Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 111

Midterm Exam #3

December 5, 2014

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or webpages, or other aids are permitted.

• You need *not* simplify algebraically complicated answers for the derivative section. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, e^0 should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		20
2		25
3		10
4		15
5		20
6		10
Total		100

1. [20 Points] Compute each of the following derivatives.

(a) $f'(x)$ where $f(x) = \int_{\cos x}^{7} \frac{1}{e^t + 5t}\, dt$

$$f'(x) = \frac{d}{dx}\int_{\cos x}^{7} \frac{1}{e^t+5t}\,dt = \frac{d}{dx}\left[-\int_{7}^{\cos x} \frac{1}{e^t+5t}\,dt\right]$$

$$= -\left(\frac{1}{e^{\cos x}+5\cos x}\right)(-\sin x) = \boxed{\frac{\sin x}{e^{\cos x}+5\cos x}}$$

(b) y' where $y = e^9 + \frac{e^{9x}}{9} + e^{\frac{9}{x}} - \frac{1}{9e^x} + \frac{9}{e^x} + \frac{e^x}{e^{9x}} + e^{9-x} + \frac{1}{9-e^x} + e^x \cdot e^{9x} + e^{\sqrt{9-x}}$.

Do **not** simplify here.

Rewrite

$$y = e^9 + \frac{1}{9}e^{9x} + e^{9/x} - \frac{1}{9}e^{-x} + 9e^{-x} + e^{-8x} + e^{9-x} + (9-e^x)^{-1} + e^{10x} + e^{\sqrt{9-x}}$$

$$y' = \boxed{0 + \frac{1}{\cancel{9}}\cancel{9}e^{9x} + e^{9/x}\left(-\frac{9}{x^2}\right) + \frac{1}{9}e^{-x} - 9e^{-x} - 8e^{-8x} + e^{9-x}(-1) - (9-e^x)^{-2}(-e^x) + 10e^{10x} + e^{\sqrt{9-x}}\left(\frac{1}{2\sqrt{9-x}}\right)(-1)}.$$

1. (Continued) Compute the following derivative.

(c) $\dfrac{dy}{dx}$ where $e^{x^2y} + e = y^2 + 1 + \tan x$. Simplify where possible.

$$\frac{d}{dx}\left[e^{x^2y} + e\right] = \frac{d}{dx}\left[y^2 + 1 + \tan x\right]$$

$$e^{x^2y}\left[x^2\frac{dy}{dx} + y(2x)\right] + 0 = 2y\frac{dy}{dx} + 0 + \sec^2 x$$

$$x^2e^{x^2y}\frac{dy}{dx} + 2xye^{x^2y} = 2y\frac{dy}{dx} + \sec^2 x$$

$$x^2e^{x^2y}\frac{dy}{dx} - 2y\frac{dy}{dx} = \sec^2 x - 2xye^{x^2y}$$

$$\left[x^2e^{x^2y} - 2y\right]\frac{dy}{dx} = \sec^2 x - 2xye^{x^2y}$$

Solve $\dfrac{dy}{dx} = \boxed{\dfrac{\sec^2 x - 2xye^{x^2y}}{x^2e^{x^2y} - 2y}}$

2. [25 Points] Compute each of the following integrals. Simplify your answers.

(a) $\int \left(e^{3x} + \frac{1}{e^{2x}}\right)\left(e^{x} + \frac{1}{e^{5x}}\right)\, dx = \int e^{4x} + e^{-2x} + e^{-x} + e^{-7x}\, dx$

$$= \frac{e^{4x}}{4} + \frac{e^{-2x}}{-2} + \frac{e^{-x}}{-1} + \frac{e^{-7x}}{-7} + C$$

$$= \boxed{\frac{1}{4}e^{4x} - \frac{1}{2e^{2x}} - \frac{1}{e^{x}} - \frac{1}{7e^{7x}} + C}$$

(b) $\int \frac{1}{e^{4x}(1+e^{-4x})^4}\, dx = -\frac{1}{4}\int \frac{1}{u^4}\, du = -\frac{1}{4}\int u^{-4}\, du = -\frac{1}{4}\left(\frac{u^{-3}}{-3}\right) + C$

$$\boxed{\begin{aligned} u &= 1+e^{-4x} \\ du &= -4e^{-4x}\,dx \\ -\tfrac{1}{4}\,du &= e^{-4x}\,dx \end{aligned}}$$

$$= \frac{1}{12u^3} + C = \boxed{\frac{1}{12(1+e^{-4x})^3} + C}$$

(c) $\int_{\pi}^{3\pi} \sin\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)\, dx = 3\int_{\sqrt{3}/2}^{0} u\, du = 3\frac{u^2}{2}\Big|_{\sqrt{3}/2}^{0} = \frac{3}{2}\left[0 - \left(\frac{\sqrt{3}}{2}\right)^2\right]$

$$\boxed{\begin{aligned} u &= \sin\left(\tfrac{x}{3}\right) \\ du &= \cos\left(\tfrac{x}{3}\right)\left(\tfrac{1}{3}\right)dx \\ 3\,du &= \cos\left(\tfrac{x}{3}\right)dx \end{aligned}}
\qquad
\boxed{\begin{aligned} x = \pi &\Rightarrow u = \sin(\pi/3) = \sqrt{3}/2 \\ x = 3\pi &\Rightarrow u = \sin\pi = 0 \end{aligned}}$$

$$= \frac{3}{2}\left[-\frac{3}{4}\right]$$

$$= \boxed{-\frac{9}{8}}$$

2. [Continued] Compute each of the following integrals. Simplify your answers.

(d) $\int_{-2}^{-1} x\,(x+1)^7\,dx$ $= \int_{-1}^{0} (u-1)u^7\,du = \int_{-1}^{0} u^8 - u^7\,du = \frac{u^9}{9} - \frac{u^8}{8}\Big|_{-1}^{0}$

$U = X+1 \overset{\text{invert}}{\Rightarrow} X = u-1$

$du = dx$

$X = -2 \Rightarrow u = -1$

$X = -1 \Rightarrow u = 0.$

$= (0-0) - \left(\frac{(-1)^9}{9} - \frac{(-1)^8}{8}\right)$

$= -\left(-\frac{1}{9} - \frac{1}{8}\right) = \frac{1}{9} + \frac{1}{8} = \frac{8}{72} + \frac{9}{72} = \boxed{\frac{17}{72}}$

(e) $\int \frac{x^{\frac{3}{5}} - x^{\frac{1}{4}}}{\sqrt{x}}\,dx$ $= \int \frac{X^{3/5}}{\sqrt{X}} - \frac{X^{1/4}}{\sqrt{X}}\,dx = \int \frac{X^{6/10}}{X^{5/10}} - \frac{X^{1/4}}{X^{1/2}}\,dX$

$= \int X^{1/10} - X^{-1/4}\,dx$

$= \frac{X^{11/10}}{11/10} - \frac{X^{3/4}}{3/4} + C$

$= \boxed{\frac{10}{11} X^{11/10} - \frac{4}{3} X^{3/4} + C}$

3. [10 Points] Find the function $f(x)$ that satisfies $f'(x) = \dfrac{\sec^2 x}{\sqrt{3+\tan x}}$ and $f\left(\frac{\pi}{4}\right) = 9$.

$$f(x) = \int \frac{\sec^2 x}{\sqrt{3+\tan x}}\,dx = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = 2\sqrt{u} + C$$

$$\boxed{\begin{aligned} u &= 3+\tan x \\ du &= \sec^2 x\,dx \end{aligned}}$$

$$= 2\sqrt{3+\tan x} + C.$$

Use Condition.

$$f(\pi/4) = 2\sqrt{3+\underbrace{\tan(\pi/4)}_{1}} + C \overset{\text{set}}{=} 9.$$

$$2\sqrt{4} + C = 9$$

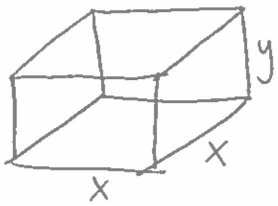
$$4 + C = 9 \Rightarrow C = 5.$$

Finally, $f(x) = \boxed{2\sqrt{3+\tan x} + 5}$

4. [15 Points] You need to construct a box with a square base with a fixed volume of 24 cubic feet. The material for the bottom and top costs \$3 per square foot, and the material for the sides costs \$1 per square foot. What are the **dimensions** that minimize the cost required to build such a box? What is that **minimum cost**?

(Don't forget to state the common sense bounds, that is, the domain of the function that you are maximizing or minimizing.)

• Diagram



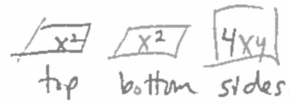
• Variables: Let x = length of base of box

y = height of box

C = Cost of Materials.

V = Volume.

• Equations



$$V = x^2 y \overset{set}{=} 24 \Rightarrow y = 24/x^2$$

$$C = \$3 \cdot x^2 + \$3 x^2 + \$1 \cdot 4xy$$

$$= 6x^2 + 4xy$$

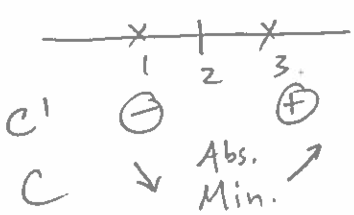
$$= 6x^2 + 4x\left(24/x^2\right) = 6x^2 + \frac{96}{x}$$ Domain $\{x \mid x > 0\}$

• Differentiate.

$$C' = 12x - 96/x^2 \overset{set}{=} 0.$$

$$12x = 96/x^2 \Rightarrow x^3 = 96/12 = 8 \Rightarrow x = 2$$ Critical #.

• Sign Testing into C'



$x = 2 \Rightarrow y = \frac{24}{4} = 6.$

$C(2) = 6(2)^2 + \frac{96}{2}$

$= 24 + 48 = 72.$

• Answer: The Dimensions that minimize Cost are $\boxed{2 \times 2 \times 6}$ feet with the Minimal Cost $\boxed{\$72}$

5. [20 Points] Compute $\int_1^3 5 - x^2\,dx$ using each of the following two different methods:

(a) Fundamental Theorem of Calculus.

(b) Limit definition of the definite integral ***.

*** Recall $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, and $\sum_{i=1}^n 1 = n$

(a) $\int_1^3 5-x^2\,dx = 5x - x^3/3 \Big|_1^3 = 15-9-(5-1/3) = 6-5+1/3 = 1+1/3 = \boxed{\frac{4}{3}}$

(b) Here $a=1,\ b=3,\ \Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n},\quad x_i = a+i\Delta x = 1+i\left(\frac{2}{n}\right) = 1+2i/n.$

$$\int_1^3 5-x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^n f(1+2i/n)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^n\left[5-(1+2i/n)^2\right]\frac{2}{n}$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^n 5-\left(1+4i/n+4i^2/n^2\right) = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^n \underbrace{5-1}_{4}-4i/n-4i^2/n^2$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^n 4 - \frac{2}{n}\sum_{i=1}^n 4i/n - \frac{2}{n}\sum_{i=1}^n 4i^2/n^2$$

$$= \lim_{n\to\infty} 8/n\sum_{i=1}^n 1 - 8/n^2\sum_{i=1}^n i - 8/n^3\sum_{i=1}^n i^2$$

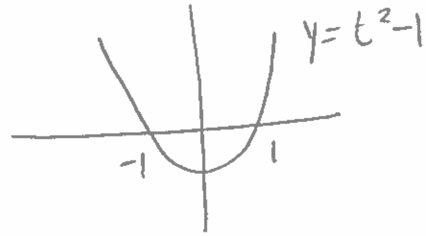
$$= \lim_{n\to\infty}\frac{8}{\cancel{n}}(\cancel{n}) - \frac{8}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right].$$

$$= \lim_{n\to\infty} 8 - 4\left(\cancel{\frac{n}{n}}\right)\left(\frac{n+1}{n}\right) - \frac{8}{6}\left(\cancel{\frac{n}{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} 8-4(1+\cancel{1/n}^{\,0}) - 4/3\,(1+\cancel{1/n}^{\,0})(2+\cancel{1/n}^{\,0})$$

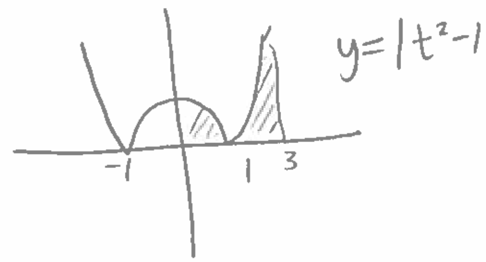
$$= 8-4-4/3(2) = 8-4-8/3 = 4-8/3 = 12/3-8/3 = \boxed{4/3}$$ Match!

6. [10 Points] A moving object has velocity $v(t) = t^2 - 1$ feet per second, at time t seconds. Compute the **Total Distance** travelled by this object from time $t = 0$ to $t = 3$ seconds.



$$\text{Total Distance} = \int_0^3 |t^2-1|\, dt$$

$$= \int_0^1 |t^2-1|\, dt + \int_1^3 |t^2-1|\, dt$$



$$= \int_0^1 \underbrace{-(t^2-1)}_{1-t^2}\, dt + \int_1^3 t^2-1\, dt$$

OR

$$|t^2-1| = \begin{cases} t^2-1 & \text{if } t^2-1 \geq 0. \\ -(t^2-1) & \text{if } t^2-1 < 0. \end{cases}$$

$$= \begin{cases} t^2-1 & \text{if } t \leq -1 \text{ or } t \geq 1 \\ -(t^2-1) & \text{if } -1 < t < 1 \end{cases}$$

$$= t - t^3/3 \Big|_0^1 + t^3/3 - t \Big|_1^3$$

$$= 1 - 1/3 - (0-0) + (9-3) - \underbrace{\left(\tfrac{1}{3} - 1\right)}_{-2/3}$$

$$= 2/3 + 6 + 2/3$$

$$= 4/3 + 6$$

$$= 4/3 + 18/3 = \boxed{\frac{22}{3}}$$

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute the following limit:

$$\lim_{n\to\infty} \frac{5}{n}\left(\sqrt{4+\frac{5}{n}}+\sqrt{4+\frac{10}{n}}+\sqrt{4+\frac{15}{n}}+\sqrt{4+\frac{20}{n}}+\ldots+\sqrt{4+5}\right)$$

$\Delta x = 5/n = \frac{b-a}{n}$

$x_i = a + i\,\Delta x = 4 + i\left(5/n\right).$

$\Rightarrow a = 4 \Rightarrow b = 9.$

$f(x) = \sqrt{x}.$

$= \int_4^9 \sqrt{x}\, dx$

$= \frac{2}{3} x^{3/2} \Big|_4^9$

$= \frac{2}{3}\left[(9)^{3/2} - (4)^{3/2}\right]$

$= 2/3\left[(\underbrace{\sqrt{9}}_{3})^3 - (\underbrace{\sqrt{4}}_{2})^3\right]$

$= 2/3\left[27 - 8\right]$

$= 2/3\,(19) = \boxed{\frac{38}{3}}$