Math 111 ANSWER KEY (Extra Long) Practice Exam #3 Fall 2014

1. Compute $\int_1^5 5-2x-x^2\ dx$ using two different methods:

(a) Fundamental Theorem of Calculus

$$\int_1^5 5-2x-x^2\ dx = 5x-x^2-\frac{x^3}{3}\Big|_1^5 = 25-25-\frac{125}{3}-\left(5-1-\frac{1}{3}\right) = -\frac{124}{3}-4 = \boxed{-\frac{136}{3}}$$

(b) Limit Definition of the Definite Integral.

Here $a=1, b=5, \Delta x = \frac{b-a}{n} = \frac{4}{n}$, and $x_i = a+i\Delta x = 1+\frac{4i}{n}$

$$\int_1^5 5-2x-x^2\ dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x \quad = \lim_{n\to\infty}\sum_{i=1}^n f\left(1+\frac{4i}{n}\right)\frac{4}{n}$$

$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^n 5-2\left(1+\frac{4i}{n}\right)-\left(1+\frac{4i}{n}\right)^2$$

$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^n 5-2-\frac{8i}{n}-1-\frac{8i}{n}-\frac{16i^2}{n^2}$$

$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^n 2-\frac{16i}{n}-\frac{16i^2}{n^2}$$

$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^n 2-\frac{4}{n}\sum_{i=1}^n\frac{16i}{n}-\frac{4}{n}\sum_{i=1}^n\frac{16i^2}{n^2}$$

$$= \lim_{n\to\infty}\frac{8}{n}\sum_{i=1}^n 1-\frac{64}{n^2}\sum_{i=1}^n i-\frac{64}{n^3}\sum_{i=1}^n i^2$$

$$= \lim_{n\to\infty}\frac{8}{n}(n)-\frac{64}{n^2}\left(\frac{n(n+1)}{2}\right)-\frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} 8-\frac{64}{2}\left(\frac{n(n+1)}{n^2}\right)-\frac{64}{6}\left(\frac{n(n+1)(2n+1)}{n^3}\right)$$

$$= \lim_{n\to\infty} 8-32\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)-\frac{32}{3}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} 8-32(1)\left(1+\frac{1}{n}\right)-\frac{32}{3}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= 8-32-\frac{64}{3} = -24-\frac{64}{3} = \boxed{-\frac{136}{3}} \quad \text{Match}$$

2. Compute each of the following derivatives.

(a) $g'(x)$ where $g(x) = \displaystyle\int_x^4 \frac{\sin t}{e^t}\, dt$

$$g'(x) = \frac{d}{dx}\int_x^4 \frac{\sin t}{e^t}\, dt = -\frac{d}{dx}\int_4^x \frac{\sin t}{e^t}\, dt = \boxed{-\frac{\sin x}{e^x}} \text{ using FTC part 1.}$$

(b) $f''(x)$, where $f(x) = \dfrac{x^4}{e^x}$. Simplify here.

$$f'(x) = \frac{e^x(4x^3) - x^4 e^x}{(e^x)^2} = \frac{e^x(4x^3 - x^4)}{e^{2x}} = \frac{4x^3 - x^4}{e^x}$$

$$f''(x) = \frac{e^x(12x^2 - 4x^3) - (4x^3 - x^4)e^x}{e^{2x}} = \boxed{\frac{12x^2 - 8x^3 + x^4}{e^x}}$$

(c) $g'(x)$, where $g(x) = \dfrac{1}{\sin\sqrt{e^x + e^7}} + \dfrac{1}{e^{\sqrt{x^2 + 7\sin x}}} + \dfrac{1}{\sqrt{e^{x^2} + 7\sin x}}$. Do not simplify here.

$$g(x) = (\sin\sqrt{e^x + e^7})^{-1} + e^{-\sqrt{x^2+7\sin x}} + (e^{x^2} + 7\sin x)^{-\frac{1}{2}}$$

$$g'(x) = \boxed{-(\sin\sqrt{e^x + e^7})^{-2}\cos\sqrt{e^x + e^7}\left(\frac{1}{2\sqrt{e^x + e^7}}\right)e^x} \text{ (continued)}$$

$$\boxed{+e^{-\sqrt{x^2+7\sin x}}\left(-\frac{1}{2\sqrt{x^2 + 7\sin x}}\right)(2x + 7\cos x) - \frac{1}{2}(e^{x^2} + 7\sin x)^{-\frac{3}{2}}(e^{x^2}(2x) + 7\cos x)}$$

(d) $\dfrac{dy}{dx}$, if $\sin y + e^x = \sec x + \cos(e^9) - e^{xy}$.

Implicit differentiation on both sides.

$$\frac{d}{dx}(\sin y + e^x) = \frac{d}{dx}(\sec x + \cos(e^9) - e^{xy})$$

$$\cos y\frac{dy}{dx} + e^x = \sec x \tan x - e^{xy}\left[x\frac{dy}{dx} + y\right]$$

$$\cos y\frac{dy}{dx} + e^x = \sec x \tan x - xe^{xy}\frac{dy}{dx} - ye^{xy}$$

Isolate

$$\cos y\frac{dy}{dx} + xe^{xy}\frac{dy}{dx} = \sec x \tan x - ye^{xy} - e^x$$

Factor

$$(\cos y + xe^{xy})\frac{dy}{dx} = \sec x \tan x - ye^{xy} - e^x$$

Solve

$$\frac{dy}{dx} = \boxed{\frac{\sec x \tan x - ye^{xy} - e^x}{\cos y + xe^{xy}}}$$

3. Evaluate each of the following integrals. Simplify.

(a) $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\left(\frac{x}{2}\right)\, dx = 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \sin u\; du = -2\cos u\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{4}}$

$\displaystyle = -2\cos\left(\frac{\pi}{4}\right) + 2\cos\left(-\frac{\pi}{6}\right) = -2\left(\frac{\sqrt{2}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) = \boxed{\sqrt{3}-\sqrt{2}}$

Here $\boxed{\begin{aligned} u &= \frac{x}{2} \\ du &= \frac{1}{2}dx \\ 2du &= dx \end{aligned}}$ and $\boxed{\begin{aligned} x = -\frac{\pi}{3} &\implies u = -\frac{\pi}{6} \\ x = \frac{\pi}{2} &\implies u = \frac{\pi}{4} \end{aligned}}$

(b) $\displaystyle\int \frac{\sqrt{2}\ \sec^2(3x+4)}{\tan^2(3x+4)}\, dx = \frac{\sqrt{2}}{3}\int \frac{1}{u^2}\, du = \frac{\sqrt{2}}{3}\int u^{-2}\, du = \frac{\sqrt{2}}{3}\left(\frac{u^{-1}}{-1}\right) + C = \boxed{-\frac{\sqrt{2}}{3\tan(3x+4)} + C}$

Here $\boxed{\begin{aligned} u &= \tan(3x+4) \\ du &= \sec^2(3x+4)(3)dx \\ \tfrac{1}{3}du &= \sec^2(3x+4)dx \end{aligned}}$

(c) $\displaystyle\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}\ (1+\sin\sqrt{x}\ )^3}\, dx = 2\int_2^1 \frac{1}{u^3}\, du = 2\int_2^1 u^{-3}\, du = \frac{2u^{-2}}{-2}\Big|_2^1 = -\frac{1}{u^2}\Big|_2^1 = -1 + \frac{1}{4} = \boxed{-\frac{3}{4}}$

Here $\boxed{\begin{aligned} u &= 1+\sin\sqrt{x} \\ du &= \cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) dx \\ 2du &= \frac{\cos\sqrt{x}}{\sqrt{x}} dx \end{aligned}}$ and $\boxed{\begin{aligned} x = \tfrac{\pi^2}{4} &\implies u = 1+\sin\tfrac{\pi}{2} = 1+1 = 2 \\ x = \pi^2 &\implies u = 1+\sin\pi = 1+0 = 1 \end{aligned}}$

(d) $\displaystyle\int \frac{\cos x + \sin x}{\sqrt{\cos x - \sin x}}\, dx = -\int \frac{1}{\sqrt{u}}\, du = -\int u^{-\frac{1}{2}}\, du = -2\sqrt{u} + C = \boxed{-2\sqrt{\cos x - \sin x} + C}$

Here $\boxed{\begin{aligned} u &= \cos x - \sin x \\ du &= -\sin x - \cos x dx \\ -du &= \sin x + \cos x dx \end{aligned}}$

(e) $\displaystyle\int \frac{x^{\frac{7}{4}} + x^{-\frac{1}{3}}}{\sqrt{x}}\, dx = \int \frac{x^{\frac{7}{4}}}{x^{\frac{1}{2}}} + \frac{x^{-\frac{1}{3}}}{x^{\frac{1}{2}}}\, dx = \int \frac{x^{\frac{7}{4}}}{x^{\frac{2}{4}}} + \frac{x^{-\frac{2}{6}}}{x^{\frac{3}{6}}}\, dx = \int x^{\frac{5}{4}} + x^{-\frac{5}{6}}\, dx = \boxed{\frac{4}{9}x^{\frac{9}{4}} + 6x^{\frac{1}{6}} + C}$

(f) $\displaystyle\int \frac{5}{x^2\left(5+\frac{3}{x}\right)^{\frac{5}{3}}}\, dx = -\frac{5}{3}\int \frac{1}{u^{\frac{5}{3}}}\, du = -\frac{5}{3}\left(-\frac{3}{2}\right)u^{-\frac{2}{3}} + C = \boxed{\frac{5}{2\left(5+\frac{3}{x}\right)^{\frac{2}{3}}} + C}$

Here $\boxed{\begin{aligned} u &= 5 + \frac{3}{x} \\ du &= -\frac{3}{x^2} dx \\ -\frac{1}{3} du &= \frac{1}{x^2} dx \end{aligned}}$

(g) $\displaystyle\int_{-2}^{-1} \left(x - \frac{5}{x^3}\right)^2 \, dx = \int_{-2}^{-1} x^2 - \frac{10}{x^2} + \frac{25}{x^6} \, dx$

$\displaystyle = \frac{x^3}{3} + \frac{10}{x} - \frac{5}{x^5} \Big|_{-2}^{-1}$

$\displaystyle = -\frac{1}{3} - 10 + 5 - \left(\frac{8}{3} - 5 + \frac{5}{32}\right) = -\frac{1}{3} - 10 + 5 + \frac{8}{3} + 5 - \frac{5}{32} = \frac{7}{3} - \frac{5}{32} = \frac{224}{96} - \frac{15}{96} = \boxed{\frac{209}{96}}$

(h) $\displaystyle\int_{-3}^{-2} x \, (x+2)^7 \, dx = \int_{-1}^{0} (u-2) \, u^7 \, du = \int_{-1}^{0} u^8 - 2u^7 \, du = \frac{u^9}{9} - \frac{u^8}{4} \Big|_{-1}^{0}$

$\displaystyle = (0 - 0) - \left(-\frac{1}{9} - \frac{1}{4}\right) = \frac{1}{9} + \frac{1}{4} = \boxed{\frac{13}{36}}$

Here $\boxed{\begin{aligned} u &= x + 2 \Rightarrow x = u - 2 \\ du &= dx \end{aligned}}$ and $\boxed{\begin{aligned} x = -3 &\implies u = -1 \\ x = -2 &\implies u = 0 \end{aligned}}$

(i) $\displaystyle\int \frac{\sec(e^{-x}) \tan(e^{-x})}{e^x} \, dx = \int \sec u \tan u \, du = \sec u + C = \boxed{\sec(e^{-x}) + C}$

Here $\boxed{\begin{aligned} u &= e^{-x} \\ du &= -e^{-x} dx \end{aligned}}$

(j) $\displaystyle\int_1^4 \frac{1}{\sqrt{x} \, e^{1+\sqrt{x}}} \, dx = 2 \int_2^3 \frac{1}{e^u} \, du = 2 \int_2^3 e^{-u} \, du = -2e^{-u} \Big|_2^3$

$\displaystyle = -2e^{-3} - \left(-2e^{-2}\right) = -\frac{2}{e^3} + \frac{2}{e^2} = \boxed{\frac{2}{e^2} - \frac{2}{e^3}}$

Here $\boxed{\begin{aligned} u &= 1 + \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}} dx \\ 2du &= \frac{1}{\sqrt{x}} dx \end{aligned}}$ and $\boxed{\begin{aligned} x = 1 &\implies u = 2 \\ x = 4 &\implies u = 3 \end{aligned}}$

(k) $\displaystyle\int \frac{1}{e^{3x} \left(1 + e^{-3x}\right)^{\frac{2}{9}}} \, dx = -\frac{1}{3} \int \frac{1}{u^{\frac{2}{9}}} \, dx = -\frac{1}{3} \int u^{-\frac{2}{9}} \, dx = \frac{1}{3} \left(\frac{9}{7}\right) u^{\frac{7}{9}} = \boxed{\frac{3}{7} \left(1 + e^{-3x}\right)^{\frac{7}{9}} + C}$

Here $\boxed{\begin{aligned} u &= 1+e^{-3x} \\ du &= -3e^{-3x}dx \\ -\tfrac{1}{3}du &= \tfrac{1}{e^{3x}}dx \end{aligned}}$

(l) $\displaystyle\int e^x + \frac{1}{e^x} + x^e + \frac{1}{x^e} + \frac{x}{e} + \frac{e}{x^2} + ex + \frac{1}{e^3x^3}\ dx$

$\displaystyle= \int e^x + e^{-x} + x^e + x^{-e} + \frac{x}{e} + ex^{-2} + ex + \frac{1}{e^3}x^{-3}\ dx$

$\displaystyle= \boxed{e^x - e^{-x} + \frac{x^{e+1}}{e+1} + \frac{x^{-e+1}}{-e+1} + \frac{x^2}{2e} - ex^{-1} + e\frac{x^2}{2} + \frac{x^{-2}}{e^3(-2)} + C}$

(m) $\displaystyle\int \frac{e^x}{(1+e^x)^2}\ dx = \int \frac{1}{u^2}\ du = \int u^{-2}\ du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = \boxed{-\frac{1}{1+e^x} + C}$

Here $\boxed{\begin{aligned} u &= 1+e^x \\ du &= e^x dx \end{aligned}}$

(n) $\displaystyle\int \frac{(1+e^x)^2}{e^x}\ dx = \int \frac{1+2e^x+e^{2x}}{e^x}\ dx = \int \frac{1}{e^x} + \frac{2e^x}{e^x} + \frac{e^{2x}}{e^x}\ dx$

$\displaystyle= \int e^{-x} + 2 + e^x\ dx = \boxed{-e^{-x} + 2x + e^x + C}$

4. Consider an object travelling with velocity $v(t) = 3t - 9$ meters per second.

(a) Compute the **displacement** for the object from time $t = 1$ to $t = 4$.

Displacement$\displaystyle= \int_1^4 3t - 9\ dt = \frac{3t^2}{2} - 9t\Big|_1^4 = (24-36) - \left(\frac{3}{2} - 9\right)$

$\displaystyle= 24 - 36 + 9 - \frac{3}{2} = -3 - \frac{3}{2} = \boxed{-\frac{9}{2}}$

(b) Compute the **total distance** travelled by the object from time $t = 1$ to $t = 4$.

Total Distance$\displaystyle= \int_1^4 |3t-9|\ dt = \int_1^3 |3t-9|\ dt + \int_3^4 |3t-9|\ dt$

$\displaystyle= \int_1^3 -(3t-9)\ dt + \int_3^4 3t-9\ dt = \int_1^3 9-3t\ dt + \int_3^4 3t-9\ dt = 9t - \frac{3t^2}{2}\Big|_1^3 + \frac{3t^2}{2} - 9t\Big|_3^4$

$\displaystyle= 27 - \frac{27}{2} - \left(9 - \frac{3}{2}\right) + 24 - 36 - \left(\frac{27}{2} - 27\right) = 27 - \frac{27}{2} - 9 + \frac{3}{2} - 12 - \frac{27}{2} + 27$

$\displaystyle= -\frac{51}{2} + 33 = \boxed{\frac{15}{2}}$

5. Let R be the region bounded between $y = 9 - x^2$ and the x-axis. Find the area of the largest rectangle that can be inscribed in the region R. Two vertices of the rectangle lie on the x-axis. Its other two vertices above the x-axis lie on the parabola $y = 9 - x^2$.

(Remember to state the domain of the function you are computing extreme values for.)

Diagram: Draw a diagram. See me for a sketch.

Consider the point (x, y) on the parabola.

Variables:

Let x=x-coordinate of the point on the parabola. Let y=y-coordinate of the point on the parabola. Let A=Area of the inscribed rectangle.

Equations: $y = 9 - x^2$ Fixed!

Area $A = 2xy = 2x(9 - x^2) = 18x - 2x^3$ must be maximized.

The (common-sense-bounds)domain of A is $\{x : 0 \leq x \leq 3\}$ to keep $x \geq 0$ and $9 - x^2 \geq 0$.

Differentiate:

Compute $A' = 18 - 6x^2$. Setting $A' = 0$ we solve for $x^2 = 3$ or $x = \sqrt{3}$. (We take the positive square root here because we're talking distance.)

Sign-Testing:

Sign-testing the critical number does indeed yield a maximum for the area function.

A'	$\oplus$	$\ominus$
A	$\nearrow \sqrt{3} \searrow$	

MAX

Since $x = \sqrt{3}$ then $y = 9 - (\sqrt{3})^2 = 6$.

Answer:

As a result, the largest area that occurs is $A = 2xy = 2\sqrt{3}(6) = 12\sqrt{3}$ square units.

6. Compute the area bounded between $y = e^x$, $y = x$, $x = 0$ and $x = 1$.

See me for a sketch.

$$\text{Area}=\int_0^1 e^x - x dx = e^x - \frac{x^2}{2}\Bigg|_0^1 = e - \frac{1}{2} - (e^0 - 0) = e - \frac{1}{2} - 1 = \boxed{e - \frac{3}{2}}$$

7. A ball is thrown upwards from the top edge of a building with initial velocity 128 feet per second. The velocity of the ball at impact with the ground is -160 feet per second. How tall is the building?

Using the equations of motion:

$a(t) = -32$ ft/sec^2

$v(t) = -32t + v_0 = -32t + 128$ ft/sec

$s(t) = -16t^2 + v_0 t + s_0 = -16t^2 + 128t + s_0$

We are looking to solve for the initial position s_0.

We know $v\left(t_{\text{impact}}\right) = -32t_{\text{impact}} + 128 \stackrel{\text{set}}{=} -160$

Solving this for t_{impact} we find

$32t_{\text{impact}} = 288 \Rightarrow t_{\text{impact}} = 9$ seconds.

Next we have $s\left(t_{\text{impact}}\right) = s(9) = -16(9)^2 + 128(9) + s_0 = 0 \Rightarrow -1296 + 1152 + s_0 = 0$. Finally, $\boxed{s_0 = 144}$ feet above the ground.

The height of the building was 144 feet above the ground.