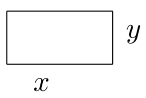
Answer Key for the Review Packet for Exam #3

Professor Danielle Benedetto – Math 11

**Max-Min Problems**

1. Show that of all rectangles with a given area, the one with the smallest perimeter is a square.
   - Diagram:



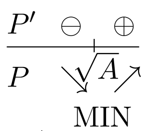

   - Variables:
     Let $x$ =length of the rectangle.
     Let $y$ =width of the rectangle.
     Let $A$ =area of rectangle.
     Let $P$ =perimeter of rectangle.
   - Equations:

     We know $A = xy$ is fixed, so that $y = \dfrac{A}{x}$.

     Then the perimeter $P = 2x + 2y = 2x + 2\dfrac{A}{x}$ must be minimized. The (common-sense-bounds)domain of $P$ is $\{x : x > 0\}$.
   - Minimize:

     Next $P' = 2 - \dfrac{2A}{x^2}$. Setting $P' = 0$ we solve for $x = +\sqrt{A}$. (We take the positive square root here since we are talking about positive lengths).

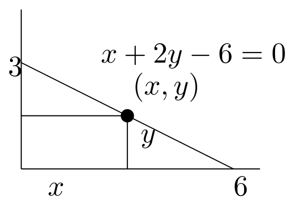
     Sign-testing the critical number does indeed yield a minimum for the perimeter function.




   - Answer:

     Since $x = \sqrt{A}$ then $y = \dfrac{A}{\sqrt{A}} = \sqrt{A}$. As a result, $x = y$ and the smallest perimeter occurs when the rectangle is a $\boxed{\text{square}}$.

2. A rectangle lies in the first quadrant, with one vertex at the origin, two sides along the coordinate axes, and the fourth vertex on the line $x + 2y - 6 = 0$. Find the maximum area of the rectangle.
   - Diagram:



- Variables:
  Let $x = x$-coordinate of point on the line.
  Let $y = y$-coordinate of point on the line.
  Let $A =$area of inscribed rectangle.
- Equations:

The vertex point $(x, y)$ lies on the line $x + 2y - 6 = 0$. The fixed line $x + 2y - 6 = 0$ can be rewritten as $y = 3 - \frac{x}{2}$.

The area of the rectangle is given as $A = xy = x\left(3 - \frac{x}{2}\right) = 3x - \frac{x^2}{2}$ and must be maximized.

The (common-sense-bounds)domain of $A$ is $\{x : 0 \le x \le 6\}$.

- Maximize:

Next, $A' = 3 - x$. Setting $A' = 0$ we solve for $x = 3$ as the critical number.

Sign-testing the critical number does indeed yield a maximum for the area function.

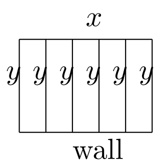
| | | | |
|---|---|---|---|
| $A'$ | $\oplus$ | | $\ominus$ |
| $A$ | $\nearrow$ | 3 | $\searrow$ |
| | | MAX | |

- Answer:

Finally, $x = 3 \Longrightarrow y = \frac{3}{2}$. As a result the maximum area is Area$=xy = 3 \cdot \frac{3}{2} = \boxed{\frac{9}{2}}$ square units.

3. A farmer wants to use a fence to surround a rectangular field, using an existing stone wall as one side of the plot. She also wants to divide the field into 5 equal pieces using fence parallel to the sides that are perpendicular to the stone wall (see diagram). The farmer must use exactly 1200 feet of fence. What is the maximum area possible for this field?

- Diagram:




- Variables:
  Let $x =$length of side parallel to wall.
  Let $y =$width of side(s) perpendicular to wall.
  Let $L =$length of fence.
  Let $A =$area of enclosed field.
- Equations:

We know that the length of fence used $L = x + 6y = 1200$ is fixed so that $x = 1200 - 6y$.

Then the area $A = xy = (1200 - 6y)y = 1200y - 6y^2$ must be maximized.

The (common-sense-bounds)domain of $A$ is $\{y : 0 \le y \le 200\}$.

- Maximize:

Next $A' = 1200 - 12y$. Setting $A' = 0$ we solve for $y = 100$.

Sign-testing the critical number does indeed yield a maximum for the area function.

$A'$ $\oplus$ | $\ominus$

$A$ $\nearrow$ 100 $\searrow$

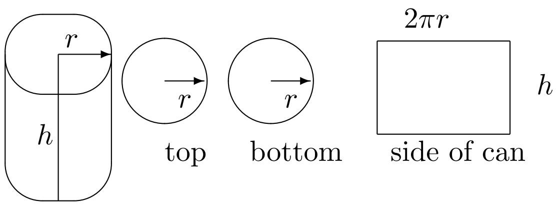
MAX

- Answer:

Since $y = 100$ then $x = 1200 - 6(100) = 600$. As a result, the maximum area possible is $\boxed{60,000}$ square feet.

4. You work for a soup manufacturing company. Your assignment is to design the newest can in the shape of a cylinder. You are given a fixed amount of material, 600 cm$^2$, to make your can. What are the dimensions of your can which will hold the maximum volume of soup?

- Diagram:




- Variables:

Let $r$ =radius of can.
Let $h$ =height of can.
Let $M$ =amount of material (surface area).
Let $V$ =volume of can.

- Equations:

We know that the amount of material used $M = 2\pi r^2 + 2\pi rh = 600$ is fixed so that $h = \dfrac{600 - 2\pi r^2}{2\pi r}$.

Then the volume $V = \pi r^2 h = \pi r^2 \left(\dfrac{600 - 2\pi r^2}{2\pi r}\right) = r\left(\dfrac{600 - 2\pi r^2}{2}\right) = 300r - \pi r^3$ must be maximized.

The (common-sense-bounds)domain of $V$ is $\{r : 0 < r \leq \sqrt{\frac{300}{\pi}}\}$.

- Maximize:

Next $V' = 300 - 3\pi r^2$. Setting $V' = 0$ yields $r = \dfrac{10}{\sqrt{\pi}}$ as the critical number.

Sign-testing the critical number does indeed yield a maximum for the volume function.

$V'$ $\oplus$ | $\ominus$

$V$ $\nearrow$ $\frac{10}{\sqrt{\pi}}$ $\searrow$

MAX

- Answer:

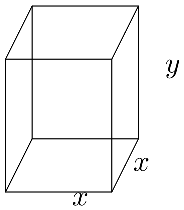
Since $r = \dfrac{10}{\sqrt{\pi}}$ then $h = \dfrac{600 - 2\pi(\frac{10}{\sqrt{\pi}})^2}{2\pi(\frac{10}{\sqrt{\pi}})} = \dfrac{600-200}{20\sqrt{\pi}} = \dfrac{20}{\sqrt{\pi}}$.

The dimensions of the can with the largest volume are $\boxed{r = \dfrac{10}{\sqrt{\pi}}}$ and $\boxed{h = \dfrac{20}{\sqrt{\pi}}}$ in cm.

A bit of extra work does indeed check that the amount of material used is 600 square cm. $M = 2\pi r^2 + 2\pi rh = 2\pi(\frac{10}{\sqrt{\pi}})^2 + 2\pi(\frac{10}{\sqrt{\pi}})(\frac{20}{\sqrt{\pi}}) = 200 + 400 = 600$.

5. A rectangular box with square base cost \$ 2 per square foot for the bottom and \$ 1 per square foot for the top and sides. Find the box of largest volume which can be built for \$36.

• Diagram:




• Variables:
Let $x$ =length of side on base of box.
Let $y$ =height of box.
Let Cost=Cost for amount of material (surface area).
Let $V$ =volume of box.

• Equations:

Then the Cost of materials, which is fixed is given as

$$\begin{aligned} \text{Cost} &= \text{cost of base} + \text{cost of top} + \text{cost of 4 sides} \\ &= x^2(\$2) + x^2(\$1) + 4xy(\$1) \\ &= 3x^2 + 4xy = 36 \\ &\implies y = \frac{36 - 3x^2}{4x} \end{aligned}$$

Then the volume of the box $V = x^2 y = x^2\left(\dfrac{36 - 3x^2}{4x}\right) = 9x - \dfrac{3}{4}x^3$ must be maximized.

The (common-sense-bounds)domain of $V$ is $\{x : 0 < x \leq \sqrt{12}\}$.

• Maximize:

Next $V' = 9 - \dfrac{9}{4}x^2$. Setting $V' = 0$ we solve for $x = 2$ as the critical number.

Sign-testing the critical number does indeed yield a maximum for the volume function.

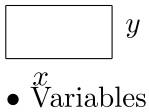
V′ ⊕ ⊖
V ↗ 2 ↘
MAX

• Answer:

Since $x = 2$ then $y = \dfrac{36 - 12}{8} = 3$. As a result, the box of largest volume will measure $\boxed{\text{2x2x3}}$, each in feet.

6. Among all the rectangles with given perimeter P, find the one with the maximum area.

- Diagram:




- Variables:

 Let $x$ =length of the rectangle.
 Let $y$ =width of the rectangle.
 Let $A$ =area of rectangle.
 Let $P$ =perimeter of rectangle.

- Equations:

Let $P$ equal the given perimeter. We know $2x + 2y = P$ is fixed, so that $y = \dfrac{P-2x}{2}$.

Then the area $A = xy = x\left(\dfrac{P-2x}{2}\right) = \dfrac{P}{2}x - x^2$ must be maximized.

The (common-sense-bounds)domain of $A$ is $\{x : 0 \le x \le \frac{P}{2}\}$.

- Maximize:

Next $A' = \dfrac{P}{2} - 2x$. Setting $A' = 0$ we solve for $x = \dfrac{P}{4}$.

Sign-testing the critical number does indeed yield a maximum for the area function.

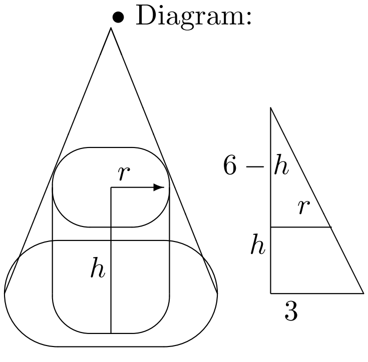
$P'$ ⊕ ⊖
$P$ ↗ $\frac{P}{4}$ ↘
MAX

- Answer:

Since $x = \dfrac{P}{4}$ then $y = \dfrac{P - 2(\frac{P}{4})}{2} = \dfrac{P}{4}$. As a result, $x = y$ and the largest area occurs when the rectangle is a $\boxed{\text{square}}$.

7. Consider a cone such that the height is 6 inches high and its base has diameter 6 in. Inside this cone we inscribe a cylinder whose base lies on the base of the cone and whose top intersects the cone in a circle. What is the maximum volume of the cylinder?

- Diagram:




- Variables:

Let $r$ =radius of cylinder.
Let $h$ =height of the cylinder.
Let $V$ =volume of inscribed cylinder.

- Equations:

Using similar triangles, for the cross slice of the cone and cylinder, we see $\frac{r}{3} = \frac{6-h}{6}$ which implies that $6r = 18 - 3h \implies h = 6 - 2r$.

Then the volume of the cylinder, given by, $V = \pi r^2 h = \pi r^2(6-2r) = 6\pi r^2 - 2\pi r^3$ must be maximized.

The (common-sense-bounds)domain of $V$ is $\{r : 0 \leq r \leq 3\}$.

- Maximize:

Next $V' = 12\pi r - 6\pi r^2$. Setting $V' = 0$ we see $6\pi r(2-r) = 0$ and solve for $r = 0$ or $r = 2$ as critical numbers. Of course $r = 0$ will not lead to a maximum since no cylinder exists there.

Sign-testing the critical number $r = 2$ does indeed yield a maximum for the volume function.

$V'$ $\oplus$ | $\ominus$

$V$ $\nearrow$ 2 $\searrow$

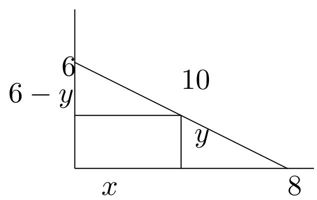
MAX

- Answer:

Since $r = 2$, then $h = 2$, and the maximum volume $V = \pi(2)^2 2 = 8\pi$. As a result the maximum volume of the cylinder is $\boxed{8\pi}$ cubic inches.

8. Consider the right triangle with sides 6, 8 and 10. Inside this triangle, we inscribe a rectangle such that one corner of the rectangle is the right angle of the triangle and the opposite corner of the rectangle lies on the hypotenuse. What is the maximum area of the rectangle?

- Diagram:




- Variables:

Let $x$ = $x$-coordinate of point on the triangle line.
Let $y$ = $y$-coordinate of point on the triangle line.
Let $A$ =area of inscribed rectangle.

- Equations:

Using similar triangles we find the relationship $\frac{6-y}{x} = \frac{6}{8}$ which implies that $48 - 8y = 6x \implies 8y = 48 - 6x \implies y = 6 - \frac{6}{8}x$. Another way to think about this is that this picture corresponds to the line with $y$-intercept 6 with slope $-\frac{6}{8}$ If you draw the triangle with width 6 and height 8 instead, the math all still works out the same in the end.

The area $A = xy = x\left(6 - \frac{6}{8}x\right) = 6x - \frac{6}{8}x^2$ must be maximized.

The (common-sense-bounds)domain of $A$ is $\{x : 0 \le x \le 8\}$.

• Maximize:

Next $A' = 6 - \frac{3}{2}x$. Setting $A' = 0$ we solve for $x = 4$ as the critical number.

Sign-testing the critical number does indeed yield a maximum for the area function.

$A'$ ⊕ ⊖

$A$ ↗ 4 ↘

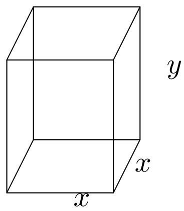
MAX

• Answer:

Finally, $x = 4 \Longrightarrow y = 6 - \frac{6}{8}(4) = 3$ As a result the maximum area is Area=$xy = 4(3) = \boxed{12}$ square units.

9. A toolshed with a square base and a flat roof is to have volume of 800 cubic feet. If the floor costs \$6 per square foot, the roof \$2 per square foot, and the sides \$5 per square foot, determine the dimensions of the most economical shed.

• Diagram:




• Variables:

Let $x$ =length of base of toolshed.
Let $y$ =height of the toolshed.
Let $V$ =volume of toolshed.
Let Cost=cost of amount of material to make toolshed.

• Equations:

We know the volume of the toolshed is given by $V = x^2y = 800$ is fixed, so that $y = \dfrac{800}{x^2}$.

Then the Cost of materials, which must be minimized, is given as

$$\begin{aligned}
\text{Cost} &= \text{cost of floor} + \text{cost of top} + \text{cost of 4 sides} \\
&= x^2(\$6) + x^2(\$2) + 4xy(\$5) \\
&= 8x^2 + 20xy \\
&= 8x^2 + 20x\left(\frac{800}{x^2}\right) \\
&= 8x^2 + \left(\frac{16000}{x}\right)
\end{aligned}$$

The (common-sense-bounds)domain of Cost is $\{x : x > 0\}$.

• Minimize:

Next $\text{Cost}' = 16x - \dfrac{16000}{x^2}$. Setting $\text{Cost}' = 0$ we solve $x^3 = 1,000 \Longrightarrow x = 10$.

Sign-testing the critical number does indeed yield a maximum for the area function.

Cost' ⊖ ⊕

Cost ↘ 10 ↗

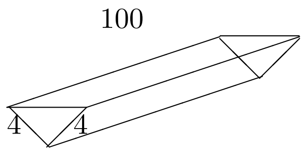
MIN

- Answer:

Since $x = 10$ then $y = \dfrac{800}{(10)^2} = 8$. As a result, the most economical shed has dimensions $\boxed{\text{10x10x8}}$, each in feet.

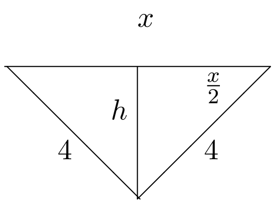
10. A rectangular sheet of metal 8 inches wide and 100 inches long is folded along the center to form a triangular trough. Two extra pieces of metal are attached to the ends of the trough. The trough is filled with water.

- Diagram:




(a). How deep should the trough be to maximize the capacity of the trough?

(b). What is the maximum capacity?




- Variables:

Let $x$ =width of endcap of triangular trough.
Let $y$ =height(depth) of trough.
Let $V$ =volume of trough.
Let $A$ =area of endcap of triangular trough.

- Equations:

We will maximize the volume of the trough, by maximizing the area of the metal pieces on the ends of the trough.

By the Pythagorean Thrm., $h = \sqrt{16 - \left(\frac{x}{2}\right)^2} = \sqrt{16 - \frac{x^2}{4}}$

Then the area, which must be maximized, is given as

$$\begin{aligned} A &= \frac{1}{2}\text{base} \cdot \text{height} \\ &= \frac{1}{2}xh \\ &= \frac{1}{2}x\sqrt{16 - \frac{x^2}{4}} \end{aligned}$$

The (common-sense-bounds)domain of $A$ is $\{x : 0 \leq x \leq 8\}$.

• Maximize:

Next,

$$\begin{aligned} A' &= \frac{1}{2}x\frac{1}{2\sqrt{16-\frac{x^2}{4}}}\left(-\frac{x}{2}\right) + \sqrt{16-\frac{x^2}{4}}\left(\frac{1}{2}\right) \\ &= -\frac{x^2}{8\sqrt{16-\frac{x^2}{4}}} + \left(\frac{1}{2}\right)\sqrt{16-\frac{x^2}{4}} \end{aligned}$$

Setting $A' = 0$ implies

$$\frac{x^2}{8\sqrt{16-\frac{x^2}{4}}} = \left(\frac{1}{2}\right)\sqrt{16-\frac{x^2}{4}}$$

so that

$$x^2 = 4\left(\sqrt{16-\frac{x^2}{4}}\right)^2$$

which implies

$$x^2 = 4\left(16-\frac{x^2}{4}\right) = 64 - x^2.$$

As a result, $x^2 = 32$ and finally $x = \sqrt{32} = 4\sqrt{2}$.

Sign-testing the critical number does indeed yield a maximum for the area function.

| $A'$ | $\oplus$ | $\ominus$ |
|---|---|---|
| $A$ | $\nearrow$ $\sqrt{32}$ | $\searrow$ |

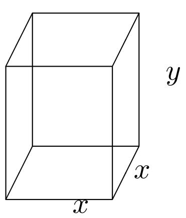
MAX

• Answer:

Since $x = \sqrt{32}$ maximizes the area, then the corresponding height is $h = \sqrt{16 - \frac{(\sqrt{32})^2}{4}} = \sqrt{8} = 2\sqrt{2}$. The trough should be $\boxed{2\sqrt{2}}$ inches deep. The maximum capacity is 800 cubic inches because Max Volume=(Max Area of end panel)(100)= $\frac{1}{2}(4\sqrt{2})(2\sqrt{2})(100) = 50(8)(2) = \boxed{800}$.

11. A manufacturer wishes to produce rectangular containers with square bottoms and tops, each container having a capacity of 250 cubic inches. The material costs \$2 per square inch for the sides. If the material used for the top and bottom costs twice as much per square inch as the material for the sides, what dimensions will minimize the cost?

• Diagram:




• Variables:

Let $x$ =length of base of container.
Let $y$ =height of container.
Let $V$ =volume of container.
Let Cost=cost of producing containers.

- Equations:

We know the volume of the toolshed is given by $V = x^2 y = 250$ is fixed, so that $y = \dfrac{250}{x^2}$.

Then the Cost of materials, which must be minimized, is given as

$$\begin{aligned} \text{Cost} &= \text{cost of floor} + \text{cost of top} + \text{cost of 4 sides} \\ &= x^2(\$4) + x^2(\$4) + 4xy(\$2) \\ &= 8x^2 + 8xy \\ &= 8x^2 + 8x\left(\frac{250}{x^2}\right) \\ &= 8x^2 + \left(\frac{2000}{x}\right) \end{aligned}$$

The (common-sense-bounds)domain of Cost is $\{x : x > 0\}$.

- Minimize:

Next $\text{Cost}' = 16x - \dfrac{2000}{x^2}$. Setting $\text{Cost}' = 0$ we solve $x^3 = \dfrac{2000}{16} = 125 \Longrightarrow x = 5$.

Sign-testing the critical number does indeed yield a minimum for the Cost function.

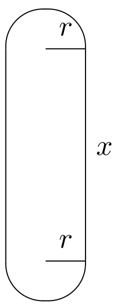
Cost′ ⊖ ⊕
Cost ↘ 5 ↗
MIN

- Answer:

Since $x = 5$ then $y = \dfrac{250}{(5)^2} = 10$. As a result, the dimension that will minimize the cost are $\boxed{\text{5x5x10}}$ in inches.

12. An outdoor track is to be created in the shape shown and is to have perimeter of 440 yards. Find the dimensions for the track that maximize the area of the rectangular portion of the field enclosed by the track.

- Diagram:




- Variables:

Let $r$ =radius of semicircle ends of track.
Let $x$ =length of side of rectangular portion of track.
Let $A$ =area of rectangular portion of track.
Let $P$ =perimeter of track.

• Equations:

We know the perimeter of the track is given by $P = 2x + 2\pi r = 440$ is fixed, so that $x = \dfrac{440 - 2\pi r}{2} = 220 - \pi r$.

Then the Area of the rectangular portion of the field, which must be minimized, is given as

$$\begin{aligned} A &= 2rx \\ &= 2r(220 - \pi r) \\ &= 440r - 2\pi r^2 \end{aligned}$$

The (common-sense-bounds)domain of $A$ is $\{r : 0 \le r \le \frac{220}{\pi}\}$.

• Maximize:

Next, $A' = 440 - 4\pi r$. Setting $A' = 0$, we solve for $r = \dfrac{110}{\pi}$.

Sign-testing the critical number does indeed yield a maximum for the area function.

| $A'$ | $\oplus$ | | $\ominus$ |
|---|---|---|---|
| $A$ | $\nearrow$ | $\frac{110}{\pi}$ | $\searrow$ |
| | | MAX | |

• Answer:

Since $r = \dfrac{110}{\pi}$, then $x = 220 - \pi\left(\frac{110}{\pi}\right) = 110$.

The dimensions that maximize the area of the rectangular portion are $\boxed{x = 110}$ and $\boxed{r = \dfrac{110}{\pi}}$ in yards.

13. Show that the entire region enclosed by the outdoor track in the previous example has maximum area if the track is circular.

• Equations:

From the previous problem, we still have $x = 220 - \pi r$.

Now the entire area enclosed by the track, which must be maximized, is given by

$$\begin{aligned} A &= \pi r^2 + 2rx \\ &= \pi r^2 + 2r(220 - \pi r) \\ &= \pi r^2 + 440r - 2\pi r^2 \\ &= -\pi r^2 + 440r \end{aligned}$$

The (common-sense-bounds)domain of $A$ is $\{r : 0 \le r \le \frac{200}{\pi}\}$.

• Maximize:

Next, $A' = 440 - 2\pi r$. Setting $A' = 0$, we solve for $r = \dfrac{220}{\pi}$.

Sign-testing the critical number does indeed yield a maximum for the area function.

| $A'$ | $\oplus$ | | $\ominus$ |
|---|---|---|---|
| $A$ | $\nearrow$ | $\frac{220}{\pi}$ | $\searrow$ |
| | | MAX | |

• Answer:

Since $\boxed{r = \dfrac{220}{\pi}}$, then $x = 220 - \pi\left(\frac{220}{\pi}\right) = \boxed{0}$, in yards, resulting in a $\boxed{\text{circular}}$ track.

**Initial Valued Differential Equations**

14. Find a function $f(x)$ that satisfies $f''(x) = 12x^2 + 5, f'(1) = 5$ and passes through the point $(1, 3)$.

    Antidifferentiating yields $f'(x) = 4x^3 + 5x + C_1$. The initial condition $f'(1) = 5$ yields $4 + 5 + C_1 = 5 \Longrightarrow C_1 = -4 \Longrightarrow f'(x) = 4x^3 + 5x - 4$.

    Antidifferentiating one last time yields $f(x) = x^4 + \dfrac{5}{2}x^2 - 4x + C_2$. The other initial condition with $f$ passing through the point $(1, 3)$ implies $f(1) = 3$. As a result, $1 + \dfrac{5}{2} - 4 + C_2 = 3 \Longrightarrow C_2 = 6 - \dfrac{5}{2} = \dfrac{7}{2}$. Finally, $\boxed{f(x) = x^4 + \dfrac{5}{2}x^2 - 4x + \dfrac{7}{2}}$

15. Find a function $f(x)$ that satisfies $f''(x) = x + \sin x, f'(0) = 6$ and $f(0) = 4$.

    Antidifferentiating yields $f'(x) = \dfrac{x^2}{2} - \cos x + C_1$. The initial condition $f'(0) = 6$ implies $0 - \cos 0 + C_1 = 6 \Longrightarrow C_1 = 7$. Then, $f'(x) = \dfrac{x^2}{2} - \cos x + 7$

    Antidifferentiating one last time yields $f(x) = \dfrac{x^3}{6} - \sin x + 7x + C_2$. The other initial condition with $f(0) = 4$ implies $0 - \sin 0 + 0 + C_2 = 4 \Longrightarrow C_2 = 4$. As a result, $\boxed{f(x) = \dfrac{x^3}{6} - \sin x + 7x + 4}$.

16. Find a function $f(x)$ that satisfies $f''(x) = 2 - 12x, f(0) = 9$ and $f(2) = 15$.

    Antidifferentiating yields $f'(x) = 2x - 6x^2 + C_1$. Antidifferentiating one last time yields $f(x) = x^2 - 2x^3 + C_1x + C_2$. The first initial condition $f(0) = 9$ implies $C_2 = 9$ so $f(x) = x^2 - 2x^3 + C_1x + 9$. The second initial condition $f(2) = 15$ implies $4 - 16 + 2C_1 + 9 = 15 \Longrightarrow 2C_1 = 18 \Longrightarrow C_1 = 9$. As a result $\boxed{f(x) = x^2 - 2x^3 + 9x + 9}$.

17. Find a function $f(x)$ that satisfies $f''(x) = 20x^3 + 12x^2 + 4, f(0) = 8$ and $f(1) = 5$.

    Antidifferentiating yields $f'(x) = 5x^4 + 4x^3 + 4x + C_1$. Antidifferentiating one last time yields $f(x) = x^5 + x^4 + 2x^2 + C_1x + C_2$. The first initial condition $f(0) = 8$ implies $C_2 = 8$ so $f(x) = x^5 + x^4 + 2x^2 + C_1x + 8$. The second initial condition $f(1) = 5$ implies $1 + 1 + 2 + C_1 + 8 = 5 \Longrightarrow C_1 = -7$. As a result $\boxed{f(x) = x^5 + x^4 + 2x^2 - 7x + 8}$.

**Area and Riemann Sums**

18. Evaluate $\displaystyle\int_{-1}^{1} x\, dx$ using Riemann Sums.

    Here $a = -1, b = 1, \Delta x = \dfrac{1 - (-1)}{n} = \dfrac{2}{n}$ and $x_i = -1 + i\left(\dfrac{2}{n}\right) = -1 + \dfrac{2i}{n}$.

$$\begin{aligned}
\int_{-1}^{1} x\ dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\frac{2i}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(-1+\frac{2i}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty}\left(\frac{2}{n}\sum_{i=1}^{n}-1+\frac{2}{n}\sum_{i=1}^{n}\frac{2i}{n}\right) \\
&= \lim_{n\to\infty}\left(\frac{2}{n}\cdot(-n)+\frac{4}{n^2}\sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty}\left(\frac{2}{n}\cdot(-n)+\frac{4}{n^2}\frac{n(n+1)}{2}\right) \\
&= \lim_{n\to\infty}\left(-2+\frac{4}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(-2+2\cdot 1\cdot\left(1+\frac{1}{n}\right)\right) \\
&= -2+2=\boxed{0}
\end{aligned}$$

19. Evaluate $\int_0^2 x^2 - 5x\ dx$ using Riemann Sums.

Here $a = 0, b = 2, \Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\left(\dfrac{2}{n}\right) = \dfrac{2i}{n}$.

$$\int_0^2 x^2 - 5x\, dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f\left(\frac{2i}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^n \left(\left(\frac{2i}{n}\right)^2 - 5\left(\frac{2i}{n}\right)\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \left(\frac{2}{n}\sum_{i=1}^n \frac{4i^2}{n^2} - \frac{2}{n}\sum_{i=1}^n \frac{10i}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{8}{n^3}\sum_{i=1}^n i^2 - \frac{20}{n^2}\sum_{i=1}^n i\right)$$

$$= \lim_{n\to\infty} \left(\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{20}{n^2}\frac{n(n+1)}{2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{8}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - \frac{20}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\right)$$

$$= \lim_{n\to\infty} \left(\frac{4}{3}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 10(1)\left(1+\frac{1}{n}\right)\right)$$

$$= \frac{8}{3} - 10 = \frac{8}{3} - \frac{30}{3} = \boxed{-\frac{22}{3}}$$

20. Use Riemann Sums to estimate $\int_0^1 x^2 + 1\, dx$ using 4 equal-length subintervals and right endpoints.

$$\int_0^1 x^2 + 1\, dx \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$= f\left(\frac{1}{4}\right)\frac{1}{4} + f\left(\frac{1}{2}\right)\frac{1}{4} + f\left(\frac{3}{4}\right)\frac{1}{4} + f(1)\cdot\frac{1}{4}$$

$$= \left(\frac{17}{16}\right)\frac{1}{4} + \left(\frac{5}{4}\right)\frac{1}{4} + \left(\frac{25}{16}\right)\frac{1}{4} + 2\cdot\frac{1}{4}$$

$$= \frac{1}{4}\left(\frac{17}{16} + \frac{20}{16} + \frac{25}{16} + \frac{32}{16}\right)$$

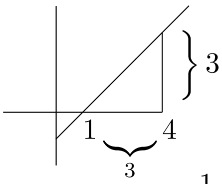
$$= \frac{1}{4}\left(\frac{94}{16}\right) = \frac{94}{64} = \boxed{\frac{47}{32}}$$

21. Sand is added to a pile at a rate of $10 + t^2$ cubic feet per hour for $0 \le t \le 8$. Compute the Riemann Sum to estimate $\int_0^8 10 + t^2\, dt$ using 4 subintervals and the left endpoint of each subinterval. Finally, what two things does this Riemann Sum approximate?

$$\int_0^8 10+t^2\,dt \approx f(t_0)\Delta t+f(t_1)\Delta t+f(t_2)\Delta t+f(t_3)\Delta t$$
$$= f(0)\cdot 2+f(2)\cdot 2+f(4)\cdot 2+f(6)\cdot 2$$
$$= 2(10+14+26+46) = 2(96) = \boxed{192}$$

This Riemann sum approximates at least two things: the area under the curve $y = 10+t^2$ from $t = 0$ to $t = 8$, that is the definite integral. By the Net Change Thrm, it also approximates the net change of sand from time $t = 0$ to time $t = 8$. Recall, $\int_0^8$ rate of change $dt =$ Amount Sand$(t = 8)$ − Amount Sand$(t = 0)$.

22. Compute $\int_1^4 x-1\,dx$ using three different methods: (a) using Area interpretations of the definite integral, (b) Fundamental Theorem of Calculus, and (c) Riemann Sums.




(a) Area$= \frac{1}{2}$(base) (height) $= \frac{1}{2}(3)(3) = \boxed{\frac{9}{2}}$

(b) $\int_1^4 x-1\,dx = \left.\frac{x^2}{2}-x\right|_1^4 = (8-4)-(\frac{1}{2}-1) = 4+\frac{1}{2} = \boxed{\frac{9}{2}}$

(c)

Here $a = 1, b = 4, \Delta x = \frac{4-1}{n} = \frac{3}{n}$ and $x_i = 1+i\left(\frac{3}{n}\right) = 1+\frac{3i}{n}$.

$$\int_1^4 x-1\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^n f\left(1+\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^n \left(\left(1+\frac{3i}{n}\right)-1\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\left(\frac{3}{n}\sum_{i=1}^n \frac{3i}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{9}{n^2}\sum_{i=1}^n i\right)$$

$$= \lim_{n\to\infty}\frac{9}{n^2}\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty}\frac{9}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty}\frac{9}{2}(1)\left(1+\frac{1}{n}\right)$$

$$= \boxed{\frac{9}{2}}$$

**Differentiation** Answer each of the following questions regarding derivatives:

23. Suppose that $e^{xy}+xy=2$. Compute $\frac{dy}{dx}$.

Differentiating implicitly $\frac{d}{dx}(e^{xy}+xy)=\frac{d}{dx}(2)$ yields $e^{xy}\left(x\frac{dy}{dx}+y\right)+\left(x\frac{dy}{dx}+y\right)$. Finally, expanding and solving for $\boxed{\frac{dy}{dx}=\frac{-y-ye^{xy}}{xe^{xy}+x}}$

24. $\frac{d}{dx}\int_1^x -t-1\,dt = \boxed{-x-1}$

25. $\frac{d}{dx}\int_x^7 1-\sin t\,dt = -\frac{d}{dx}\int_7^x 1-\sin t\,dt = -(1-\sin x) = \boxed{\sin x - 1}$

26. $\frac{d}{dx}\int_0^{\sin x}\sqrt{1-t^2}\,dt = \boxed{\sqrt{1-\sin^2 x}(\cos x)}$

27. $\frac{d}{dx}\int_{3x}^2 \cos t\,dt = -\frac{d}{dx}\int_2^{3x}\cos t\,dt = -\cos(3x)(3) = \boxed{-3\cos(3x)}$

28. Find $g''(x)$ if $g(x)=\int_{3x}^7 7t^2+\sin t\,dt$.

$g'(x) = \dfrac{d}{dx}\displaystyle\int_{3x}^{7} 7t^2 + \sin t\ dt = -\dfrac{d}{dx}\int_{7}^{3x} 7t^2 + \sin t\ dt = -(7(3x)^2 + \sin(3x))(3) = -189x^2 - 3\sin(3x)$

$g''(x) = \boxed{-378x - 9\cos(3x)}$

29. $\dfrac{d}{dx}\left(\sqrt{\tan x}\cdot\displaystyle\int_0^x t^3 - t\ dt\right) = \sqrt{\tan x}\ (x^3 - x) + \int_0^x t^3 - t\ dt\left(\dfrac{\sec^2 x}{2\sqrt{\tan x}}\right) = \sqrt{\tan x}\ (x^3 - x) + \left(\dfrac{t^4}{4} - \dfrac{t^2}{2}\right)\Bigg|_0^x\left(\dfrac{\sec^2 x}{2\sqrt{\tan x}}\right) = \boxed{\sqrt{\tan x}\ (x^3 - x) + \left(\dfrac{x^4}{4} - \dfrac{x^2}{2}\right)\left(\dfrac{\sec^2 x}{2\sqrt{\tan x}}\right)}$

30. $\dfrac{d}{dx}\left(\displaystyle\int_0^x t + \sin t\ dt\right)^3 = 3\left(\int_0^x t + \sin t\ dt\right)^2 (x + \sin x) = 3\left(\dfrac{t^2}{2} - \cos t\Bigg|_0^x\right)^2 (x + \sin x) = 3\left(\left(\dfrac{x^2}{2} - \cos x\right) - (0 - \cos 0)\right)^2 (x + \sin x) = \boxed{3\left(\dfrac{x^2}{2} - \cos x + 1\right)^2 (x + \sin x)}$

31. Find $\dfrac{d}{dx}\displaystyle\int_1^{x^2} 3t - 1\ dt$ using two different methods.

(a). $\dfrac{d}{dx}\displaystyle\int_1^{x^2} 3t - 1\ dt = (3x^2 - 1)(2x) = \boxed{6x^3 - 2x}$ We used the FTC Part I.

(b). $\dfrac{d}{dx}\displaystyle\int_1^{x^2} 3t - 1\ dt = \dfrac{d}{dx}\left(\dfrac{3}{2}t^2 - t\Bigg|_1^{x^2}\right) = \dfrac{d}{dx}\left(\left(\dfrac{3}{2}x^4 - x^2\right) - \left(\dfrac{3}{2} - 1\right)\right) = \boxed{6x^3 - 2x}$

We used FTC Part II.

32. Find $\dfrac{d}{dx}\displaystyle\int_x^0 t + 3\sin t\ dt$ using two different methods.

(a). $\dfrac{d}{dx}\displaystyle\int_x^0 t + 3\sin t\ dt = -\dfrac{d}{dx}\int_0^x t + 3\sin t\ dt = \boxed{-(x + 3\sin x)}$ We used the FTC Part I.

(b). $\dfrac{d}{dx}\displaystyle\int_x^0 t + 3\sin t\ dt = \dfrac{d}{dx}\left(\dfrac{t^2}{2} - 3\cos t\Bigg|_x^0\right) = \dfrac{d}{dx}\left((0 - 3\cos 0) - \left(\dfrac{x^2}{2} - 3\cos x\right)\right) = \boxed{-x - 3\sin x}$ We used FTC Part II.

33. Find $f(x)$ if $\displaystyle\int_1^x f(t)dt = 2x - 2$

Differentiating both sides yields: $\dfrac{d}{dx}\displaystyle\int_1^x f(t)dt = \dfrac{d}{dx}(2x - 2)$. That is, $\boxed{f(x) = 2}$.

34. Find $f(x)$ if $\displaystyle\int_x^0 f(t)dt = x^2$.

Differentiating both sides yields: $\dfrac{d}{dx}\displaystyle\int_x^0 f(t)dt = \dfrac{d}{dx}x^2$ or

$-\dfrac{d}{dx}\displaystyle\int_0^x f(t)dt = \dfrac{d}{dx}x^2$. That is, $-f(x) = 2x \Rightarrow \boxed{f(x) = -2x}$.

35. Differentiate $y = (\sin x)e^{\sqrt{x+2}}$

$$\boxed{y' = (\sin x)e^{\sqrt{x+2}}\frac{1}{2\sqrt{x+2}} + e^{\sqrt{x+2}}(\cos x)}$$

36. Differentiate $y = (x - e^{-\cos x}) \cdot e^{\frac{x}{2}}$

$$\boxed{y' = (x - e^{-\cos x}) \cdot e^{\frac{x}{2}}\left(\frac{1}{2}\right) + e^{\frac{x}{2}}\left(1 - e^{-\cos x}(\sin x)\right)}$$

37. Differentiate $y = e^{e^x} \cdot \cos(e^{\sqrt{x}})$

$$\boxed{y' = e^{e^x}\left(-\sin(e^{\sqrt{x}})e^{\sqrt{x}}\frac{1}{2\sqrt{x}}\right) + \cos(e^{\sqrt{x}})e^{e^x}e^x}$$

38. Differentiate $y = \dfrac{1 - e^{-3x}}{x}$

$$\boxed{y' = \frac{x(-e^{-3x}(-3)) - (1 - e^{-3x})(1)}{x^2} = \frac{3xe^{-3x} - 1 + e^{-3x}}{x^2}}$$

39. Differentiate $y = \dfrac{1 + e^{-2x}}{1 - e^{7x}}$

$$y' = \frac{(1 - e^{7x})e^{-2x}(-2) - (1 + e^{-2x})(-e^{7x})(7)}{(1 - e^{7x})^2} = \frac{-2e^{-2x} + 2e^{5x} + 7e^{7x} + 7e^{5x}}{(1 - e^{7x})^2} = \boxed{\frac{-2e^{-2x} + 9e^{5x} + 7e^{7x}}{(1 - e^{7x})^2}}$$

40. Differentiate $y = \sin(e^x)\cos(e^{-x})$

$$y' = \sin(e^x)(-\sin(e^{-x}))e^{-x}(-1) + \cos(e^{-x})\cos(e^x)e^x = \boxed{\sin(e^x)\sin(e^{-x})e^{-x} + \cos(e^{-x})\cos(e^x)e^x}$$

41. Differentiate $y = (e^{2x} - e^{-3x})^7$

$$y' = 7(e^{2x} - e^{-3x})^6(e^{2x}(2) - e^{-3x}(-3)) = \boxed{7(e^{2x} - e^{-3x})^6(2e^{2x} + 3e^{-3x})}$$

**Integration** Evaluate each of the following integrals:

42. $$\int \frac{1}{\sqrt[3]{(7-5z)^2}}\, dz = \int \frac{1}{(7-5z)^{\frac{2}{3}}}\, dz = -\frac{1}{5}\int u^{-\frac{2}{3}}\, du = -\frac{1}{5}(3)u^{\frac{1}{3}} + C = \boxed{-\frac{3}{5}(7-5z)^{\frac{1}{3}} + C}$$

Here $\begin{cases} u &= 7 - 5z \\ du &= -5dz \\ -\frac{1}{5}du &= dz \end{cases}$

43. $$\int_0^{\frac{\pi}{3}} \sec^2\theta\, d\theta = \tan\theta\Big|_0^{\frac{\pi}{3}} = \tan\frac{\pi}{3} - \tan 0 = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} - 0 = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \boxed{\sqrt{3}}$$

44. $$\int_{-3}^{3} |x^2 - 1|\, dx$$

We can solve this one two different ways. We will detail both: First

$$\int_{-3}^{3} |x^2 - 1| \ dx$$
$$= \int_{-3}^{-1} x^2 - 1 \ dx + \int_{-1}^{1} 1 - x^2 \ dx + \int_{1}^{3} x^2 - 1 \ dx$$
$$= \frac{x^3}{3} - x\Big|_{-3}^{-1} + x - \frac{x^3}{3}\Big|_{-1}^{1} + \frac{x^3}{3} - x\Big|_{1}^{3}$$
$$= \left(-\frac{1}{3} + 1\right) - (-9 + 3) + \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) + (9 - 3) - \left(\frac{1}{3} - 1\right)$$
$$= \frac{2}{3} + 6 + \frac{2}{3} + \frac{2}{3} + 6 + \frac{2}{3}$$
$$= \frac{8}{3} + 12$$
$$= \frac{8}{3} + \frac{36}{3}$$
$$= \boxed{\frac{44}{3}}$$

Second, we could use symmetry

$$\int_{-3}^{3} |x^2 - 1| \ dx$$
$$= 2\left[\int_{0}^{1} 1 - x^2 \ dx + \int_{1}^{3} x^2 - 1 \ dx\right]$$
$$= 2\left[x - \frac{x^3}{3}\Big|_{0}^{1} + \frac{x^3}{3} - x\Big|_{1}^{3}\right]$$
$$= 2\left[\left(1 - \frac{1}{3}\right) - 0 + (9 - 3) - \left(\frac{1}{3} - 1\right)\right]$$
$$= 2\left[\frac{2}{3} + 6 + \frac{2}{3}\right] = 2\left[\frac{4}{3} + \frac{18}{3}\right] = 2\left[\frac{22}{3}\right] = \boxed{\frac{44}{3}}$$

45. $\displaystyle\int_{-1}^{2} (|x| - 4) \ dx$

$$= \int_{-1}^{0} (-x-4)\ dx + \int_{0}^{2} (x-4)\ dx$$

$$= -\frac{x^2}{2} - 4x\Big|_{-1}^{0} + \frac{x^2}{2} - 4x\Big|_{0}^{2}$$

$$= 0 - \left(-\frac{1}{2} + 4\right) + (2-8) - 0 = -\frac{7}{2} - 6 = -\frac{7}{2} - \frac{12}{2} = \boxed{-\frac{19}{2}}$$

46. $\displaystyle\int \frac{(x+1)(x+2)}{\sqrt{x}}\ dx$

$$= \int \frac{(x^2+3x+2}{\sqrt{x}}\ dx$$
$$= \int x^{\frac{3}{2}} + 3x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}\ dx$$
$$= \frac{2}{5}x^{\frac{5}{2}} + 3\left(\frac{2}{3}\right)x^{\frac{3}{2}} + 2(2)x^{\frac{1}{2}} + C$$
$$= \boxed{\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + C}$$

47. $\displaystyle\int \frac{u+\sqrt{u}+7}{u^3}\ du$

$$= \int u^{-2} + u^{-\frac{5}{2}} + 7u^{-3}\ du$$
$$= -u^{-1} - \frac{2}{3}u^{-\frac{3}{2}} - \frac{7}{2}u^{-2} + C$$
$$= \boxed{-\frac{1}{u} - \frac{2}{3u^{\frac{3}{2}}} - \frac{7}{2u^2} + C}$$

48. $\displaystyle\int_{-3}^{3} x|x|\ dx = \int_{-3}^{0} -x^2\ dx + \int_{0}^{3} x^2\ dx = -\frac{x^3}{3}\Big|_{-3}^{0} + \frac{x^3}{3}\Big|_{0}^{3} = (0-9) + (9-0) = \boxed{0}$

49. $\displaystyle\int_{0}^{2\pi} |\sin x|\ dx$

$$= \int_{0}^{\pi} \sin x\ dx + \int_{\pi}^{2\pi} -\sin x\ dx$$
$$= -\cos x\Big|_{0}^{\pi} + \cos x\Big|_{\pi}^{2\pi}$$
$$= -\cos\pi - (-\cos 0) + \cos(2\pi) - \cos\pi$$
$$= -(-1) + 1 + 1 - (-1) = \boxed{4}$$

50. $\displaystyle\int_{0}^{\frac{\pi}{6}} \frac{\cos x}{(1+6\sin x)^2}\ dx = \frac{1}{6}\int_{u=1}^{u=4} u^{-2}\ du = -\frac{1}{6}u^{-1}\Big|_{u=1}^{u=4} = -\frac{1}{24} + \frac{1}{6} = \frac{3}{24} = \boxed{\frac{1}{8}}$

Here $\begin{cases} u &= 1 + 6\sin x \\ du &= 6\cos x dx \\ \frac{1}{6}du &= \cos x dx \end{cases}$ and $\begin{cases} x = 0 & \Longrightarrow u = 1 \\ x = \frac{\pi}{6} & \Longrightarrow u = 1 + 6\sin(\frac{\pi}{6}) = 1 + 6(\frac{1}{2}) = 4 \end{cases}$

51. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin(2x)\ dx = \frac{1}{2}\int_{u=\frac{\pi}{2}}^{u=\frac{2\pi}{3}} \sin u\ du = -\frac{1}{2}\cos u\Big|_{u=\frac{\pi}{2}}^{u=\frac{2\pi}{3}} = -\frac{1}{2}\cos\frac{2\pi}{3} + \frac{1}{2}\cos\frac{\pi}{2} = -\frac{1}{2}\left(-\frac{1}{2}+0\right)$
$= \boxed{\frac{1}{4}}$

Here $\begin{cases} u &= 2x \\ du &= 2dx \\ \frac{1}{2}du &= dx \end{cases}$ and $\begin{cases} x = \frac{\pi}{4} & \Longrightarrow u = \frac{\pi}{2} \\ x = \frac{\pi}{3} & \Longrightarrow u = \frac{2\pi}{3} \end{cases}$

52. $\displaystyle\int_0^{\pi} \sin^2\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right)\ dx = 6\int_{u=0}^{u=\frac{1}{2}} u^2\ du = 2u^3\Big|_{u=0}^{u=\frac{1}{2}} = \boxed{\frac{1}{4}}$

Here $\begin{cases} u &= \sin\left(\frac{x}{6}\right) \\ du &= \frac{1}{6}\cos\left(\frac{x}{6}\right)dx \\ 6du &= \cos\left(\frac{x}{6}\right)dx \end{cases}$ and $\begin{cases} x = 0 & \Longrightarrow u = 0 \\ x = \pi & \Longrightarrow u = \frac{1}{2} \end{cases}$

53. $\displaystyle\int_0^{\frac{\pi}{8}} \tan^3(2x)\sec^2(2x)\ dx = \frac{1}{2}\int_{u=0}^{u=1} u^3\ du = \frac{1}{2}\frac{u^4}{4}\Big|_{u=0}^{u=1} = \frac{1}{8} - 0 = \boxed{\frac{1}{8}}$

Here $\begin{cases} u &= \tan(2x) \\ du &= 2\sec^2(2x)dx \\ \frac{1}{2}du &= \sec^2(2x)dx \end{cases}$ and $\begin{cases} x = 0 & \Longrightarrow u = 0 \\ x = \frac{\pi}{8} & \Longrightarrow u = \tan\frac{\pi}{4} = 1 \end{cases}$

54. $\displaystyle\int_{\frac{\pi^2}{36}}^{\frac{\pi^2}{4}} \frac{\cos\sqrt{t}}{\sqrt{t\sin\sqrt{t}}}\ dt = \int_{\frac{\pi^2}{36}}^{\frac{\pi^2}{4}} \frac{\cos\sqrt{t}}{\sqrt{t}\sqrt{\sin\sqrt{t}}}\ dt = 2\int_{u=\frac{1}{2}}^{1} u^{-\frac{1}{2}}\ du = 2(2)u^{\frac{1}{2}}\Big|_{u=\frac{1}{2}}^{u=1} = \boxed{4 - \frac{4}{\sqrt{2}}}$

Here $\begin{cases} u &= \sin\sqrt{t} \\ du &= \cos\sqrt{t}\frac{1}{2\sqrt{t}}dt \\ 2du &= \frac{\cos\sqrt{t}}{\sqrt{t}}dt \end{cases}$ and $\begin{cases} x = \frac{\pi^2}{36} & \Longrightarrow u = \sin\sqrt{\frac{\pi^2}{36}} = \sin\frac{\pi}{6} = \frac{1}{2} \\ x = \frac{\pi^2}{4} & \Longrightarrow u = \sin\sqrt{\frac{\pi^2}{4}} = \sin\frac{\pi}{2} = 1 \end{cases}$

55. $\displaystyle\int_0^{\frac{\pi}{3}} \frac{3\sin x\cos x}{(1+3\sin^2 x)^2}\ dx = \frac{1}{2}\int_{u=1}^{u=\frac{13}{4}} \frac{1}{u^2}\ du = -\frac{1}{2u}\Big|_{u=1}^{u=\frac{13}{4}} = -\frac{1}{2}\left(\frac{4}{13}\right) - \left(-\frac{1}{2}(1)\right) = -\frac{2}{13} + \frac{1}{2} =$
$\boxed{\frac{9}{26}}$

Here $\begin{cases} u &= 1+3\sin^2 x \\ du &= 6\sin x\cos x dx \\ \frac{1}{2}du &= 3\sin x\cos x dx \end{cases}$ and $\begin{cases} x = 0 & \Longrightarrow u = 1 \\ x = \frac{\pi}{3} & \Longrightarrow u = 1 + 3\sin^2(\frac{\pi}{3}) = 1 + 3(\frac{3}{4}) = \frac{13}{4} \end{cases}$

56. $\displaystyle\int_5^5 3x^5\sqrt{x}\sin(3x)\ dx = 0$ why?

57. $\displaystyle\int (w^3\cos w^4 + 2009)\ dw = \int w^3\cos w^4\ dw + \int 2009\ dw = \frac{1}{4}\int \cos u\ du + \int 2009\ dw =$
$\frac{1}{4}\sin u + 2009w + C = \boxed{\frac{1}{4}\sin w^4 + 2009w + C}$

Here $\begin{cases} u &= w^4 \\ du &= 4w^3dw \\ \frac{1}{4}du &= w^3dw \end{cases}$

58. $\displaystyle\int \frac{y^3 - 9y\sin y + 26y^{-1}}{y}\ dy = \int y^2 - 9\sin y + 26y^{-2}\ dy = \boxed{\frac{y^3}{3} + 9\cos y - 26y^{-1} + C}$

59. $\displaystyle\int \sqrt{x}\cos(x\sqrt{x})\ dx = \frac{2}{3}\int \cos u\ du = \frac{2}{3}\sin u + C = \boxed{\frac{2}{3}\sin(x^{\frac{3}{2}}) + C}$

Here $\begin{cases} u &= x^{\frac{3}{2}} \\ du &= \frac{3}{2}x^{\frac{1}{2}}dx \\ \frac{2}{3}du &= \sqrt{x}dx \end{cases}$

60. $\displaystyle\int \frac{1}{t^2}\sin\left(\frac{1}{t}\right)\ dt = -\int \sin u\ du = \cos u + C = \boxed{\cos\tfrac{1}{t} + C}$

Here $\begin{cases} u &= \frac{1}{t} \\ du &= -\frac{1}{t^2}dt \\ -du &= \frac{1}{t^2}dt \end{cases}$

61. $\displaystyle\int \frac{1}{u^2}\sqrt[3]{1 - \frac{1}{u}}\ du = \int w^{\frac{1}{3}}\ dw = \frac{3}{4}w^{\frac{4}{3}} + C = \boxed{\frac{3}{4}\left(1 - \frac{1}{u}\right)^{\frac{4}{3}} + C}$

Here $\begin{cases} w &= 1 - \frac{1}{u} \\ dw &= \frac{1}{u^2}du \end{cases}$

62. $\displaystyle\int_{-2}^{2} |1 - x|\ dx = \int_{-2}^{1} 1 - x\ dx + \int_{1}^{2} x - 1\ dx = x - \frac{x^2}{2}\Big|_{-2}^{1} + \frac{x^2}{2} - x\Big|_{1}^{2} = (1 - \frac{1}{2}) - (-2 - 2) + (2 - 2) - (\frac{1}{2} - 1) = \frac{1}{2} + 4 + \frac{1}{2} = \boxed{5}$

63. $\displaystyle\int_{0}^{2\pi} |\cos x - \sin x|\ dx$

$\displaystyle= \int_{0}^{\frac{\pi}{4}} \cos x - \sin x\ dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x\ dx + \int_{\frac{5\pi}{4}}^{2\pi} \cos x - \sin x\ dx$

$\displaystyle= \sin x + \cos x\Big|_{0}^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + \sin x + \cos x\Big|_{\frac{5\pi}{4}}^{2\pi}$

$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - (\sin 0 + \cos 0) + (-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}) - (-\cos\frac{\pi}{4} - \sin\frac{\pi}{4})$

$+ \sin 2\pi + \cos 2\pi - (\sin\frac{5\pi}{4} + \cos\frac{5\pi}{4})$

$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1) + (-(-\frac{\sqrt{2}}{2}) - (-\frac{\sqrt{2}}{2})) - (-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) + 0 + 1 - (-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2})$

$= \sqrt{2} - 1 + \sqrt{2} + \sqrt{2} + 1 + \sqrt{2}$

$= \boxed{4\sqrt{2}}$

64. $\displaystyle\int \frac{2t^5 + 1}{t^6 + 3t}\ dt = \frac{1}{3}\int \frac{1}{u}\ du = \frac{1}{3}\ln|u| + C = \boxed{\tfrac{1}{3}\ln|t^6 + 3t| + C}$ NOTE: THIS WILL NOT BE ON EXAM #3

Here $\begin{cases} u &= t^6 + 3t \\ du &= 6t^5 + 3dt \\ \frac{1}{3}du &= 2t^5 + 1dt \end{cases}$

65. $\displaystyle\int x(x+1)^{14}\ dx = \int (u-1)u^{14}\ du = \int u^{15} - u^{14}\ du = \frac{1}{16}u^{16} - \frac{1}{15}u^{15} + C = \boxed{\frac{(x+1)^{16}}{16} - \frac{(x+1)^{15}}{15} + C}$

Here $\begin{cases} u &= x+1 \Longrightarrow x = u-1 \\ du &= dx \end{cases}$

66. $\displaystyle\int 7\cos(5x) - 5\sin(7x)\ dx = \frac{7}{5}\int \cos u\ du - \frac{5}{7}\int \sin w\ dw = \frac{7}{5}\sin u + \frac{5}{7}\cos w + C = \boxed{\frac{7}{5}\sin(5x) + \frac{5}{7}\cos(7x) + C}$

Here $\begin{cases} u &= 5x \\ du &= 5dx \\ \frac{1}{5}du &= dx \end{cases}$ and $\begin{cases} w &= 7x \\ dw &= 7dx \\ \frac{1}{7}dw &= dx \end{cases}$

67. $\displaystyle\int x\sqrt{2-3x^2}\ dx = -\frac{1}{6}\int \sqrt{u}\ du = -\frac{1}{6}\left(\frac{2}{3}\right)u^{\frac{3}{2}} + C = -\frac{1}{9}u^{\frac{3}{2}} + C = \boxed{-\frac{1}{9}(2-3x^2)^{\frac{3}{2}} + C}$

Here $\begin{cases} u &= 2-3x^2 \\ du &= -6xdx \\ -\frac{1}{6}du &= xdx \end{cases}$

68. $\displaystyle\int x\sqrt{2-3x}\ dx = -\frac{1}{3}\int \left(\frac{2-u}{3}\right)\sqrt{u}\ du = -\frac{1}{9}\int 2u^{\frac{1}{2}} - u^{\frac{3}{2}}\ du = -\frac{1}{9}\left(2\left(\frac{2}{3}\right)u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + C = \boxed{-\frac{4}{27}(2-3x)^{\frac{3}{2}} + \frac{2}{45}(2-3x)^{\frac{5}{2}} + C}$

Here $\begin{cases} u &= 2-3x \Longrightarrow x = \dfrac{2-u}{3} \\ du &= -3dx \\ -\frac{1}{3}du &= dx \end{cases}$

69. $\displaystyle\int x(3x-1)^{\frac{5}{7}}\ dx = \frac{1}{3}\int \left(\frac{u+1}{3}\right)u^{\frac{5}{7}}\ du = \frac{1}{9}\left(\int u^{\frac{12}{7}} + u^{\frac{5}{7}}\ du\right) = \frac{1}{9}\left(\frac{7}{19}u^{\frac{19}{7}} + \frac{7}{12}u^{\frac{12}{7}}\right) + C = \frac{7}{171}u^{\frac{19}{7}} + \frac{7}{108}u^{\frac{12}{7}} + C = \boxed{\frac{7}{171}(3x-1)^{\frac{19}{7}} + \frac{7}{108}(3x-1)^{\frac{12}{7}} + C}$

Here $\begin{cases} u &= 3x-1 \Longrightarrow x = \dfrac{u+1}{3} \\ du &= 3dx \\ \frac{1}{3}du &= dx \end{cases}$

70. $\displaystyle\int (x^{\frac{7}{2}} + x^{-\frac{1}{3}})\sqrt{x}\ dx = \int x^4 + x^{\frac{1}{6}}\ dx = \boxed{\frac{x^5}{5} + \frac{6}{7}x^{\frac{7}{6}} + C}$

71. $\displaystyle\int x^2e^{-x^3}\ dx = -\frac{1}{3}\int e^u\ du = -\frac{1}{3}e^u + C = \boxed{-\frac{1}{3}e^{-x^3} + C}$

Here $\begin{cases} u &= -x^3 \\ du &= -3x^2dx \\ -\frac{1}{3}du &= x^2dx \end{cases}$

72. $\displaystyle\int_1^3 xe^{-3x^2}\ dx = -\frac{1}{6}\int_{u=-3}^{u=-27} e^u\ du = -\frac{1}{6}e^u\Big|_{u=-3}^{u=-27} = -\frac{1}{6}(e^{-27} - e^{-3}) = \boxed{\frac{1}{6e^3} - \frac{1}{6e^{27}}}$

Here $\begin{cases} u &= -3x^2 \\ du &= -6x dx \\ -\frac{1}{6}du &= x\ dx \end{cases}$ and $\begin{cases} x = 1 & \Longrightarrow u = -3 \\ x = 3 & \Longrightarrow u = -27 \end{cases}$

73. $\displaystyle\int \frac{e^{-\frac{1}{x}}}{7x^2}\ dx = \frac{1}{7}\int e^u\ du = \frac{1}{7}e^u + C = \boxed{\frac{1}{7}e^{-\frac{1}{x}} + C}$

Here $\begin{cases} u &= -\frac{1}{x} \\ du &= \frac{1}{x^2}dx \end{cases}$

74. $\displaystyle\int \frac{e^x}{(e^x-1)^2}\ dx = \int \frac{1}{u^2}\ du = -u^{-1} + C = \boxed{-\frac{1}{e^x-1} + C}$

Here $\begin{cases} u &= e^x - 1 \\ du &= e^x dx \end{cases}$

75. $\displaystyle\int \frac{3e^{7x}}{\sqrt{1-e^{7x}}}\ dx = -\frac{3}{7}\int \frac{1}{\sqrt{u}}\ du = -\frac{3}{7}(2)u^{\frac{1}{2}} + C = -\frac{6}{7}\sqrt{u} + C = \boxed{-\frac{6}{7}\sqrt{1-e^{7x}} + C}$

Here $\begin{cases} u &= 1 - e^{7x} \\ du &= -7e^{7x}dx \\ -\frac{1}{7}du &= e^{7x}dx \end{cases}$

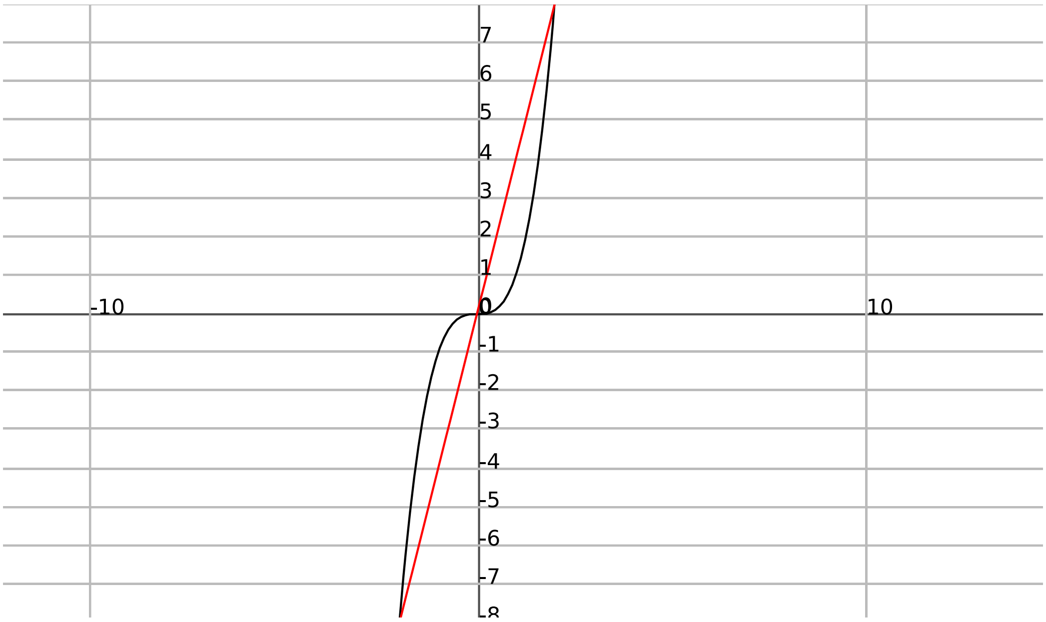
76. $\displaystyle\int e^{3x}e^{e^{3x}}\ dx = \frac{1}{3}\int e^u\ du = \frac{1}{3}e^u + C = \boxed{\frac{1}{3}e^{e^{3x}} + C}$

Here $\begin{cases} u &= e^{3x} \\ du &= 3e^{3x}dx \\ \frac{1}{3}du &= e^{3x}dx \end{cases}$

**Area between Curves**

77. Compute the area bounded by $y = x^3$ and $y = 4x$.

You can solve this problem two different ways. I will detail both. First we will use symmetry to integrate one side and then double the value to capture the entire area. Note that $y = x^3$ and $y = 4x$ intersect when $x^3 = 4x \Longrightarrow x(x^2 - 4) = 0$ or when $x = 0$ or $x = \pm 2$.




$$\begin{aligned}
\text{Area} &= 2\left[\int_0^2 \text{top} - \text{bottom } dx\right] \\
&= 2\left[\int_0^2 4x - x^3 \ dx\right] \\
&= 2\left[2x^2 - \frac{x^4}{4}\bigg|_0^2\right] \\
&= 2\left[(8-4) - 0\right] = \boxed{8}
\end{aligned}$$

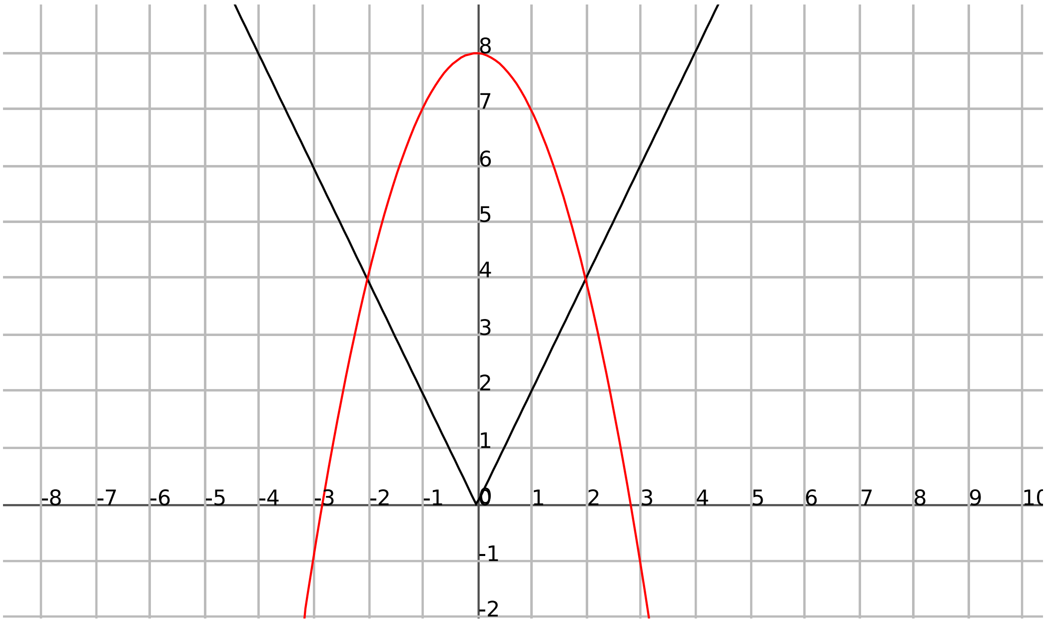
If you don't use symmetry we see that

$$\begin{aligned}
\text{Area} &= \int_{-2}^0 \text{top} - \text{bottom } dx + \int_0^2 \text{top} - \text{bottom } dx \\
&= \int_{-2}^0 x^3 - 4x \ dx + \int_0^2 4x - x^3 \ dx \\
&= \left(\frac{x^4}{4} - 2x^2\right)\bigg|_{-2}^0 + \left(2x^2 - \frac{x^4}{4}\right)\bigg|_0^2 \\
&= 0 - (4-8) + (8-4) + (8-4) - 0 \\
&= 4 + 4 = \boxed{8}
\end{aligned}$$

78. Compute the area bounded by $y = 2|x|$ and $y = 8 - x^2$.

You can solve this problem two different ways. I will detail both. First we will use symmetry to integrate one side and then double the value to capture the entire area. Note that $y = 2|x|$ and $y = 8 - x^2$ intersect (for $x > 0$) when $2x = 8 - x^2 \implies x^2 + 2x - 8 = 0 \implies (x+4)(x-2) = 0$

or when $x = -4$ or $x = 2$. We will ignore $x = -4$ here since we are considering the side with $x > 0$.



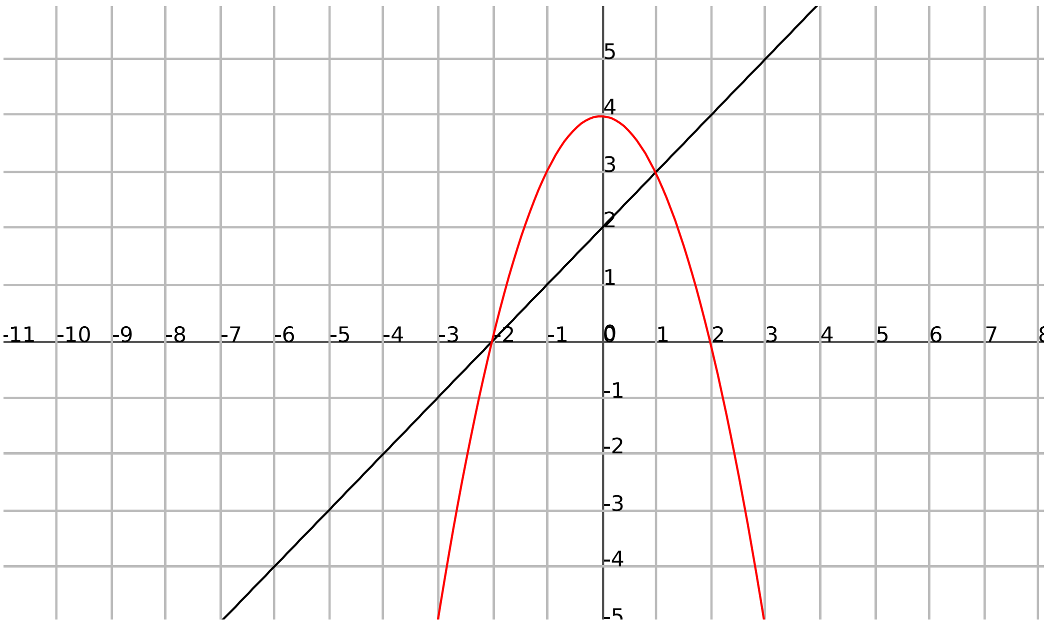

$$\begin{aligned}
\text{Area} &= 2\left[\int_0^2 \text{top} - \text{bottom}\ dx\right] \\
&= 2\left[\int_0^2 (8 - x^2) - 2x\ dx\right] \\
&= 2\left[\int_0^2 -x^2 - 2x + 8\ dx\right] \\
&= 2\left[-\frac{x^3}{3} - x^2 + 8x\Big|_0^2\right] \\
&= 2\left[\left(-\frac{8}{3} - 4 + 16\right) - 0\right] \\
&= 2\left[12 - \frac{8}{3}\right] \\
&= 2\left[\frac{36}{3} - \frac{8}{3}\right] \\
&= 2\left[\frac{28}{3}\right] \\
&= \boxed{\frac{56}{3}}
\end{aligned}$$

If you don't use symmetry we see that

$$\begin{aligned}
\text{Area} &= \int_{-2}^{0} \text{top} - \text{bottom}\ dx + \int_{0}^{2} \text{top} - \text{bottom}\ dx \\
&= \int_{-2}^{0} (8 - x^2) - 2(-x)\ dx + \int_{0}^{2} (8 - x^2) - 2x\ dx \\
&= \int_{-2}^{0} -x^2 + 2x + 8\ dx + \int_{0}^{2} -x^2 - 2x + 8\ dx \\
&= \left(-\frac{x^3}{3} + x^2 + 8x\right)\Big|_{-2}^{0} + \left(-\frac{x^3}{3} - x^2 + 8x\right)\Big|_{0}^{2} \\
&= 0 - \left(-\frac{8}{3} + 4 - 16\right) + \left(-\frac{8}{3} - 4 + 16\right) - 0 \\
&= -\frac{8}{3} + 12 - \frac{8}{3} + 12 \\
&= -\frac{16}{3} + 24 \\
&= -\frac{16}{3} + \frac{72}{3} \\
&= \boxed{\frac{56}{3}}
\end{aligned}$$

79. Compute the area bounded by $y = 4 - x^2$, $y = x + 2$, $x = -3$, and $x = 0$.

Note that these two curves intersect when $4 - x^2 = x + 2 \implies x^2 + x - 2 = 0 \implies (x+2)(x-1) = 0 \implies x = -2$ or $x = 1$.

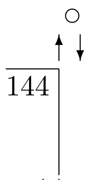


$$
\begin{aligned}
\text{Area} &= \int_{-3}^{-2} \text{top} - \text{bottom}\ dx + \int_{-2}^{0} \text{top} - \text{bottom}\ dx \\
&= \int_{-3}^{-2} (x+2) - (4-x^2)\ dx + \int_{-2}^{0} (4-x^2) - (x+2)\ dx \\
&= \int_{-3}^{-2} x^2 + x - 2\ dx + \int_{-2}^{0} -x^2 - x + 2\ dx \\
&= \left(\frac{x^3}{3} + \frac{x^2}{2} - 2x\right)\Bigg|_{-3}^{-2} + \left(-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right)\Bigg|_{-2}^{0} \\
&= \left(-\frac{8}{3} + 2 + 4\right) - \left(-9 + \frac{9}{2} + 6\right) + 0 - \left(\frac{8}{3} - 2 - 4\right) \\
&= -\frac{8}{3} + 6 + 3 - \frac{9}{2} - \frac{8}{3} + 6 \\
&= -\frac{16}{3} + 15 - \frac{9}{2} \\
&= -\frac{32}{6} + \frac{90}{6} - \frac{27}{6} \\
&= \boxed{\frac{31}{6}}
\end{aligned}
$$

**Position, Velocity, Acceleration**

80. A ball is thrown upward with a speed of 128 ft/sec from the edge of a cliff 144 ft above the ground. Find its height above the ground $t$ seconds later. When does it reach its maximum height? When does it hit the ground?

    Note $v_0 = 128\frac{\text{ft}}{\text{sec}}, s_0 = 144\text{ft}$.

    $a(t) = -32$
    $v(t) = -32t + v_0 \Longrightarrow v(t) = -32t + 128$
    $s(t) = -16t^2 + 128t + s_0 \Longrightarrow s(t) = -16t^2 + 128t + 144$

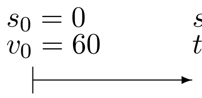
    Max height occurs when $v(t) = 0$. That is, $-32t + 128 = 0$ or when $t = 4$ seconds.

    The ball hits the ground when $s(t) = -16t^2 + 128t + 144 = 0$ or when $-16(t^2 - 8t - 9) = 0$ so that $-16(t-9)(t+1) = 0$. Then $t = 9$ or $t = -1$. We will ignore the negative time here. The ball hits the ground after 9 seconds.

81. The skid marks made by an automobile indicate that its brakes were fully applied for a distance of 90 ft before it came to a stop. Suppose that it is known that the car in question has a constant deceleration of 20 ft/sec$^2$ under the conditions of the skid. Suppose also that

the car was travelling at 60 ft/sec when the brakes were first applied. How long did it take for the car to come to a complete stop?

Note $v_0 = 60\frac{\text{ft}}{\text{sec}}, s_0 = 0\text{ft}$.



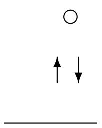

$$\begin{aligned} a(t) &= -20 \\ v(t) &= -20t + v_0 \Longrightarrow v(t) = -20t + 60 \\ s(t) &= -10t^2 + 60t + s_0 \Longrightarrow s(t) = -10t^2 + 128t + 0 \end{aligned}$$

You can solve this two ways. First, the car stops when $v(t) = 0$. Set $v(t) = -20t + 60 = 0$ and solve for the stopping time. That is, $t = 3$ seconds.

Second, the car stops when $s(t) = 90$. Set $s(t) = -10t^2 + 60t = 90$ and solve for the stopping time. We see that $-10(t^2 - 6t + 9) = 0$ implies $-10(t - 3)(t - 3) = 0$ or the car stops when $t = 3$ seconds.

82. Suppose that a bolt was fired vertically upward from a powerful crossbow at ground level, and that it struck the ground 48 seconds later. If air resistance may be neglected, find the initial velocity of the bolt and the maximum height it reached.

Note $v_0 =?\frac{\text{ft}}{\text{sec}}, s_0 = 0 \text{ ft}, t_{\text{impact}} = 48\text{sec}$.



$$\begin{aligned} a(t) &= -32 \\ v(t) &= -32t + v_0 \\ s(t) &= -16t^2 + v_0 t + s_0 \Longrightarrow s(t) = -16t^2 + v_0 t + 0 \Longrightarrow s(t) = -16t^2 + v_0 t \end{aligned}$$

First we use the impact information, $s(48) = 0$.

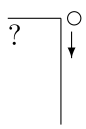
$$s(48) = 0$$

$$-16(48)^2 + v_0(48) = 0 \Longrightarrow 48v_0 = 16(48)^2 \Longrightarrow v_0 = \frac{16(48)^2}{48} = 16(48) = 768\frac{\text{ft}}{\text{sec}},$$

As a result $v(t) = -32t + 768$ and the max height occurs when $v(t) = 0 \Longrightarrow t = \dfrac{768}{32} = 24$ seconds. Finally the max height is $s(24) = -16(24)^2 + 768(24) = -16(576) + 18432 = -9216 + 18432 = 9216$ feet.

83. Jack throws a baseball straight downward from the top of a tall building. The initial speed of the ball is 25 feet per second. It hits the ground with a speed of 153 feet per second. How tall is the building?

Note $v_0 = -25\frac{\text{ft}}{\text{sec}}, s_0 =?\text{ft}, v_{\text{impact}} = -153\frac{\text{ft}}{\text{sec}}$



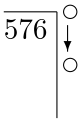
$a(t) = -32$
$v(t) = -32t + v_0 \Longrightarrow v(t) = -32t - 25$
$s(t) = -16t^2 - 25t + s_0$

The ball hits the ground when $v(t) = -32t - 25 = -153$ or when $32t = 153 - 25 = 128$ which is when $t_{\text{impact}} = 4$ seconds.

Finally, we solve $s(4) = 0$ for $s_0$. The ball hits the ground when $-16(4)^2 - 25(4) + s_0 = 0$ or when $-256 - 100 + s_0 = 0$ which is when $s_0 = 356$ feet. As a result, the building is 356 feet tall.

84. A ball is dropped from the top of the building 576 feet high. With what velocity should a second ball be thrown straight downward 3 seconds later so that the two balls hit the ground simultaneously?

Note $v_0 = 0 \ \frac{\text{ft}}{\text{sec}}, s_0 = 576\text{ft}$.




Ball 1 has the following motion equations:

$a(t) = -32$
$v(t) = -32t + v_0 \Longrightarrow v(t) = -32t$
$s(t) = -16t^2 + s_0 \Longrightarrow s(t) = -16t^2 + 576$

The *first* ball hits the ground when $s(t) = -16t^2 + 576 = 0$ or when $16t^2 = 576$ which is when $t^2 = \dfrac{576}{16}$ or when $t_{\text{impact}} = 6$ seconds.

For the *second* ball to be thrown 3 seconds later and hit the ground at the same time as the *first* ball, the *second* ball must travel just 3 seconds before hitting the ground.

Ball 2 has the following motion equations:

$a(t) = -32$
$v(t) = -32t + v_0$
$s(t) = -16t^2 + v_0t + s_0 \Longrightarrow s(t) = -16t^2 + v_0t + 576$

We solve $s(3) = 0$ to find the second ball's initial velocity. Set $-16(3)^2 + 3v_0 + 576 = 0 \Longrightarrow 3v_0 = -576 + 144 = -432 \Longrightarrow v_0 = -144$. Finally, the second ball must have an initial velocity of $-144$ feet per second, or we can say the second ball must be thrown straight down with a speed of 144 feet per second.

85. A particle starts from rest at the point $x = 10$ and moves along the $x$-axis with acceleration function $a(t) = 12t$. Find its resulting position function.

Note $s_0 = 10, v_0 = 0$.

$a(t) = 12t$
$v(t) = 6t^2 + v_0 \Longrightarrow v(t) = 6t^2$
$s(t) = 2t^3 + s_0 \Longrightarrow s(t) = 2t^3 + 10$

As a result, the resulting position function is given as $s(t) = 2t^3 + 10$.

86. The skid marks made by an automobile indicate that its brakes were fully applied for a distance of 160 ft before it came to a stop. Suppose that it is known that the car in question

has a constant deceleration of 20 ft/sec$^2$ under the conditions of the skid. How fast was the car travelling when its brakes were first applied?

Note $v_0 = ?\frac{\text{ft}}{\text{sec}}$, $s_0 = 0\text{ft}$, $s_{\text{stop}} = 160\text{ft}$.

$s_0 = 0$ $\quad$ $s_{\text{stop}} = 160$
$v_0 = ?$ $\quad$ $v_{\text{stop}} = 0$

$$\begin{aligned}
a(t) &= -20\\
v(t) &= -20t + v_0\\
s(t) &= -10t^2 + v_0 t + s_0 \Longrightarrow s(t) = -10t^2 + v_0 t + 0
\end{aligned}$$

You can solve this in two parts. First, the car stops when $v(t) = 0$. Set $v(t) = -20t + v_0 = 0$ and solve for the stopping time in terms of the initial velocity $v_0$. That is, $t_{\text{stop}} = \frac{v_0}{20}$ seconds.

Second, the car stops when $s(t_{\text{stop}}) = 160$ or $s\left(\frac{v_0}{20}\right) = 160$. Set $-10\left(\frac{v_0}{20}\right)^2 + v_0\left(\frac{v_0}{20}\right) = 160$ and solve for the unknown initial velocity. We see that

$$\begin{aligned}
-10\left(\frac{v_0}{20}\right)^2 + v_0\left(\frac{v_0}{20}\right) &= 160\\
\frac{-10v_0^2}{400} + \frac{v_0^2}{20} &= 160\\
\frac{-10v_0^2}{400} + \frac{20v_0^2}{400} &= 160\\
\frac{10v_0^2}{400} &= 160\\
10v_0^2 &= 160(400)\\
v_0^2 &= 16(400) = 6400\\
v_0 &= 80
\end{aligned}$$

Finally, the initial velocity $v_0 = 80$ feet per second.

**Displacement–Total Distance–Net Change**

87. Suppose that the velocity of a moving particle is $v(t) = t^2 - 11t + 24$ feet per second. Find both the displacement and total distance it travels between time $t = 0$ and $t = 10$ seconds.

Displacement$= \displaystyle\int_0^{10} t^2 - 11t + 24\ dt = \frac{t^3}{3} - \frac{11t^2}{2} + 24t\Big|_0^{10} = \left(\frac{1000}{3} - \frac{1100}{2} + 240\right) - 0 = \frac{1000}{3} - 550 + 240 = \frac{1000}{3} - 310 = \frac{1000}{3} - \frac{930}{3} = \boxed{\frac{70}{3}}$

The displacement in question is $\boxed{\frac{70}{3}}$.

Note that if you sketch the parabola $y = t^2 - 11t + 24$, it passes below the $x$-axis between $x = 3$ and $x = 8$.

Total Distance$=\int_0^{10} |t^2-11t+24|\ dt$

$$= \int_0^3 t^2-11t+24\ dt + \int_3^8 -(t^2-11t+24)\ dt + \int_8^{10} t^2-11t+24\ dt$$
$$= \frac{t^3}{3}-\frac{11t^2}{2}+24t\Big|_0^3 + \left(-\frac{t^3}{3}+\frac{11t^2}{2}-24t\right)\Big|_3^8 + \frac{t^3}{3}-\frac{11t^2}{2}+24t\Big|_8^{10}$$
$$= \left(9-\frac{99}{2}+72\right)-0+\left(-\frac{512}{3}+\frac{704}{2}-192\right)-\left(-9+\frac{99}{2}-72\right)$$
$$+\left(\frac{1000}{3}-\frac{1100}{2}+240\right)-\left(\frac{512}{3}-\frac{704}{2}+192\right)$$
$$= 81-\frac{99}{2}-\frac{512}{3}+352-192+81-\frac{99}{2}+\frac{1000}{3}-550+240-\frac{512}{3}+350-192$$
$$= 170-\frac{198}{2}-\frac{24}{3}$$
$$= 170-99-8=\boxed{63}$$

The total distance in question is $\boxed{63}$.

88. Supppose that water is pumped into an initially empty tank. The rate of water flow into the tank at time $t$ (in seconds) is $50-t$ liters per second. How much water flows into the tank during the first 30 seconds?

Using the Net Change Theorem, the amount of water in the tank between time $t=0$ and time $t=30$ is $W(30)-W(0)=\int_0^{30}$ rate of water flow $dt$.

Using the FTC, we simply integrate

$$\int_0^{30} 50-t\ dt = 50t-\frac{t^2}{2}\Big|_0^{30} = 1500-\frac{900}{2}=1500-450=1050.$$

The amount of water that flows into the tank during the first 30 seconds is $\boxed{1050}$ liters.

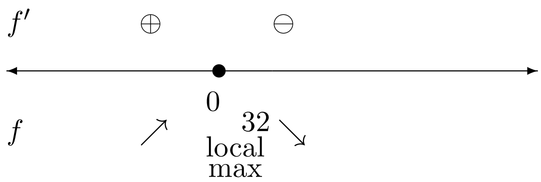
**Curve Sketching**

89. Use curve sketching techniques to present a detailed sketch for $f(x)=e^{-\frac{x^2}{2}}$.

- Domain: $f(x)$ has domain $(-\infty,\infty)$.
- Symmetry: $f(x)$ is an **even** function since $f(-x)=f(x) \Longrightarrow$ symmetry about $y$-axis.
- Vertical asymptotes:none.
- Horizontal asymptotes: There are horizontal asymptotes for this $f$ at $y=0$ since $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x)=0$.
- First Derivative Information:

Here $f'(x)=e^{-\frac{x^2}{2}}(-x)$.

The critical points occur where $f'$ is undefined (never here) or zero ($x=0$). Recall that the exponential is never zero. As a result, $x=0$ is the critical number. Using sign testing/analysis for $f'$,



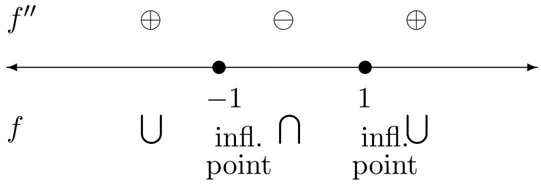
Therefore, $f$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. There is a local max at the point $(0, 1)$.

- Second Derivative Information

Next, $f''(x) = e^{-\frac{x^2}{2}}(-1) + (-x)e^{-\frac{x^2}{2}}(-x) = e^{-\frac{x^2}{2}}(x^2 - 1)$.

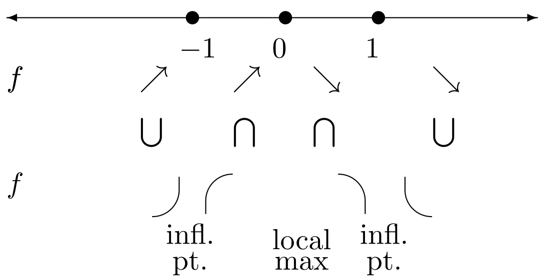
Possible inflection points occur when $f''$ is undefined (never here) or zero ($x = \pm 1$).

Using sign testing/analysis for $f''$,

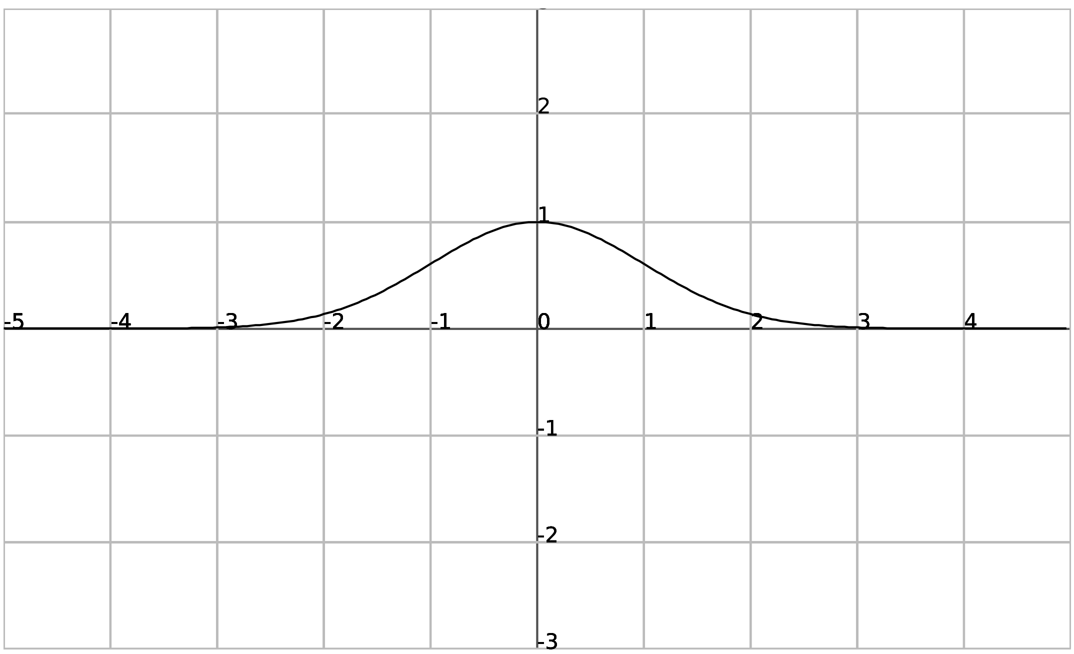



Therefore, $f$ is concave up on $(-\infty, -1)$ and $(1, \infty)$, whereas $f$ is concave down on $(-1, 1)$. There are inflection points at $(\pm 1, e^{-\frac{1}{2}})$.

- Piece the first and second derivative information together



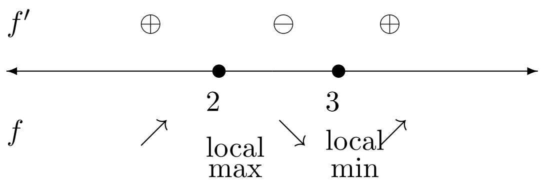

- Sketch:



90. Use curve sketching techniques to present a detailed sketch for $f(x) = (-x^2 + 3x - 3)e^{-x}$.

- Domain: $f(x)$ has domain $(-\infty, \infty)$.
- Vertical asymptotes:none.
- Horizontal asymptotes: There is horizontal asymptote for this $f$ at $y = 0$ since $\lim_{x\to\infty} f(x) = 0$ (will see why in Math 12) and $\lim_{x\to-\infty} f(x) = -\infty$.
- First Derivative Information:

Here $f'(x) = (-x^2 + 3x - 3)e^{-x}(-1) + e^{-x}(-2x + 3) = e^{-x}(x^2 - 5x + 6) = e^{-x}(x - 3)(x - 2)$. The critical points occur where $f'$ is undefined (never here) or zero ($x = 2$ and $x = 3$). Recall that the exponential is never zero. As a result, $x = 2$ and $x = 3$ are the critical numbers. Using sign testing/analysis for $f'$,



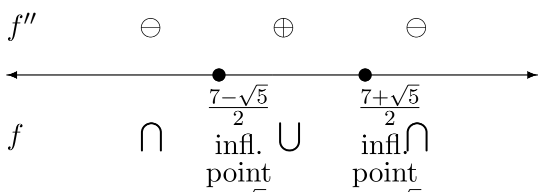

Therefore, $f$ is increasing on $(-\infty, 2)$ and $(3, \infty)$, and decreasing on $(2, 3)$. There is a local max at the point $\left(2, -e^{-2}\right)$ and local min at the point $\left(3, -3e^{-3}\right)$.

- Second Derivative Information

Next, $f''(x) = e^{-x}(2x - 5) + (x^2 - 5x + 6)e^{-x}(-1) = e^{-x}(-x^2 + 7x - 11)$.

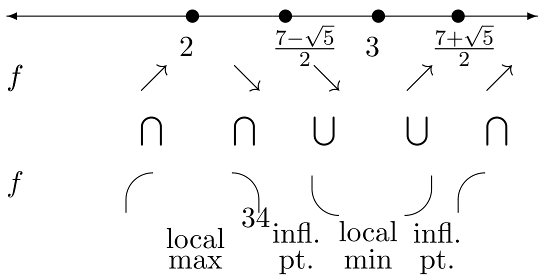
Possible inflection points occur when $f''$ is undefined (never here) or zero ($x = \frac{7\pm\sqrt{5}}{2}$).

Using sign testing/analysis for $f''$,




Therefore, $f$ is concave down on $(-\infty, \frac{7-\sqrt{5}}{2})$ and $(\frac{7+\sqrt{5}}{2}, \infty)$, whereas $f$ is concave down on $(\frac{7-\sqrt{5}}{2}, \frac{7+\sqrt{5}}{2})$. There are inflection points at $\left(\frac{7-\sqrt{5}}{2}, \left(-\left(\frac{7-\sqrt{5}}{2}\right)^2 + 3\left(\frac{7-\sqrt{5}}{2}\right) - 3\right)e^{-\frac{7-\sqrt{5}}{2}}\right) = \left(\frac{7-\sqrt{5}}{2}, -6 + 2\sqrt{5}\right)$ and $\left(\frac{7+\sqrt{5}}{2}, \left(-\left(\frac{7+\sqrt{5}}{2}\right)^2 + 3\left(\frac{7+\sqrt{5}}{2}\right) - 3\right)e^{-\frac{7+\sqrt{5}}{2}}\right) = \left(\frac{7+\sqrt{5}}{2}, -6 - 2\sqrt{5}\right)$.

- Piece the first and second derivative information together



• Sketch: