Math 111 Midterm Exam #3 November 30, 2012

Answer Key

1. [20 Points] **Differentiate** each of the following functions. You **do not** need to simplify your answers.

(a) $f(x) = \int_{\sec x}^{7} \sqrt{\cos t + 7e^t}\, dt$

$$f'(x) = \frac{d}{dx}\int_{\sec x}^{7} \sqrt{\cos t + 7e^t}\, dt = -\frac{d}{dx}\int_{7}^{\sec x} \sqrt{\cos t + 7e^t}\, dt = \boxed{-\sqrt{\cos(\sec(x)) + 7e^{\sec x}} \cdot \sec x \tan x}$$

(b) $f(x) = \tan\left(e^x + \sqrt{x}\right) + e^{\tan\sqrt{x}} + \sqrt{e^x + \tan x}$

$$f'(x) = \boxed{\sec^2\left(e^x + \sqrt{x}\right)\left(e^x + \frac{1}{2\sqrt{x}}\right) + e^{\tan\sqrt{x}}\sec^2\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) + \frac{1}{2\sqrt{e^x + \tan x}}\left(e^x + \sec^2 x\right)}$$

(c) $f(x) = e^x + x^e + ex + e^e + e^{(e^x)} + (x^e)^e + e^{\frac{1}{x}} - \frac{1}{e^x}$

$= e^x + x^e + ex + e^e + e^{(e^x)} + x^{e^2} + e^{\frac{1}{x}} - e^{-x}.$

$$f'(x) = \boxed{e^x + ex^{e-1} + e + 0 + e^{(e^x)}e^x + e^2x^{e^2-1} + e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) - e^{-x}(-1)}$$

2. [25 Points] Compute each of the following integrals. Simplify your answers.

(a) $$\int\left(e^{7x} + \frac{1}{e^{4x}}\right)^2 dx = \int\left(e^{7x} + \frac{1}{e^{4x}}\right)\left(e^{7x} + \frac{1}{e^{4x}}\right) dx = \int e^{14x} + 2e^{3x} + e^{-8x}\, dx$$

$$= \boxed{\frac{1}{14}e^{14x} + \frac{2}{3}e^{3x} - \frac{1}{8}e^{-8x} + C}$$

(b) $$\int_0^1 \frac{e^x}{\sqrt{e^x + 8}}\, dx = \int_9^{e+8} \frac{1}{\sqrt{u}}\, du = \int_9^{e+8} u^{-\frac{1}{2}}\, du = 2\sqrt{u}\Big|_9^{e+8} = 2\sqrt{e+8} - 2\sqrt{9} = \boxed{2\sqrt{e+8} - 6}$$

Here $\boxed{\begin{aligned} u &= e^x + 8 \\ du &= e^x dx \end{aligned}}$ and $\boxed{\begin{aligned} x = 0 &\implies u = e^0 + 8 = 1 + 8 = 9 \\ x = 1 &\implies u = e^1 + 8 = e + 8 \end{aligned}}$

(c) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{\sin^3 x}\, dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{u^3}\, du = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} u^{-3}\, du = \frac{u^{-2}}{-2}\Bigg|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = -\frac{1}{2u^2}\Bigg|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= -\frac{1}{2\left(\frac{\sqrt{3}}{2}\right)^2} - \left(-\frac{1}{2\left(\frac{1}{2}\right)^2}\right) = -\frac{1}{2\left(\frac{3}{4}\right)} + \left(\frac{1}{2\left(\frac{1}{4}\right)}\right) = -\frac{2}{3} + 2 = \boxed{\frac{4}{3}}$$

Here $\boxed{\begin{aligned} u &= \sin x \\ du &= \cos x dx \end{aligned}}$ and $\boxed{\begin{aligned} x = \frac{\pi}{6} &\implies u = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\ x = \frac{\pi}{3} &\implies u = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \end{aligned}}$

(d) $\displaystyle\int x(x-1)^{\frac{5}{7}}\, dx = \int (u+1)u^{\frac{5}{7}}\, du = \int u^{\frac{12}{7}} + u^{\frac{5}{7}}\, du = \frac{7}{19}u^{\frac{19}{7}} + \frac{7}{12}u^{\frac{12}{7}} + C$

$= \boxed{\frac{7}{19}(x-1)^{\frac{19}{7}} + \frac{7}{12}(x-1)^{\frac{12}{7}} + C}$

Here $\boxed{\begin{aligned} u &= x-1 \implies x = u+1 \\ du &= dx \end{aligned}}$ Inverted Substitution

3. [10 Points] Find the function $f(x)$ that satisfies $f'(x) = \dfrac{e^{\sqrt{\tan x}}\ \sec^2 x}{\sqrt{\tan x}}$ and $f\left(\frac{\pi}{4}\right) = 1$.

$$f(x) = \int \frac{e^{\sqrt{\tan x}}\ \sec^2 x}{\sqrt{\tan x}}\, dx = 2\int e^u\, du = 2e^u + C = 2e^{\sqrt{\tan x}} + C$$

now use the initial condition:

$$f\left(\frac{\pi}{4}\right) = 2e^{\sqrt{\tan\left(\frac{\pi}{4}\right)}} + C \stackrel{\text{set}}{=} 1 \implies 2e + C = 1 \implies C = 1 - 2e$$

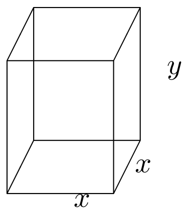
Finally, $f(x) = \boxed{2e^{\sqrt{\tan x}} + (1-2e)}$

Here $\boxed{\begin{aligned} u &= \sqrt{\tan x} \\ du &= \frac{1}{2\sqrt{\tan x}}\sec^2 x dx \\ 2du &= \frac{1}{\sqrt{\tan x}}\sec^2 x dx \end{aligned}}$

4. [15 Points] You need to construct a box with a square base with a fixed volume of 24 cubic feet. The material for the bottom and top costs \$3 per square foot, and the material for the sides costs \$1 per square foot. What are the **dimensions** that minimize the cost required to build such a box? What is that **minimum cost**?

(Don't forget to state the common sense bounds, that is, the domain of the function that you are maximizing or minimizing.)

- Diagram:



• Variables:

Let x =length of side on base of box.
Let y =height of box.
Let C =Cost for amount of material (surface area).
Let V =volume of box.

• Equations:

Volume $V = x^2y = 24$. (Fixed!) $\implies y = \dfrac{24}{x^2}$

Then the Cost of materials is given by

$$\begin{aligned} C &= \text{cost of base} + \text{cost of top} + \text{cost of 4 sides} \\ &= x^2(\$3) + x^2(\$3) + 4xy(\$1) \\ &= 6x^2 + 4xy \end{aligned}$$

Substitute,

$C = 6x^2 + 4x\left(\dfrac{24}{x^2}\right) = 6x^2 + \dfrac{96}{x}$ (Minimize!)

The (common-sense-bounds)domain of V is $\{x : 0 < x < \infty\}$.

• Minimize:

Next $C' = 12x - \dfrac{96}{x^2}$. Setting $C' = 0$ we solve $x^3 = 8$ for $x = 2$ as the critical number.

Sign-testing the critical number does indeed yield a minimum for the cost function.

C′ ⊖ ⊕
C ↘ 2 ↗
MIN

Here $x = 2 \implies y = \dfrac{24}{4} = 6$ and $C(4) = 6(2)^2 + \dfrac{96}{2} = 24 + 48 = 72$.

• Answer: As a result, the dimensions that minimize the cost are 2 feet by 2 feet by 6 feet. The minimum cost will be \$72.

5. [20 Points] Compute $\displaystyle\int_1^3 x^2 - 3x\ dx$ using each of the following **two** different methods:

(a) Fundamental Theorem of Calculus.

$$\int_1^3 x^2 - 3x\ dx = \frac{x^3}{3} - \frac{3x^2}{2}\bigg|_1^3 = \frac{3^3}{3} - \frac{3(3)^2}{2} - \left(\frac{1}{3} - \frac{3}{2}\right) = 9 - \frac{27}{2} - \frac{1}{3} + \frac{3}{2} = 9 - \frac{24}{2} - \frac{1}{3} = 9 - 12 - \frac{1}{3}$$

$$= -3 - \frac{1}{3} = \boxed{-\frac{10}{3}}$$

(b) Riemann Sums and the limit definition of the definite integral ***.

*** Recall $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, and $\sum_{i=1}^{n} 1 = n$

(b) Here $a = 1, b = 3, \Delta x = \frac{b-a}{n} = \frac{2}{n}$, and $x_i = a + i\Delta x = 1 + \frac{2i}{n}$

$$
\begin{aligned}
\int_1^3 x^2 - 3x \, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1 + \left(\frac{2i}{n}\right)\right)^2 - 3\left(1 + \frac{2i}{n}\right)\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} 1 + \frac{4i}{n} + \frac{4i^2}{n^2} - 3 - \frac{6i}{n} \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \frac{4i^2}{n^2} - \frac{2i}{n} - 2 \\
&= \lim_{n\to\infty} \left(\frac{2}{n} \sum_{i=1}^{n} \frac{4i^2}{n^2} - \frac{2}{n} \sum_{i=1}^{n} \frac{2i}{n} - \frac{2}{n} \sum_{i=1}^{n} 2\right) \\
&= \lim_{n\to\infty} \left(\frac{8}{n^3} \sum_{i=1}^{n} i^2 - \frac{4}{n^2} \sum_{i=1}^{n} i - \frac{4}{n} \sum_{i=1}^{n} 1\right) \\
&= \lim_{n\to\infty} \left(\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2}\frac{n(n+1)}{2} - \frac{4}{n} \cdot n\right) \\
&= \lim_{n\to\infty} \left(\frac{8}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - \frac{4}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) - 4\right) \\
&= \lim_{n\to\infty} \left(\frac{8}{6}(1)\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \frac{4}{2}(1)\left(1 + \frac{1}{n}\right) - 4\right) \\
&= \frac{8}{3} - 2 - 4 \\
&= \frac{8}{3} - 6 = \boxed{-\frac{10}{3}} \quad \text{Match}
\end{aligned}
$$

6. [10 Points] A moving object has velocity $v(t) = 2t - 6$ feet per second, at time t seconds. Compute the **Total Distance** travelled by this object from time $t = 0$ to $t = 4$ seconds.

Total Distance=

$$\begin{aligned}
\int_0^4 |v(t)|\ dt &= \int_0^4 |2t-6|\ dt = \int_0^3 |2t-6|\ dt + \int_3^4 |2t-6|\ dt \\
&= \int_0^3 -(2t-6)\ dt + \int_3^4 2t-6\ dt = \int_0^3 6-2t\ dt + \int_3^4 2t-6\ dt \\
&= 6t - t^2\Big|_0^3 + t^2 - 6t\Big|_3^4 = (18-9) - (0) + (16-24) - (9-18) \\
&= 9 - 8 + 9 = \boxed{10} \text{ feet}
\end{aligned}$$

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1

Compute $\displaystyle\lim_{n\to\infty} \frac{e^{\left(1+\frac{1}{n}\right)} + e^{\left(1+\frac{2}{n}\right)} + e^{\left(1+\frac{3}{n}\right)} + \ldots + e^2}{n}$

$= \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} e^{\left(1+\frac{i}{n}\right)} \left(\frac{1}{n}\right) = \int_0^1 e^{x+1}\ dx = e^{x+1}\Big|_0^1 = \boxed{e^2 - e}$ Note: this limit/sum was the Riemann Sum for that definite integral. Or same as $\displaystyle\int_1^2 e^x\ dx$.

OPTIONAL BONUS #2

Compute $\displaystyle\int \sin^3 x\ dx = \int \sin^2 x \sin x\ dx$

$= \displaystyle\int (1-\cos^2 x)\sin x\ dx = -\int 1 - u^2\ du = -\left(u - \frac{u^3}{3}\right) + C = -\left(\cos x - \frac{\cos^3 x}{3}\right) + C$

$= \boxed{-\cos x + \dfrac{\cos^3 x}{3} + C}$

Here $\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x\ dx \\ -du &= \sin x\ dx \end{aligned}}$