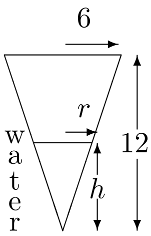
Extra Example for Related Rates

Professor D. Benedetto, Math 111

1. A conical reservoir, 12 ft. deep and also 12 ft. across the top is being filled with water at the rate of 5 cubic feet per minute. How fast is the water rising when it is 4 feet deep?

The cross section (with water level drawn in) looks like:

• Diagram



• Variables
Let r = radius of the water level at time t
Let h = height of the water level at time t
Let V = volume of the water in the tank at time t
Find $\frac{dh}{dt}$ =? when $h = 4$ feet
and $\frac{dV}{dt} = 5\frac{\text{ft}^3}{\text{sec}}$

• Equation relating the variables:

Volume= $V = \frac{1}{3}\pi r^2 h$

• Extra solvable information: Note that r is not mentioned in the problem's info. But there is a relationship, via similar triangles, between r and h. We must have

$$\frac{r}{6} = \frac{h}{12} \implies r = \frac{h}{2}$$

After substituting into our previous equation, we get:

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right) \implies \frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \cdot \frac{dh}{dt} \implies \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$5 = \frac{1}{4}\pi(4)^2\frac{dh}{dt}$

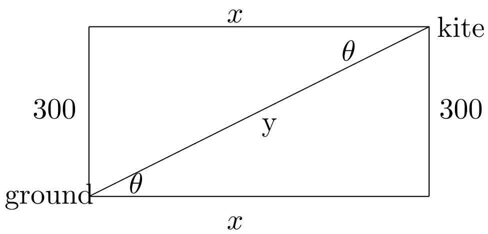
• Solve for the desired quantity:

$\frac{dh}{dt} = \frac{5 \cdot 4}{16\pi} = \frac{5}{4\pi}\frac{\text{ft}}{\text{sec}}$

• Answer the question: The water is rising at a rate of $\frac{5}{4\pi}$ feet every second at that moment.

2. A kite starts flying 300 feet directly above the ground. The kite is being blown horizontally at 10 feet per second. When the kite has blown horizontally for 40 seconds, how fast is the angle between the string and the ground changing?

• Diagram



The picture at arbitrary time t is:

• Variables

Let $x =$ distance kite has travelled horizontally at time t

Let $y =$ length of string from ground to kite at time t

Let $\theta =$ the angle between the string/horizontal

Given $\frac{dx}{dt} = 10\frac{\text{ft}}{\text{sec}}$,

find $\frac{d\theta}{dt} =$? when $x = 10\frac{\text{ft}}{\text{sec}} \cdot (40)\text{sec} = 400$ feet

• Equation relating the variables:

The trigonometry of the triangle yields $\tan\theta = \frac{300}{x}$.

Note, you can also use $\cot\theta = \frac{x}{300}$

• Differentiate both sides w.r.t. time t.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{300}{x}\right) \implies \sec^2\theta\frac{d\theta}{dt} = -\frac{300}{x^2}\frac{dx}{dt}. \text{ (Related Rates!)}$$

• Substitute Key Moment Information (now and not before now!!!):

At the key instant when $x = 400$, we have $y = 500$ by the Pythagorean Theorem. No angle was given so we read the trigonometry off the given triangle. Therefore, $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{500}{400} = \frac{5}{4}$ and

$$\left(\frac{5}{4}\right)^2\frac{d\theta}{dt} = -\frac{300}{(400)^2} \cdot (10).$$

• Solve for the desired quantity:

$$\frac{d\theta}{dt} = \frac{-300(10)(4^2)}{(400)^2(5)^2} = -\frac{3}{250}\frac{\text{rad}}{\text{sec}}$$

• Answer: The angle is decreasing at a rate of $\boxed{\frac{3}{250}}$ radians every second at that moment.

3. A swimming pool is 40 feet wide, 100 feet long, has a slanted base that, when the pool is full, starts at water level and slopes down evenly to 12 feet at the opposite end, 100 feet away. The pool is being filled. When the water is 3 feet from the top, water is being pumped in at 60 ft^3 per second. How fast is the water level rising at that moment?

- Diagram

- Variables

Let x = length of the water level at time t

Let y = height of the water level at the deep end at time t

Let V = volume of the water in the pool at time t

Find $\frac{dy}{dt}$ =? when $y = 9$ feet (3 feet from the top)

and $\frac{dV}{dt} = 60\frac{\text{ft}^3}{\text{sec}}$

- Equation relating the variables:

Volume=Base $\cdot$ Height. We can use the side view of the pool for the base. First find the area of the triangle on the side and then multiple by the width of the pool (which is 40 feet here) to get the volume.

$V = \frac{1}{2}xy \cdot 40 = 20xy$

- Extra solvable information: Note that x is not mentioned in the problem's info. But there is a relationship, via similar triangles, between x and y. We must have

$$\frac{x}{100} = \frac{y}{12} \implies x = \frac{100y}{12} = \frac{25y}{3}$$

After substituting into our previous equation, we get:

$V = 20xy = 20\left(\frac{25y}{3}\right)y = \frac{500}{3}y^2$

- Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{500}{3}y^2\right) \implies \frac{dV}{dt} = \frac{500}{3} \cdot 2y \cdot \frac{dy}{dt} \implies \frac{dV}{dt} = \frac{1000}{3}y\frac{dy}{dt}$ (Related Rates!)

- Substitute Key Moment Information (now and not before now!!!):

$60 = \frac{1000}{3}(9)\frac{dy}{dt}$

- Solve for the desired quantity:

$\frac{dy}{dt} = \frac{60}{3000} = \frac{1}{50}\frac{\text{ft}}{\text{sec}}$

- Answer the question: The water is rising at a rate of $\frac{1}{50}$ feet every second at that moment.