi}{6}$

First $g'(x) = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$

$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$

$= \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-(\sin x + 1)}{(1+\sin x)^2} = \frac{-1}{1+\sin x} = -(1+\sin x)^{-1}$

Next, $g''(x) = -(-1)(1+\sin x)^{-2}(\cos x) = \frac{\cos x}{(1+\sin x)^2}$

Finally, $g''\left(\frac{\pi}{6}\right) = \frac{\cos\left(\frac{\pi}{6}\right)}{\left(1+\sin\left(\frac{\pi}{6}\right)\right)^2} = \frac{\frac{\sqrt{3}}{2}}{\left(1+\frac{1}{2}\right)^2} = \frac{\frac{\sqrt{3}}{2}}{\frac{9}{4}} = \boxed{\frac{2\sqrt{3}}{9}}$

3. Use (quick) differentiation rules (like we did in class) to show that $\frac{d}{dx}\tan x = \sec^2 x$.

$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 = \sec^2 x.$

4. Use (quick) differentiation rules (like we did in class) to show that $\frac{d}{dx}\sec x = \sec x \tan x$.

$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{d}{dx}(\cos x)^{-1} = -(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x}$

$$= \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x.$$

OR you can use the Quotient Rule

$$\frac{d}{dx}\sec x = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x$$

5. Let $W(x) = \cos^2(2x) + \tan(2x) + 3\sec x$. Compute $W'\left(\frac{\pi}{6}\right)$. Simplify your answer completely.

$$W'(x) = 2\cos(2x)(-\sin(2x))(2) + \sec^2(2x)(2) + 3\sec x \tan x$$
$$= -4\cos(2x)\sin(2x) + 2\sec^2(2x) + 3\sec x \tan x$$

$$W'\left(\frac{\pi}{6}\right) = -4\cos\left(2\left(\frac{\pi}{6}\right)\right)\sin\left(2\left(\frac{\pi}{6}\right)\right) + 2\sec^2\left(2\left(\frac{\pi}{6}\right)\right) + 3\sec\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{6}\right)$$
$$= -4\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) + 2\sec^2\left(\frac{\pi}{3}\right) + 3\sec\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{6}\right)$$
$$= -4\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 2(2)^2 + 3\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)$$
$$= -\sqrt{3} + 8 + 2 = \boxed{10 - \sqrt{3}}.$$

6. For each function below, find the equation of the tangent line to the curve $f(x)$ at the given x-coordinate.

(a) $f(x) = \sin x$ at $x = 0$. (We did this in class.)

First, $f'(x) = \cos x$. Next $f'(0) = \cos 0 = 1$.
The point is given by $(0, f(0)) = (0, \sin 0) = (0, 0)$.
Using point-slope form, we get
$y - 0 = 1(x - 0)$ or $\boxed{y = x}$.

(b) $f(x) = \cos x$ at $x = \frac{\pi}{6}$.

First, $f'(x) = -\sin x$. Next $f'\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$.

The point is given by $\left(\frac{\pi}{6}, f\left(\frac{\pi}{6}\right)\right) = \left(\frac{\pi}{6}, \left(\frac{\sqrt{3}}{2}\right)\right)$.

Using point-slope form, we get

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(x - \frac{\pi}{6}\right) \text{ or } \boxed{y = -\frac{1}{2}x + \frac{\pi}{12} + \frac{\sqrt{3}}{2}}.$$

(c) $f(x) = \tan x$ at $x = \frac{\pi}{3}$.

First, $f'(x) = \sec^2 x$. Next $f'\left(\frac{\pi}{3}\right) = \sec^2\left(\frac{\pi}{3}\right) = \frac{1}{\cos^2\left(\frac{\pi}{3}\right)} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4.$

The point is given by $\left(\frac{\pi}{3}, f\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \sqrt{3}\right)$.

Using point-slope form, we get

$$y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right) \text{ or } \boxed{y = 4x - \frac{4\pi}{3} + \sqrt{3}}$$

7. Simplify the expression $6(x+1)^2(1-2x)^4 + (x+1)^3 4(1-2x)^3(-2)$. Hint: Common factors.

$= 2(x+1)^2(1-2x)^3[3(1-2x) - 4(x+1)] = 2(x+1)^2(1-2x)^3[3 - 6x - 4x - 4]$

$= \boxed{2(x+1)^2(1-2x)^3[-10x-1]}$

8. For later purposes we need to practice solving.

(a) Consider the equation $x^2 + 2xyy' = 3y - 7y'$. Solve for y'.

$2xyy' + 7y' = 3y - x^2$

$y'(2xy + 7) = 3y - x^2$

$$y' = \boxed{\frac{3y - x^2}{2xy + 7}}$$

(b) Consider the equation $3y^2\frac{dy}{dx} - 5x^3y = 4x + 7\frac{dy}{dx}$. Solve for $\frac{dy}{dx}$.

$3y^2\frac{dy}{dx} - 7\frac{dy}{dx} = 4x + 5x^3y$

$\frac{dy}{dx}(3y^2 - 7) = 4x + 5x^3y$

Finally, $\frac{dy}{dx} = \boxed{\frac{4x + 5x^3y}{3y^2 - 7}}$

9. Find **all** x-coordinates at which the graphs of the following functions have horizontal tangent lines. Please **simplify** your derivatives first. Why? You practiced doing that in #7 above.

(a) $f(x) = (7x-3)^4(5x+2)^6$

$f'(x) = (7x-3)^4 6(5x+2)^5(5) + (5x+2)^6 4(7x-3)^3(7)$
$= (7x-3)^3 2(5x+2)^5 \left((7x-3)(15) + (5x+2)(14)\right)$
$= (7x-3)^3 2(5x+2)^5 \left(105x - 45 + 70x + 28\right)$
$= (7x-3)^3 2(5x+2)^5 \left(175x - 17\right) \overset{\text{set}}{=} 0$

For a product of several terms to equal 0, then any of the terms could be equal to 0. Therefore,

$7x - 3 = 0$ OR $5x + 2 = 0$ OR $175x - 17 = 0$

Finally, solving each equation

$\boxed{x = \frac{3}{7}}$ OR $\boxed{x = -\frac{2}{5}}$ OR $\boxed{x = \frac{17}{175}}$

(b) $w(t) = t^2(1-t)^6$

$w'(t) = t^2 6(1-t)^5(-1) + (1-t)^6(2t)$
$= 2t(1-t)^5(-3t + (1-t))$
$= 2t(1-t)^5(-4t+1) \overset{\text{set}}{=} 0$

That implies $t = 0$ OR $1 - t = 0$ OR $-4t + 1 = 0$

so that $\boxed{t = 0}$ OR $\boxed{t = 1}$ OR $\boxed{t = \frac{1}{4}}$