Math 11-March 26, 2010

ANSWER KEY for Midterm Exam #2

1. [12 Points] Compute each of the following **limits**. Justify your answers.

(a) $$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \frac{3\sin(3x)}{3x} = 3\lim_{x\to 0} \frac{\sin(3x)}{3x} = 3 \cdot 1 = \boxed{3}$$

(b) $$\lim_{x\to 0} \frac{x^2 - x^3}{\cos x \cdot \sin^2(7x)} = \lim_{x\to 0} \frac{x^2(1-x)}{\cos x \cdot \sin^2(7x)} = \lim_{x\to 0} \frac{x}{\sin(7x)} \cdot \frac{x}{\sin(7x)} \cdot \frac{1-x}{\cos x}$$

$$= \lim_{x\to 0} \frac{7x}{7\sin(7x)} \cdot \frac{7x}{7\sin(7x)} \cdot \frac{1-x}{\cos x} = \lim_{x\to 0} \frac{1}{7} \cdot \frac{1}{7} = \boxed{\frac{1}{49}}$$

(c) $$\lim_{x\to\infty} \frac{x^2+9}{x^7-4x+7} = \lim_{x\to\infty} \frac{x^2+9}{x^7-4x+7} \cdot \frac{\frac{1}{x^7}}{\frac{1}{x^7}} = \lim_{x\to\infty} \frac{\dfrac{1}{x^5} + \dfrac{9}{x^7}}{1 - \dfrac{4}{x^6} + \dfrac{7}{x^7}} = \frac{0}{1} = \boxed{0}$$

(d) $$\lim_{x\to\infty} \frac{19x^7 + 4x^5 - 8}{3x^7 + 2010x} = \lim_{x\to\infty} \frac{19x^7 + 4x^5 - 8}{3x^7 + 2010x} \cdot \frac{\frac{1}{x^7}}{\frac{1}{x^7}} = \lim_{x\to\infty} \frac{19 + \dfrac{4}{x^2} - \dfrac{8}{x^7}}{3 + \dfrac{2010}{x^6}} = \boxed{\frac{19}{3}}$$

2. [18 Points] **Differentiate** each of the following functions. You **do not** need to simplify your answers. Please do not waste time simplifying your derivative.

(a) $f(x) = \sqrt{\cos\left(\frac{1}{x}\right)}$ $$f'(x) = \frac{1}{2\sqrt{\cos\left(\frac{1}{x}\right)}}\left(-\sin\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right)$$

(b) $f(x) = (9-x^2)^8(x^3-6x)^9$

$$f'(x) = (9-x^2)^8 9(x^3-6x)^8(3x^2-6) + (x^3-6x)^9 8(9-x^2)^7(-2x)$$

(c) $$f(x) = \frac{1}{\left(\tan x + \frac{1}{x^2}\right)^{\frac{5}{7}}} = \left(\tan x + \frac{1}{x^2}\right)^{-\frac{5}{7}}$$

$$f'(x) = -\tfrac{5}{7}\left(\tan x + \frac{1}{x^2}\right)^{-\frac{12}{7}}\left(\sec^2(x) - \frac{2}{x^3}\right)$$

3. [10 Points] Find the **absolute maximum** and **absolute minimum value(s)** of the function

$$G(x) = \frac{10x}{x^2+1} \quad \text{on the interval} \quad [0,2].$$

First $G'(x) = \dfrac{(x^2+1)10 - 10x(2x)}{(x^2+1)^2} = \dfrac{10x^2+10-20x^2}{(x^2+1)^2} = \dfrac{-10x^2+10}{(x^2+1)^2} = \dfrac{10-10x^2}{(x^2+1)^2}$

So critical numbers are when $10 - 10x^2 = 0$ or when $x = \pm 1$. Notice that $x = -1$ is not in our interval of interest, so we evaluate G at $x = 1$ here and at the endpoints $x = 0$ and $x = 2$ and use the Closed Interval Method:

$G(1) = \dfrac{10}{2} = 5$ ABSOLUTE MAX VALUE

$G(0) = 0$ ABSOLUTE MIN VALUE

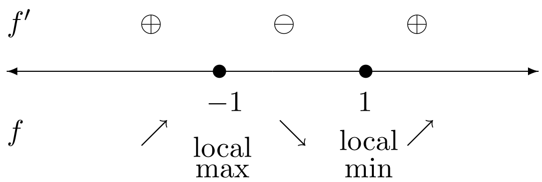
$G(2) = \dfrac{20}{5} = 4$

4. [12 Points] Let $\boldsymbol{f(x) = x^3 - 3x + 3}$.

For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

- $f(x)$ has domain $(-\infty, \infty)$
- It is a polynomial, continuous everywhere, and so has no vertical asymptotes.
- There are no horizontal asymptotes for this f since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.
- First Derivative Information

We compute $f'(x) = 3x^2 - 3$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when $x = \pm 1$. As a result, $x = \pm 1$ are the critical numbers. Using sign testing/analysis for f',



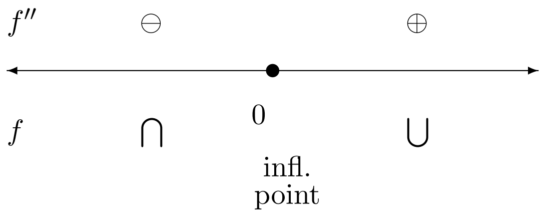
or our f' chart is

x	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
$f'(x)$	$\oplus$	$\ominus$	$\oplus$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

So f is increasing on $(-\infty,-1)$ and on $(1,\infty)$; and f is decreasing on $(-1,1)$. Moreover, f has a local max at $x = -1$ with $f(-1) = 5$, and a local min at $x = 1$ with $f(1) = 1$.

- Second Derivative Information

Meanwhile, f'' is always defined and continuous, and $f'' = 6x = 0$ only at our possible inflection point $x = 0$. Using sign testing/analysis for f'',

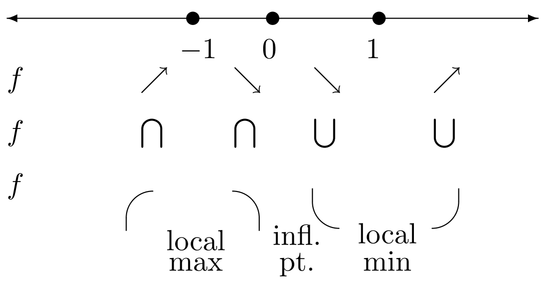


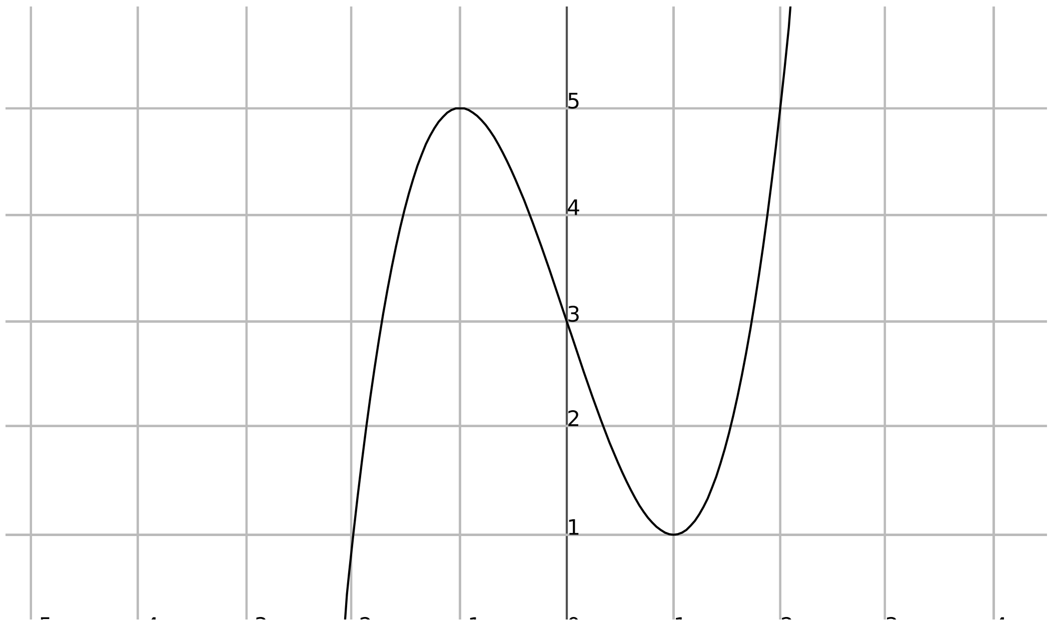
or our f'' chart is

x	$(-\infty, 0)$	$(0, \infty)$
$f''(x)$	$\ominus$	$\oplus$
$f(x)$	$\cap$	$\cup$

So f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$, with an inflection point at $x = 0$.

• Piece the first and second derivative information together





5. [18 Points] Let $\boldsymbol{f(x) = \dfrac{x^2 - 16}{x^2 - 9}}$.

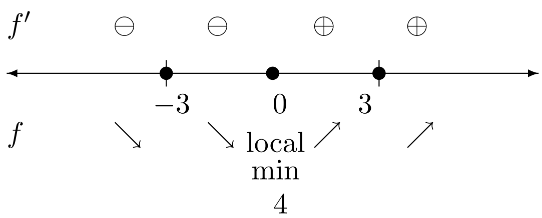
For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

Take my word for it that (you do **not** have to compute these)

$$\boldsymbol{f'(x) = \frac{14x}{(x^2-9)^2}} \quad \text{and} \quad \boldsymbol{f''(x) = \frac{-42(x^2+3)}{(x^2-9)^3}}.$$

- $f(x)$ has domain $\{x | x \neq \pm 3\}$
- Vertical asymptotes at $x = \pm 3$.
- Horizontal asymptote at $y = 1$ for this f since $\lim_{x \to \pm\infty} f(x) = 1$.
- First Derivative Information

We use $f'(x) = \dfrac{14x}{(x^2-9)^2}$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined or zero. The latter happens when $x = 0$. The derivative is undefined when $x = \pm 3$, but those values are not in the domain of the original function. As a result, $x = 0$ is the critical number. Using sign testing/analysis for f',



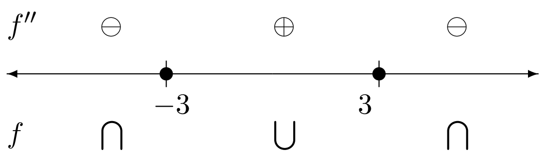
or our f' chart is

x	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	$\ominus$	$\oplus$
$f(x)$	$\searrow$	$\nearrow$

So f is decreasing on $(-\infty, 0)$; and f is increasing on $(0, \infty)$. Moreover, f has a local min at $x = 0$.

• Second Derivative Information

Meanwhile, $f'' = \dfrac{-42(x^2+3)}{(x^2-9)^3}$ is never zero. Using sign testing/analysis for f'' around the vertical asymptotes,

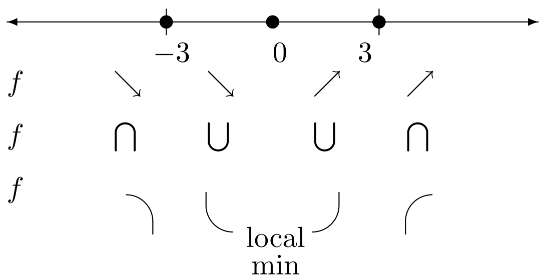


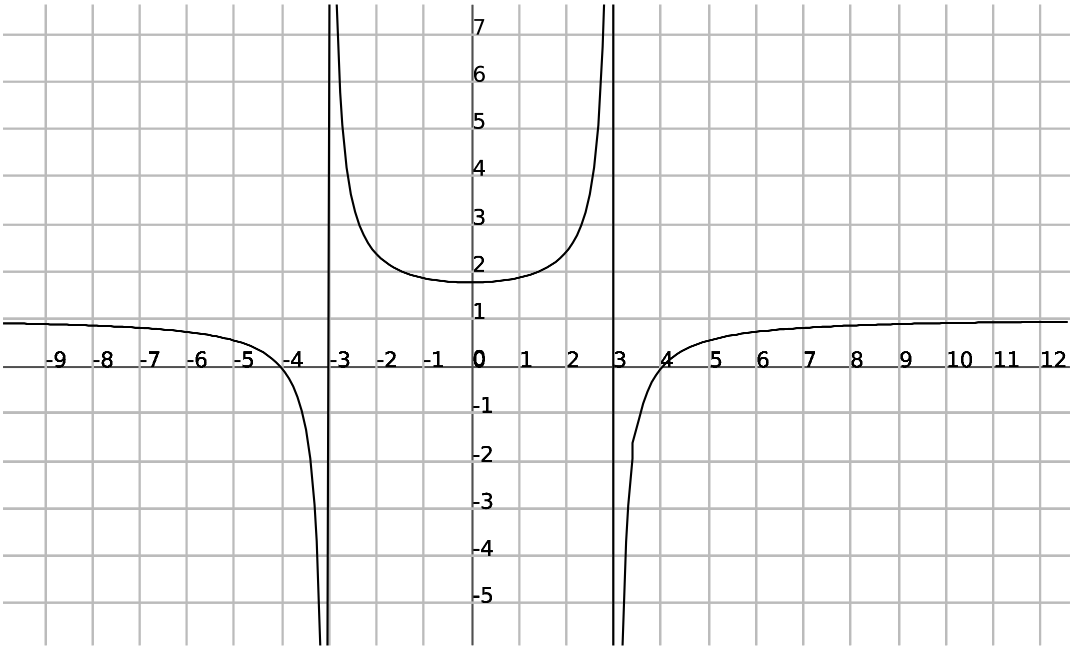
or our f'' chart is

x	$(-\infty, -3)$	$(-3, 3)$	$(3, \infty)$
$f''(x)$	$\ominus$	$\oplus$	$\ominus$
$f(x)$	$\cap$	$\cup$	$\cap$

So f is concave up on $(-3, 3)$ and concave down on $(-\infty, -3)$ and $(3, \infty)$.

• Piece the first and second derivative information together





6. [7 Points] Find the **equation of the tangent line** to the curve $\boldsymbol{x^3 + x^2y + 4y^2 = 6}$ at the point $\boldsymbol{(1, 1)}$.

First, implicit differentiate to get

$3x^2 + \left(x^2\dfrac{dy}{dx} + y2x\right)) + 8y\dfrac{dy}{dx} = 0.$

Substitute $x = 1$ and $y = 1$, our equation reduces to

$3 + \dfrac{dy}{dx} + 2 + 8\dfrac{dy}{dx} = 0$ or $9\dfrac{dy}{dx} = -5$ which implies $\dfrac{dy}{dx} = -\dfrac{5}{9}$ is our desired slope.

Finally, the equation of the tangent line is found using point-slope form:

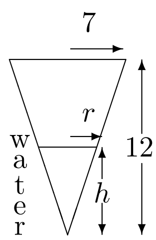
$y - 1 = -\dfrac{5}{9}(x - 1)$ or $\boxed{y = -\dfrac{5}{9}x + \dfrac{14}{9}}$.

7. [15 Points] A conical tank, 14 feet across the entire top and 12 feet deep, is leaking water. The radius of the water level is decreasing at the rate of 2 feet per minute. How fast is the water leaking out of the tank when the radius of the water level is 2 feet?

**Recall the volume of the cone is given by $V = \frac{1}{3}\pi r^2 h$

The cross section (with water level drawn in) looks like:

- Diagram



• Variables

Let r = radius of the water level at time t

Let h = height of the water level at time t

Let V = volume of the water in the tank at time t

Find $\frac{dV}{dt} = ?$ when $r = 2$ feet

and $\frac{dr}{dt} = -2\frac{\text{ft}}{\text{min}}$

• Equation relating the variables:

Volume= $V = \frac{1}{3}\pi r^2 h$

• Extra solvable information: Note that h is not mentioned in the problem's info. But there is a relationship, via similar triangles, between r and h. We must have

$$\frac{r}{7} = \frac{h}{12} \implies h = \frac{12r}{7}$$

After substituting into our previous equation, we get:

$V = \frac{1}{3}\pi r^2 \left(\frac{12r}{7}\right) = \frac{4}{7}\pi r^3$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{7}\pi r^3\right) \implies \frac{dV}{dt} = \frac{4}{7}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \implies \frac{dV}{dt} = \frac{12}{7}\pi r^2 \frac{dr}{dt}$

• Substitute Key Moment Information (now and not before now!!!):

$\frac{dV}{dt} = \frac{12}{7}\pi(2)^2(-2)$

• Solve for the desired quantity:

$\boxed{\frac{dV}{dt} = -\frac{96\pi}{7}\frac{\text{ft}^3}{\text{min}}}$

• Answer the question that was asked: The water is leaking out of the tank at a rate of $\frac{96\pi}{7}$ cubic feet every minute.

8. [8 Points] A ball is thrown straight upward from the ground with initial velocity $v_0 = 96$ feet per second. The height of the ball at time t is given by the position function $s(t) = -16t^2 + 96t$.

• Find the maximum height attained by the ball.

Maximum height is attained when $v(t) = 0$. Set $v(t) = -32t + 96 = 0$ and solve for time t. So, $t = \frac{96}{32} = 3$ seconds. That's only the time for the max height. We must plug $t = 3$ seconds into the position function to find the actual max height $= s(3) = -16(3)^2 + 96(3) = -144 + 288 = \boxed{144}$ feet.

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OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

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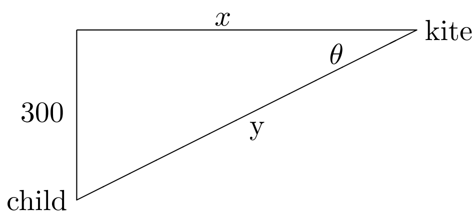
OPTIONAL BONUS A kite 300 feet high is being blown horizontally at 10 feet per second. When the kite has blown horizontally for 40 seconds, how fast is the angle between the string and the horizontal changing?

• Diagram



The picture at arbitrary time t is:

• Variables

Let x = distance kite has travelled horizontally at time t

Let y = distance between kite and child at time t

Let θ = the angle between the string/horizontal

Find $\frac{d\theta}{dt}$ =? when $x = 400$ feet (10ft/sec*40sec)

and $\frac{dx}{dt} = 10\frac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

The trigonometry of the triangle yields $\tan\theta = \frac{300}{x}$.

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{300}{x}\right) \implies \sec^2\theta\frac{d\theta}{dt} = -\frac{300}{x^2}\frac{dx}{dt}$. (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

At the key instant when $x = 400$, using the original equation, we have $y = \sqrt{(300)^2 + (400)^2} = 500$ by the Pythagorean Theorem.

Therefore, $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{500}{400} = \frac{5}{4}$

$\left(\frac{5}{4}\right)^2 \frac{d\theta}{dt} = -\frac{300}{(400)^2} \cdot 10.$

• Solve for the desired quantity:

$$\frac{d\theta}{dt} = -\frac{300}{(400)^2} \cdot 10 \cdot \left(\frac{4}{5}\right)^2 = -\frac{3}{(400)\cdot 4} \cdot 10 \cdot \left(\frac{16}{25}\right) = -\frac{3}{(40)} \cdot \left(\frac{4}{25}\right) = -\frac{3}{250}\frac{\text{rad}}{\text{sec}}$$

• Answer the question that was asked: The angle is decreasing at a rate of $= \frac{3}{250}$ radians every second.