Math 111, Fall 2014

Worksheet 2, ANSWER KEY, Tuesday, September 16, 2014

1. Compute the following limits. Justify your answers. Be clear if they equal a value, or $+\infty$, $-\infty$, or DNE.

(a) $\lim_{x\to 2} \frac{x^2+6x+8}{x+2} \overset{\text{DSP}}{=} \frac{4+12+8}{4} = \frac{24}{4} = \boxed{6}$

(b) $\lim_{x\to 2} \frac{x^2+6x+8}{x-2}^{\left(\frac{24}{0}\right)} = \boxed{\text{DNE b/c RHL} \neq LHL}$

RHL: $\lim_{x\to 2^+} \frac{x^2+6x+8}{x-2} = \frac{24}{0^+} = +\infty$

LHL: $\lim_{x\to 2^-} \frac{x^2+6x+8}{x-2} = \frac{24}{0^-} = -\infty$

(c) $\lim_{x\to 2} \frac{x^2-6x+8}{x-2}^{\left(\frac{0}{0}\right)} = \lim_{x\to 2} \frac{(x-2)(x-4)}{x-2} = \lim_{x\to 2} x-4 \overset{\text{DSP}}{=} 2-4 = \boxed{-2}$

(d) $\lim_{x\to 2} \frac{x^2+5x-14}{x^2-4x+12} \overset{\text{DSP}}{=} \frac{4+10-14}{4-8+12} = \frac{0}{8} = \boxed{0}$ TRUST THIS ANSWER.

Note: 0 in the numerator is fine (as long as the denominator is non-zero).

(e) $\lim_{x\to 2} \frac{x^2+5x-14}{x^2-8x+12}^{\left(\frac{0}{0}\right)} = \lim_{x\to 2} \frac{(x+7)(x-2)}{(x-2)(x-6)} = \lim_{x\to 2} \frac{x+7}{x-6} \overset{\text{DSP}}{=} \frac{2+7}{2-6} = \boxed{-\frac{9}{4}}$

(f) $\lim_{x\to -3} \frac{x^2+4x+3}{x^2-2x-15}^{\left(\frac{0}{0}\right)} = \lim_{x\to -3} \frac{(x+1)(x+3)}{(x-5)(x+3)}$

$= \lim_{x\to -3} \frac{x+1}{x-5} \overset{\text{DSP}}{=} \frac{-3+1}{-3-5} = \frac{-2}{-8} = \boxed{\frac{1}{4}}$

(g) $\lim_{x\to -3} \frac{x^2+4x+3}{x^2+6x+9}^{\left(\frac{0}{0}\right)} = \lim_{x\to -3} \frac{(x+1)(x+3)}{(x+3)(x+3)}$

$= \lim_{x\to -3} \frac{x+1}{x+3}^{\left(\frac{-2}{0}\right)} = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

(Note: this is a combo problem$\rightarrow$ factoring, cancel, finish with sign analysis)

RHL: $\lim_{x\to -3^+} \frac{x+1}{x+3} = \lim_{x\to -3^+} \frac{-2}{0^+} = -\infty$

LHL: $\lim_{x\to -3^-} \frac{x+1}{x+3} = \lim_{x\to -3^-} \frac{-2}{0^-} = +\infty$

(h) $\lim_{t\to 1} \dfrac{t-1}{g(t^2)-3}$ where $g(t) = 2t+1$.

$$\lim_{t\to 1} \frac{t-1}{g(t^2)-3} = \lim_{t\to 1} \frac{t-1}{(2t^2+1)-3} = \lim_{t\to 1} \frac{t-1}{2t^2-2}^{\left(\frac{0}{0}\right)}$$

$$= \lim_{t\to 1} \frac{t-1}{2(t^2-1)} = \lim_{t\to 1} \frac{t-1}{2(t-1)(t+1)} = \lim_{t\to 1} \frac{1}{2(t+1)} \overset{\text{DSP}}{=} \lim_{t\to 1} \frac{1}{2(1+1)} = \boxed{\frac{1}{4}}$$

(i) $\lim_{x\to 0} \dfrac{x+1}{x(x+2)}^{\left(\frac{1}{0}\right)} = \text{DNE b/c RHL} \neq \text{LHL}$

RHL: $\lim_{x\to 0^+} \dfrac{x+1}{x(x+2)} = \lim_{x\to 0^+} \dfrac{1}{(0^+)(2)} = +\infty$

LHL: $\lim_{x\to 0^-} \dfrac{x+1}{x(x+2)} = \lim_{x\to 0^-} \dfrac{1}{(0^-)(2)} = -\infty$

(j) $\lim_{x\to 3} \dfrac{x^2-2x-3}{x^2-6x+9}^{\left(\frac{0}{0}\right)} = \lim_{x\to 3} \dfrac{(x-3)(x+1)}{(x-3)(x-3)} = \lim_{x\to 3} \dfrac{x+1}{x-3}^{\left(\frac{4}{0}\right)} = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

(Note: this is again a combo problem$\to$ factoring, cancel, finish with sign analysis)

RHL: $\lim_{x\to 3^+} \dfrac{x+1}{x-3} = \lim_{x\to 3^+} \dfrac{4}{0^+} = +\infty$

LHL: $\lim_{x\to 3^-} \dfrac{x+1}{x-3} = \lim_{x\to 3^-} \dfrac{4}{0^-} = -\infty$

(k) $$\lim_{x\to -5} \frac{\frac{1}{4-x} - \frac{1}{9}}{x+5}^{\left(\frac{0}{0}\right)} = \lim_{x\to -5} \frac{\frac{9}{9(4-x)} - \frac{(4-x)}{9(4-x)}}{x+5} = \lim_{x\to -5} \frac{\frac{9-(4-x)}{9(4-x)}}{x+5}$$

$$= \lim_{x\to -5} \frac{\frac{9-4+x}{9(4-x)}}{x+5} = \lim_{x\to -5} \frac{\frac{5+x}{9(4-x)}}{x+5} = \lim_{x\to -5} \frac{5+x}{9(4-x)} \frac{1}{x+5}$$

$$= \lim_{x\to -5} \frac{1}{9(4-x)} \overset{\text{DSP}}{=} \frac{1}{9(4-(-5))} = \boxed{\frac{1}{81}}$$

(l) $$\lim_{x\to -3} \frac{x^2-4x-21}{\sqrt{1-x}-2}^{\left(\frac{0}{0}\right)} = \lim_{x\to -3} \frac{x^2-4x-21}{\sqrt{1-x}-2} \cdot \left(\frac{\sqrt{1-x}+2}{\sqrt{1-x}+2}\right)$$

$$= \lim_{x\to -3} \frac{(x^2-4x-21)(\sqrt{1-x}+2)}{1-x-4} = \lim_{x\to -3} \frac{(x+3)(x-7)(\sqrt{1-x}+2)}{-x-3}$$

$$= \lim_{x\to -3} \frac{(x+3)(x-7)(\sqrt{1-x}+2)}{-(x+3)} = \lim_{x\to -3} \frac{(x-7)(\sqrt{1-x}+2)}{-1}$$

$$\overset{\text{L.L.}}{=} -(-3-7)(\sqrt{1-(-3)}+2) = -(-10)(\sqrt{4}+2) = (10)(2+2) = \boxed{40}$$

(m) Let $g(x) = \sqrt{x}$. Compute $\displaystyle\lim_{s\to 1} \frac{g(s^2+8)-3}{s-1}$

$$\lim_{s\to 1} \frac{g(s^2+8)-3}{s-1}^{\left(\frac{0}{0}\right)} = \lim_{s\to 1} \frac{\sqrt{s^2+8}-3}{s-1} \cdot \left(\frac{\sqrt{s^2+8}+3}{\sqrt{s^2+8}+3}\right)$$

$$= \lim_{s\to 1} \frac{s^2+8-9}{(s-1)(\sqrt{s^2+8}+3)} = \lim_{s\to 1} \frac{s^2-1}{(s-1)(\sqrt{s^2+8}+3)}$$

$$= \lim_{s\to 1} \frac{(s-1)(s+1)}{(s-1)(\sqrt{s^2+8}+3)} = \lim_{s\to 1} \frac{s+1}{\sqrt{s^2+8}+3}$$

$$\overset{\text{L.L.}}{=} \frac{1+1}{\sqrt{9}+3} = \frac{2}{3+3} = \boxed{\frac{1}{3}}$$

(n) Let $f(x) = \dfrac{1}{x}$. Compute $\displaystyle\lim_{t\to 2} \frac{f(t-1)-2f(t)}{t^2-4}$.

$$\lim_{t\to 2} \frac{f(t-1)-2f(t)}{t^2-4}^{\left(\frac{0}{0}\right)} = \lim_{t\to 2} \frac{\frac{1}{t-1} - \frac{2}{t}}{t^2-4} = \lim_{t\to 2} \frac{\frac{t}{t(t-1)} - \frac{2(t-1)}{t(t-1)}}{t^2-4}$$

$$= \lim_{t\to 2} \frac{\frac{t-2(t-1)}{t(t-1)}}{t^2-4} = \lim_{t\to 2} \frac{\frac{t-2t+2}{t(t-1)}}{t^2-4} = \lim_{t\to 2} \frac{\left(\frac{-t+2}{t(t-1)}\right)}{t^2-4} \quad \text{(watch algebra here)}$$

$$= \lim_{t\to 2} \frac{-(t-2)}{t(t-1)} \cdot \frac{1}{t^2-4} = \lim_{t\to 2} \frac{-(t-2)}{t(t-1)} \cdot \frac{1}{(t-2)(t+2)} = \lim_{t\to 2} \frac{-1}{t(t-1)} \cdot \left(\frac{1}{t+2}\right)$$

$$= \lim_{t\to 2} \frac{-1}{t(t-1)(t+2)} \overset{\text{DSP}}{=} \frac{-1}{2(2-1)(2+2)} = \boxed{-\frac{1}{8}}$$

(o) $\displaystyle\lim_{x\to 4} \frac{|x-4|}{x-4}$ $\boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

$$\text{Recall } |x-4| = \begin{cases} x-4 & \text{if } x-4 \geq 0 \\ -(x-4) & \text{if } x-4 < 0 \end{cases} = \begin{cases} x-4 & \text{if } x \geq 4 \\ -(x-4) & \text{if } x < 4 \end{cases}$$

RHL: $\displaystyle\lim_{x\to 4^+} \frac{|x-4|}{x-4} = \lim_{x\to 4^+} \frac{x-4}{x-4} = 1$

LHL: $\displaystyle\lim_{x\to 4^-} \frac{|x-4|}{x-4} = \lim_{x\to 4^-} \frac{-(x-4)}{x-4} = -1$

WARNING: The $|x-4|$ does not just cancel with the $x-4$. You must examine the two cases for the absolute value.

(p) $\displaystyle\lim_{x\to -1} \frac{1}{|x+1|}$ $\boxed{+\infty \text{ b/c RHL} = \text{LHL}}$

Recall $|x+1| = \begin{cases} x+1 & \text{if } x+1 \geq 0 \\ -(x+1) & \text{if } x+1 < 0 \end{cases} = \begin{cases} x+1 & \text{if } x \geq -1 \\ -(x+1) & \text{if } x < -1 \end{cases}$

RHL: $\displaystyle\lim_{x\to -1^+} \frac{1}{|x+1|} = \lim_{x\to -1^+} \frac{1}{x+1} = \frac{1}{0^+} = +\infty$

LHL: $\displaystyle\lim_{x\to -1^-} \frac{1}{|x+1|} = \lim_{x\to -1^-} \frac{1}{-(x+1)} = \frac{1}{0^+} = +\infty$

2. Write out the rigorous $\varepsilon - \delta$ **Definition of the Limit** $\lim_{x\to a} f(x) = L$. (Use your notes if you need to. You must learn this statement.)

> For every $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that
>
> if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

3. See practice example.

4. Follow the example above to give an $\varepsilon - \delta$ proof that $\lim_{x\to 1} 5x - 3 = 2$.

Scratchwork: we want $|f(x) - L| = |(5x-3) - 2| < \varepsilon$; what restrictions on $|x-1|$ make that possible?

$$|f(x) - L| = |(5x-3) - 2| = |5x-5| = |5(x-1)| = |5||x-1| = 5|x-1| \text{ (want } < \varepsilon)$$

$$5|x-1| < \varepsilon \text{ means } |x-1| < \frac{\varepsilon}{5}, \qquad \text{so choose } \delta = \frac{\varepsilon}{5}.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{5}$. Given x such that $0 < |x-1| < \delta$, then

$$|f(x) - L| = |(5x-3) - 2| = |5x-5| = |5(x-1)| = |5||x-1| = 5|x-1| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon.$$

$\square$