Math 111, Fall 2014

Worksheet 3, ANSWER KEY, Tuesday, September 23, 2014

- Please *show* all of your work and *justify* all of your answers.

1. Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim_{x\to 7} \frac{x^2-4x-21}{x^2-3x} \overset{\text{DSP}}{=} \frac{0}{28} = \boxed{0}$

(b) $\lim_{x\to 4} \frac{x^2-3x-4}{|x-4|} = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 4^+} \frac{x^2-3x-4}{|x-4|} = \lim_{x\to 4^+} \frac{x^2-3x-4}{x-4} = \lim_{x\to 4^+} \frac{(x-4)(x+1)}{x-4} = \lim_{x\to 4^+} x+1 = 5$

LHL: $\lim_{x\to 4^-} \frac{x^2-3x-4}{|x-4|} = \lim_{x\to 4^-} \frac{x^2-3x-4}{-(x-4)} = \lim_{x\to 4^-} \frac{(x-4)(x+1)}{-(x-4)} = \lim_{x\to 4^+} -(x+1) = -5$

Recall $|x-4| = \begin{cases} x-4 & \text{if } x-4\geq 0 \\ -(x-4) & \text{if } x-4<0 \end{cases} = \begin{cases} x-4 & \text{if } x\geq 4 & \longleftarrow \text{RHL} \\ -(x-4) & \text{if } x<4 & \longleftarrow \text{LHL} \end{cases}$

WARNING: The $|x-4|$ does not just cancel with the $x-4$. You must examine the two cases for the absolute value, because we are approaching 7.

(c) $\lim_{x\to 4} \frac{x^2-2x-8}{x^2-5x+4}^{\left(\frac{0}{0}\right)} = \lim_{x\to 4} \frac{(x-4)(x+2)}{(x-4)(x-1)} = \lim_{x\to 4} \frac{x+2}{x-1} \overset{\text{DSP}}{=} \frac{6}{3} = \boxed{2}$

(d) $\lim_{x\to -5} \frac{\frac{1}{1-x}-\frac{1}{6}}{x^2+3x-10}^{\left(\frac{0}{0}\right)} = \lim_{x\to -5} \frac{\frac{6-(1-x)}{(1-x)6}}{x^2+3x-10} = \lim_{x\to -5} \frac{5+x}{(1-x)6}\cdot\frac{1}{x^2+3x-10}$

$= \lim_{x\to -5} \frac{5+x}{(1-x)6}\cdot\frac{1}{(x+5)(x-2)} = \lim_{x\to -5} \frac{1}{(1-x)6(x-2)} \overset{\text{DSP}}{=} \frac{1}{(6)6(-7)} = \boxed{-\frac{1}{252}}$

(e) $\lim_{x\to 3} \frac{x^2-12x+27}{x^2-6x+9}^{\left(\frac{0}{0}\right)} = \lim_{x\to 3} \frac{(x-3)(x-9)}{(x-3)(x-3)} = \lim_{x\to 3} \frac{x-9}{x-3}^{\left(\frac{-6}{0}\right)}$ $\boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 3^+} \frac{x-9}{x-3} = \frac{-6}{0^+} = -\infty$

LHL: $\lim_{x\to 3^-} \frac{x-9}{x-3} = \frac{-6}{0^-} = +\infty$

(f) $\lim_{x\to 3} \dfrac{x^2-12x+27}{x^2-9}^{\left(\frac{0}{0}\right)} = \lim_{x\to 3} \dfrac{(x-9)(x-3)}{(x-3)(x+3)} = \lim_{x\to 3} \dfrac{x-9}{x+3} \overset{\text{DSP}}{=} \dfrac{-6}{6} = \boxed{-1}$

(g) $\lim_{x\to 4} \dfrac{x+2}{4-x}^{\left(\frac{6}{0}\right)} = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 4^+} \dfrac{x+2}{4-x} = \dfrac{6}{0^-} = -\infty$

LHL: $\lim_{x\to 4^-} \dfrac{x+2}{4-x} = \dfrac{6}{0^+} = +\infty$

warning: watch the signs on the 0 piece in the denominator. We have $4-x$ here not $x-4$.

(h) $\lim_{x\to -4} \dfrac{x+2}{x+4}^{\left(\frac{-2}{0}\right)} = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to -4^+} \dfrac{x+2}{x+4} = \dfrac{-2}{0^+} = -\infty$

LHL: $\lim_{x\to -4^-} \dfrac{x+2}{x+4} = \dfrac{-2}{0^-} = +\infty$

(i) $\lim_{x\to 2} \dfrac{3-\sqrt{x+7}}{x^2-3x+2}^{\left(\frac{0}{0}\right)} = \lim_{x\to 2} \dfrac{3-\sqrt{x+7}}{x^2-3x+2} \cdot \dfrac{3+\sqrt{x+7}}{3+\sqrt{x+7}} = \lim_{x\to 2} \dfrac{9-(x+7)}{(x^2-3x+2)(3+\sqrt{x+7})}$

$= \lim_{x\to 2} \dfrac{2-x}{(x-2)(x-1)(3+\sqrt{x+7})} = \lim_{x\to 2} \dfrac{-(x-2)}{(x-2)(x-1)(3+\sqrt{x+7})}$

$= \lim_{x\to 2} \dfrac{-1}{(x-1)(3+\sqrt{x+7})} \overset{\text{L.L.}}{=} \dfrac{-1}{3+\sqrt{9}} = \boxed{-\dfrac{1}{6}}$

(j) $\lim_{x\to 1} \dfrac{G(x+2)+x-8}{G(2x)-3x^2-3x+2} =$ where $G(x) = (x-1)^2+3$

$\lim_{x\to 1} \dfrac{G(x+2)+x-8}{G(2x)-3x^2-3x+2} = \lim_{x\to 1} \dfrac{[((x+2)-1)^2+3]+x-8}{[(2x-1)^2+3]-3x^2-3x+2}$

$= \lim_{x\to 1} \dfrac{[(x+1)^2+3]+x-8}{[(2x-1)^2+3]-3x^2-3x+2} = \lim_{x\to 1} \dfrac{(x+1)^2+3)+x-8}{[(2x-1)^2+3]-3x^2-3x+2}$

$= \lim_{x\to 1} \dfrac{x^2+2x+1+3+x-8}{4x^2-4x+1+3-3x^2-3x+2} = \lim_{x\to 1} \dfrac{x^2+3x-4}{x^2-7x+6}^{\left(\frac{0}{0}\right)} = \lim_{x\to 1} \dfrac{(x+4)(x-1)}{(x-6)(x-1)}$

$= \lim_{x\to 1} \dfrac{x+4}{x-6} \overset{\text{DSP}}{=} \dfrac{5}{-5} = \boxed{-1}$

(k) $\lim\limits_{x\to 7} \dfrac{x-7}{|7-x|} =$ $\boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim\limits_{x\to 7^+} \dfrac{x-7}{-(7-x)} = \lim\limits_{x\to 7^+} \dfrac{x-7}{x-7} = 1$

LHL: $\lim\limits_{x\to 7^-} \dfrac{x-7}{|7-x|} = \lim\limits_{x\to 7^-} \dfrac{x-7}{7-x} = \lim\limits_{x\to 7^-} \dfrac{x-7}{-(x-7)} = -1$

$$\text{Recall } |7-x| = \begin{cases} 7-x & \text{if } 7-x \geq 0 \\ \\ -(7-x) & \text{if } 7-x < 0 \end{cases} = \begin{cases} 7-x & \text{if } x \leq 7 & \longleftarrow \text{LHL} \\ \\ -(7-x) & \text{if } x > 7 & \longleftarrow \text{RHL} \end{cases}$$

WARNING: The $|7-x|$ does not just cancel with the $x-7$. You must examine the two cases for the absolute value. Plus, be careful with the signs. We have $7-x$ here and not $x-7$.

(l) $\lim\limits_{x\to 5} \dfrac{f(x^2)-28}{(f(x))^2-10x-14} =$ where $f(x) = x+3$

$$\lim_{x\to 5} \frac{f(x^2)-28}{(f(x))^2-10x-14} = \lim_{x\to 5} \frac{x^2+3-28}{(x+3)^2-10x-14} = \lim_{x\to 5} \frac{x^2-25}{x^2+6x+9-10x-14}$$

$$= \lim_{x\to 5} \frac{x^2-25}{x^2-4x-5} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 5} \frac{(x-5)(x+5)}{(x-5)(x+1)} = \lim_{x\to 5} \frac{x+5}{x+1} \overset{\text{DSP}}{=} \frac{10}{6} = \boxed{\frac{5}{3}}$$

2. Prove that $\lim\limits_{x\to 3} 1-2x = -5$ using the $\varepsilon - \delta$ definition of the limit.

Scratchwork: we want $|f(x)-L| = |(1-2x)-(-5)| < \varepsilon$

$$|f(x)-L| = |1-2x+5| = |6-2x| = |-2(x-3)| = |-2||x-3| = 2|x-3| \text{ (want } < \varepsilon)$$

$$2|x-3| < \varepsilon \text{ means } |x-3| < \frac{\varepsilon}{2}$$

$$\text{So choose } \delta = \frac{\varepsilon}{2} \text{ to restrict } 0 < |x-3| < \delta. \text{ That is } 0 < |x-3| < \frac{\varepsilon}{2}.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \dfrac{\varepsilon}{2}$. Given x such that $0 < |x-3| < \delta$, then

$$|f(x)-L| = |(1-2x)-(-5)| = |6-2x| = |-2(x-3)| = |-2||x-3| = 2|x-3| < 2\cdot\frac{\varepsilon}{2} = \varepsilon.$$

$\square$

3. (a) Suppose that $f(x) = \sqrt{x^2 - 5x + 3}$. Compute $f'(x)$ using the **limit definition of the derivative**.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{\sqrt{(x+h)^2 - 5(x+h) + 3} - \sqrt{x^2 - 5x + 3}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)^2 - 5(x+h) + 3} - \sqrt{x^2 - 5x + 3}}{h}$$

$$\cdot \left(\frac{\sqrt{(x+h)^2 - 5(x+h) + 3} + \sqrt{x^2 - 5x + 3}}{\sqrt{(x+h)^2 - 5(x+h) + 3} + \sqrt{x^2 - 5x + 3}} \right)$$

$$= \lim_{h \to 0} \frac{[(x+h)^2 - 5(x+h) + 3] - [x^2 - 5x + 3]}{h(\sqrt{(x+h)^2 - 5(x+h) + 3} + \sqrt{x^2 - 5x + 3})}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 5x - 5h + 3 - x^2 + 5x - 3}{h(\sqrt{(x+h)^2 - 5(x+h) + 3} + \sqrt{x^2 - 5x + 3})}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 5h}{h(\sqrt{(x+h)^2 - 5(x+h) + 3} + \sqrt{x^2 - 5x + 3})}$$

$$= \lim_{h \to 0} \frac{h(2x + h - 5)}{h(\sqrt{(x+h)^2 - 5(x+h) + 3} + \sqrt{x^2 - 5x + 3})}$$

$$= \lim_{h \to 0} \frac{2x + h - 5}{\sqrt{(x+h)^2 - 5(x+h) + 3} + \sqrt{x^2 - 5x + 3}} = \boxed{\frac{2x - 5}{2\sqrt{x^2 - 5x + 3}}}$$

(b) Suppose that $f(x) = \dfrac{2 - 5x^2}{7 - 3x}$. Compute $f'(x)$ using the **limit definition of the derivative**.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{2 - 5(x+h)^2}{7 - 3(x+h)} - \dfrac{2 - 5x^2}{7 - 3x}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{2 - 5x^2 - 10xh - 5h^2}{7 - 3(x+h)} - \dfrac{2 - 5x^2}{7 - 3x}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{(2 - 5x^2 - 10xh - 5h^2)(7 - 3x) - (2 - 5x^2)(7 - 3x - 3h)}{(7 - 3(x+h))(7 - 3x)}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{14 - 35x^2 - 70xh - 35h^2 - 6x + 15x^3 + 30x^2h + 15xh^2 - (14 - 6x - 6h - 35x^2 + 15x^3 + 15x^2h)}{(7 - 3(x+h))(7 - 3x)}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{-70xh - 35h^2 + 15x^2h + 15xh^2 + 6h}{(7 - 3(x+h))(7 - 3x)}}{h} = \lim_{h \to 0} \frac{h(-70x - 35h + 15x^2 + 15xh + 6)}{(7 - 3(x+h))(7 - 3x)} \left(\frac{1}{h} \right)$$

$$= \lim_{h \to 0} \frac{-70x - 35h + 15x^2 + 15xh + 6}{(7 - 3(x+h))(7 - 3x)} = \boxed{\frac{15x^2 - 70x + 6}{(7 - 3x)^2}}$$

4. Suppose that $f(x) = 5 - 7x + 4x^2 - x^3$.

(a) Write the **equation of the tangent line** to the curve $y = f(x)$ when $x = 1$. **Use the limit definition of the derivative when computing the derivative.**

First option is to compute $f'(x)$. Then evaluate at $x = 1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(5 - 7(x+h) + 4(x+h)^2 - (x+h)^3) - (5 - 7x + 4x^2 - x^3)}{h}$$

$$= \lim_{h\to 0} \frac{[5 - 7(x+h) + 4(x^2 + 2xh + h^2) - (x^3 + 3x^2h + 3xh^2 + h^3)] - (5 - 7x + 4x^2 - x^3)}{h}$$

$$= \lim_{h\to 0} \frac{5 - 7x - 7h + 4x^2 + 8xh + 4h^2 - x^3 - 3x^2h - 3xh^2 - h^3 - 5 + 7x - 4x^2 + x^3}{h}$$

$$= \lim_{h\to 0} \frac{-7h + 8xh + 4h^2 - 3x^2h - 3xh^2 - h^3}{h}$$

$$= \lim_{h\to 0} \frac{h(-7 + 8x + 4h - 3x^2 - 3xh - h^2)}{h}$$

$$= \lim_{h\to 0} -7 + 8x + 4h - 3x^2 - 3xh - h^2 = -7 + 8x - 3x^2$$

Finally, $f'(1) = -7 + 8 - 3 = -2$. This is the slope of the tangent line at the point where $x = 1$.

OR compute directly using the specific slope formula:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \ldots$$

The point is given by $(1, f(1)) = (1, 1)$.	The slope is given by $f'(1) = -2$.

Using point-slope form we have

$y - 1 = -2(x - 1)$ or $y =$ $\boxed{y = -2x + 3}$

(b) Find the x-coordinate(s), if any, where the tangent line to $f(x)$ is horizontal.

The tangent line is horizontal when it has slope $f'(x) = 0$.

Here $f'(x) = -7 + 8x - 3x^2 \overset{\text{set}}{=} 0$.

Solving $3x^2 - 8x + 7 = 0$ we have $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{8 \pm \sqrt{64 - 84}}{6}$

So there is no real solution. Therefore, $\boxed{\text{there is no } x\text{-coordinate where the tangent line is horizontal.}}$

5. State the definition for a function $f(x)$ that is continuous at $x = -7$.

$g(x)$ is continuous at $x = -7$ means by definition that $\boxed{\lim_{x \to -7} g(x) = g(-7)}$

6. Suppose that f and g are functions, **and**

- $\lim_{x \to 7} g(x) = 3$
- $\lim_{x \to 2} g(x) = 6$
- $f(3) = 2$
- $g(x)$ is continuous at $x = 7$ and $x = 2$.
- $\lim_{x \to 3} f(x) = 5$.

Evaluate the following quantities and fully **justify** your answers. Do **not** just put down numbers.

(a) $g(7) = \lim_{x \to 7} g(x) = \boxed{3}$ The first equality holds because of the assumption of g being continuous at $x = 7$. The second equality was given in the assumptions.

(b) Compute $g \circ f(3) = g(f(3)) = g(2) = \lim_{x \to 2} g(x) = \boxed{6}$ The second equality holds because $f(3) = 2$ was given in the assumptions. The third equality holds because of the assumption that of g being continuous at $x = 2$. The last equality was given in the assumptions.

(c) Compute $f \circ g(7) = f(g(7)) = f(3) = 2$ The first equality holds from part (a). The last equality was given in the assumptions.

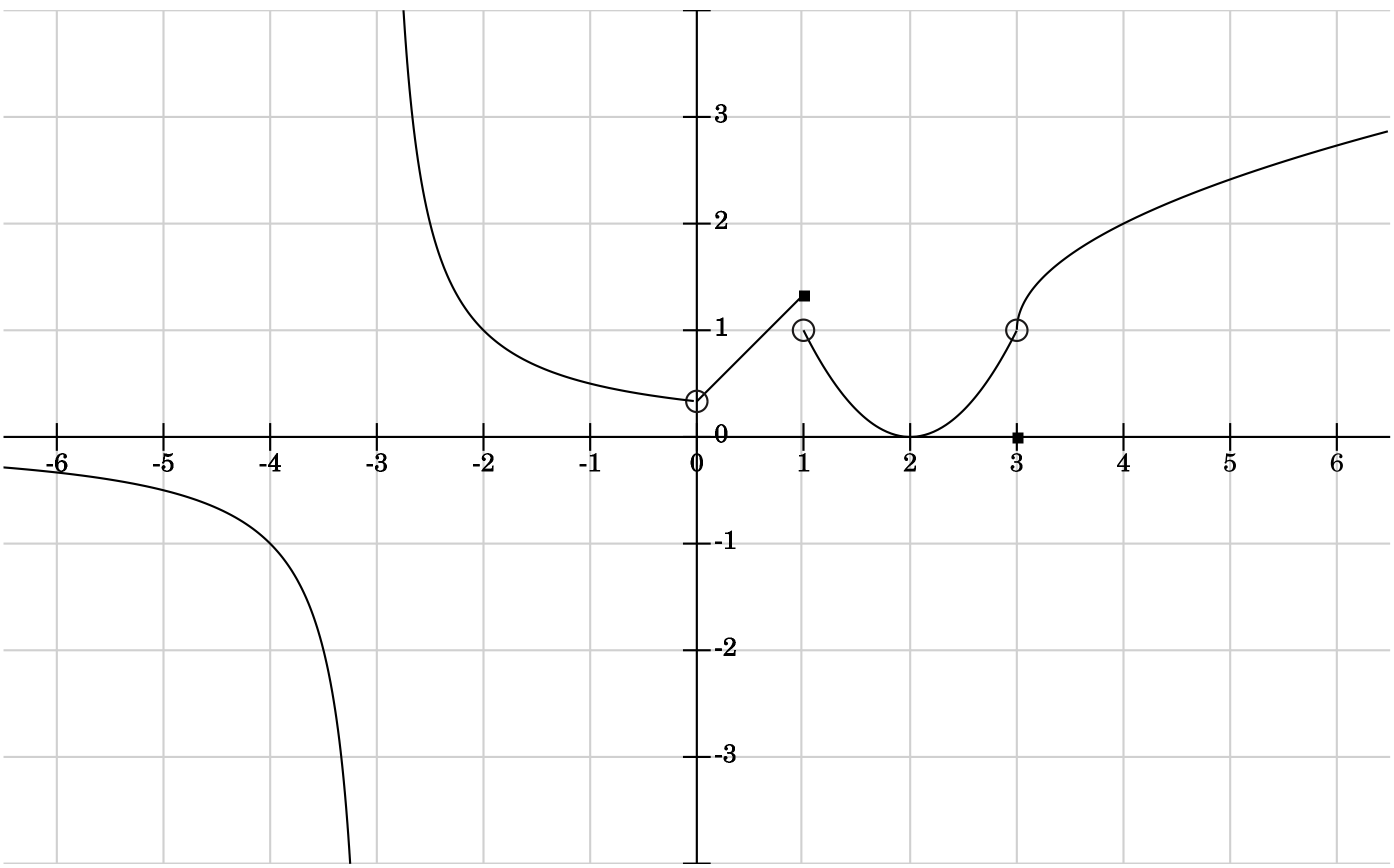
(d) Is $f(x)$ continuous at $x = 3$? Why or why not? Use math notation.

$\boxed{\text{No}}$ $f(x)$ is not continuous at $x = 3$ because $f(3) = \boxed{2} \neq \boxed{5} = \lim_{x \to 3} f(x)$

7. Consider the function defined by

$$f(x) = \begin{cases} \sqrt{x-3}+1 & \text{if } x > 3 \\ 0 & \text{if } x = 3 \\ (x-2)^2 & \text{if } 1 < x < 3 \\ x + \dfrac{1}{3} & \text{if } 0 < x \leq 1 \\ \dfrac{1}{x+3} & \text{if } x < 0 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$.



(b) State the **Domain** of the function $f(x)$. Domain= $\{x | x \neq 0, -3\}$

(c) Compute $\lim_{x \to -3} f(x) =$ DNE b/c RHL$\neq$ LHL

RHL: $\lim_{x \to -3^+} f(x) = +\infty$

LHL: $\lim_{x \to -3^-} f(x) = -\infty$

(d) Compute $\lim_{x\to 0} f(x) = \boxed{\frac{1}{3}}$ RHL=LHL

RHL: $\lim_{x\to 0^+} f(x) = \frac{1}{3}$

LHL: $\lim_{x\to 0^-} f(x) = \frac{1}{3}$

(e) Compute $\lim_{x\to 1} f(x) = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 1^+} f(x) = 1$

LHL: $\lim_{x\to 1^-} f(x) = \frac{4}{3}$

(f) Compute $\lim_{x\to 3} f(x) = \boxed{1}$ RHL=LHL

RHL: $\lim_{x\to 3^+} f(x) = 1$

LHL: $\lim_{x\to 3^-} f(x) = 1$

(g) State all the value(s) at which f is discontinuous. Justify your answer(s) using the definition of continuity.

f is discontinuous at $x = -3, 0, 1, 3$. State the reasons WHY.

- $\boxed{f \text{ is discontinuous at } x = -3 \text{ because } \lim_{x\to -3} f(x) \text{ DNE}}$

- $\boxed{f \text{ is discontinuous at } x = 0 \text{ because } f(0) \text{ is undefined}}$

- $\boxed{f \text{ is discontinuous at } x = 1 \text{ because } \lim_{x\to 1} f(x) \text{ DNE}}$

- $\boxed{f \text{ is discontinuous at } x = 3 \text{ because } \lim_{x\to 3} f(x) \neq f(3)}$ **despite** the fact that $\lim_{x\to 3} f(x)$ exists AND $f(3) = 0$ is defined, they **are <u>NOT</u> equal**.

Recall: a function f is continuous at a number $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

That is, **three** things need to hold:

(1) $\lim_{x\to a} f(x)$ exists.

(2) $f(a)$ is defined.

(3) $\lim_{x\to a} f(x) = f(a)$ meaning (1) and (2) above are equal.