Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 111

Midterm Exam #2

October 24, 2014

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted. Do not access any webpages during this exam.

• You need *not* simplify algebraically complicated answers for the derivative section. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, and $4^{\frac{3}{2}}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

• If you actually read these directions, draw a ghost at the bottom of the page.

Problem	Score	Possible Points
1		10
2		25
3		10
4		20
5		10
6		15
7		10
Total		100

1. [10 Points] Compute each of the following **limits**. Justify your answers. Show your work.

(a) $\lim_{x\to\infty} \dfrac{x^9+8x^7+6x^5+4}{3x^2+1} \dfrac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \dfrac{\underbrace{x^7}_{\to\infty} + \underbrace{8x^5}_{\to\infty} + \underbrace{6x^3}_{\to\infty} + \underbrace{\frac{4}{x^2}}_{\to 0}}{3 + \underbrace{\frac{1}{x^2}}_{\to 0}} = \boxed{\infty}$

OR

$\lim_{x\to\infty} \dfrac{x^9+8x^7+6x^5+4}{3x^2+1} \dfrac{\left(\frac{1}{x^9}\right)}{\left(\frac{1}{x^9}\right)} = \lim_{x\to\infty} \dfrac{1 + \underbrace{\frac{8}{x^2}}_{\to 0} + \underbrace{\frac{6}{x^4}}_{\to 0} + \underbrace{\frac{4}{x^9}}_{\to 0}}{\underbrace{\frac{3}{x^7}}_{\to 0} + \underbrace{\frac{1}{x^9}}_{\to 0}} = \dfrac{1}{0^+} = \boxed{\infty}$

(b) $\lim_{x\to\infty} \dfrac{x^2-x+1}{2x^5+7x^2+3} \dfrac{\left(\frac{1}{x^5}\right)}{\left(\frac{1}{x^5}\right)} = \lim_{x\to\infty} \dfrac{\underbrace{\frac{1}{x^3}}_{\to 0} - \underbrace{\frac{1}{x^4}}_{\to 0} + \underbrace{\frac{1}{x^5}}_{\to 0}}{2 + \underbrace{\frac{7}{x^3}}_{\to 0} + \underbrace{\frac{3}{x^5}}_{\to 0}} = \dfrac{0}{2} = \boxed{0}$

OR

$\lim_{x\to\infty} \dfrac{x^2-x+1}{2x^5+7x^2+3} \dfrac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \dfrac{1 - \underbrace{\frac{1}{x}}_{\to 0} + \underbrace{\frac{1}{x^2}}_{\to 0}}{\underbrace{2x^3}_{\to\infty} + 7 + \underbrace{\frac{3}{x^2}}_{\to 0}} = \boxed{0}$

2. [25 Points] **Differentiate** each of the following functions. You **do not** need to simplify your answers. Please do not waste time simplifying your derivative.

(a) $f(x) = \frac{5}{6}x + x^{\frac{5}{6}} + \frac{1}{x^{\frac{5}{6}}} + \frac{6}{5} + \frac{5}{6x^6} - \frac{6}{5x}$

$$= 5/6\,x + x^{5/6} + x^{-5/6} + \frac{6}{5} + \frac{5}{6}x^{-6} - \frac{6}{5}x^{-1}$$

$$f'(x) = \boxed{\frac{5}{6} + \frac{5}{6}x^{-1/6} - \frac{5}{6}x^{-11/6} + 0 - 5x^{-7} + \frac{6}{5}x^{-2}}$$

(b) $f(x) = \left(\frac{\overset{3x^{-2}}{\frac{3}{x^2}} + x^3}{x^{\frac{2}{3}} + \frac{3}{2}x} \right)^{\frac{2}{3}}$

$$f'(x) = \boxed{\frac{2}{3}\left(\frac{3/x^2 + x^3}{x^{2/3} + \frac{3}{2}x}\right)^{-1/3} \left[\frac{\left(x^{2/3} + 3/2\,x\right)\left(-6x^{-3}\right) - \left(\frac{3}{x^2} + x^3\right)\left(\frac{2}{3}x^{-1/3} + 3/2\right)}{\left(x^{2/3} + 3/2\,x\right)^2}\right]}$$

(c) $f(x) = \frac{\sqrt{x} + \sec\sqrt{x}}{\sqrt{1 + \sec x}}$

$$f'(x) = \boxed{\frac{\sqrt{1+\sec x}\left(\frac{1}{2\sqrt{x}} + \sec\sqrt{x}\tan\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)\right) - \left(\sqrt{x} + \sec\sqrt{x}\right)\left(\frac{1}{2\sqrt{1+\sec x}} \cdot \sec x \tan x\right)}{1+\sec x}}$$

$$\left(\sqrt{1+\sec x}\right)^2$$

2. (Continued) **Differentiate** each of the following functions. You **do not** need to simplify your answers. Please do not waste time simplifying your derivative.

(d) $f(x) = \cos\left(\tan^2\left(\frac{3}{x^5}\right)\right) + \sin^2\left(\cos\left(\frac{x^3}{5}\right)\right)$

$$f'(x) = -\sin\left(\tan^2\left(\frac{3}{x^5}\right)\right)\cdot 2\tan\left(\frac{3}{x^5}\right)\cdot\sec^2\left(\frac{3}{x^5}\right)\left(-15x^{-6}\right)$$

$$+\ 2\sin\left(\cos\left(\frac{x^3}{5}\right)\right)\cdot\cos\left(\cos\left(\frac{x^3}{5}\right)\right)\cdot\left(-\sin\left(\frac{x^3}{5}\right)\right)\cdot\frac{3}{5}x^2$$

(e) $f(x) = \left(\frac{1}{x^3} + \pi\right)^{\frac{5}{7}} \cdot \left(x^4 - \frac{1}{x^7}\right)^{-5}$

x^{-3} x^{-7}

$$f'(x) = \left(\frac{1}{x^3} + \pi\right)^{5/7}(-5)\left(x^4 - \frac{1}{x^7}\right)^{-6}\left(4x^3 + 7x^{-8}\right)$$

$$+$$

$$\left(x^4 - \frac{1}{x^7}\right)^{-5}\frac{5}{7}\left(\frac{1}{x^3} + \pi\right)^{-2/7}\left(-3x^{-4} + 0\right)$$

3. [10 Points] Find the **absolute maximum** and **absolute minimum value(s)** of the function

$$F(x) = x\sqrt{4 - x^2} \qquad \text{on the interval} \qquad [-1, 2].$$

$$F'(x) = x \frac{1}{2\sqrt{4-x^2}}(-2x) + \sqrt{4-x^2}$$

$$= \frac{-2x^2}{2\sqrt{4-x^2}} + \sqrt{4-x^2} \cdot \frac{2\sqrt{4-x^2}}{2\sqrt{4-x^2}}$$

$$= \frac{-2x^2 + 2(4-x^2)}{2\sqrt{4-x^2}}$$

$$= \frac{-4x^2+8}{2\sqrt{4-x^2}} \overset{\text{set}}{=} 0 \Rightarrow -4x^2+8=0$$

$$4x^2 = 8$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

undefined at $x = \pm 2$.

Note: $x = -2$ and $x = -\sqrt{2}$ not in interval $[-1, 2]$

$F(\sqrt{2}) = \sqrt{2}\sqrt{4-(\sqrt{2})^2} = \sqrt{2} \cdot \sqrt{4-2} = \sqrt{2} \cdot \sqrt{2} = \boxed{2}$ ← Abs. Max Value.

$F(-1) = -1\sqrt{4-1} = \boxed{-\sqrt{3}}$ ← Abs. Min Value.

$F(2) = 2 \cdot 0 = 0$

$f(6) = \frac{-36+18}{36-24+4} = \frac{-18}{16} = \frac{-9}{8}$

$f(8) = \frac{-64+24}{64-32+4} = \frac{-40}{36} = \frac{-10}{9}$

4. [20 Points] Let $f(x) = \dfrac{-x^2+3x}{(x-2)^2} = \dfrac{-x^2+3x}{x^2-4x+4}$.

For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve. **Hint:**

Take my word for it that (you do **not** have to compute these)

$$f'(x) = \frac{x-6}{(x-2)^3} \quad \text{and} \quad f''(x) = \frac{-2x+16}{(x-2)^4}.$$

- Domain $\{x \mid x \neq \pm 2\}$
- V.A. $x = \pm 2$.
- H.A. $\lim_{x \to \pm\infty} \frac{-x^2+3x}{x^2-4x+4} \frac{(1/x^2)}{(\frac{1}{x^2})} = \lim_{x\to\pm\infty} \frac{-1+\cancel{3/x}^{\to 0}}{1-\cancel{4/x}_{\to 0}+\cancel{4/x^2}_{\to 0}} = -1$

H.A @ $y=1$.

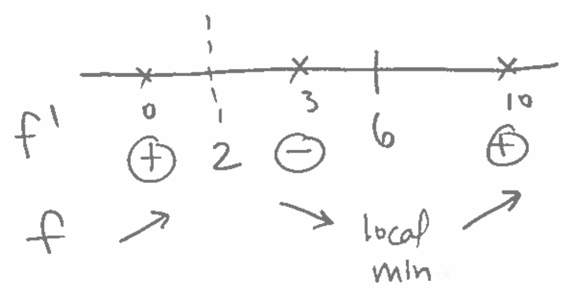
- First Derivative

$f'(x) = \frac{x-6}{(x-2)^3} \overset{set}{=} 0 \Rightarrow x = 6$.

undefined @ $x=2$, not technically critical #.

- Piece Together

- Sign Testing into 1st Derivative.



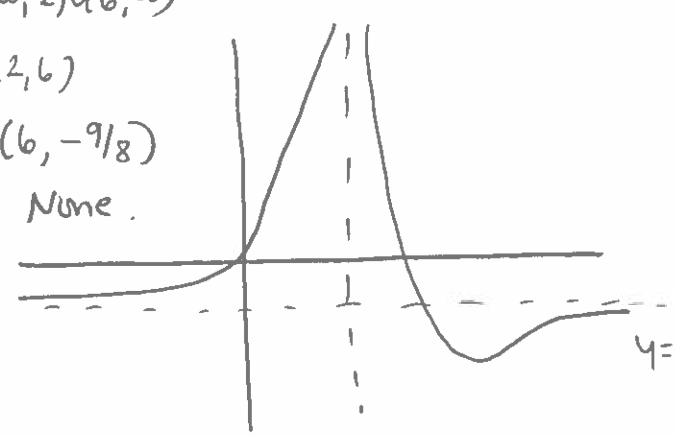
Increasing $(-\infty, 2) \cup (6, \infty)$

Decreasing $(2, 6)$

Local Min $(6, -9/8)$

Local Max None.

- Sketch



- Second Derivative.

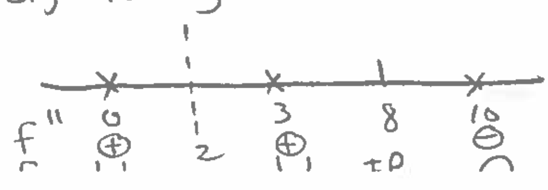
$f''(x) = \frac{-2x+16}{(x-2)^4_{\oplus}} \overset{set}{=} 0 \Rightarrow x = 8$

CU $(-\infty, 2) \cup (2, 8)$

CD $(8, \infty)$

IP $(8, -10/9)$

- Sign Testing into 2nd Derivative



5. [10 Points] Consider the equation $\cos(xy^2) + 2 = y^3 + \sin x$.

(a) Compute $\frac{dy}{dx}$.

$$-\sin(xy^2)\left[x\,2y\cdot\frac{dy}{dx} + y^2\right] = 3y^2\frac{dy}{dx} + \cos x$$

$$-2xy\sin(xy^2)\frac{dy}{dx} - y^2\sin(xy^2) = 3y^2\frac{dy}{dx} + \cos x$$

$$\left(-2xy\sin(xy^2) - 3y^2\right)\frac{dy}{dx} = y^2\sin(xy^2) + \cos x$$

$$\Rightarrow \frac{dy}{dx} = \boxed{\frac{y^2\sin(xy^2) + \cos x}{-2xy\sin(xy^2) - 3y^2}}$$

(b) Compute the equation of the tangent line to this curve at the point $(\pi, 1)$.

$$\left.\frac{dy}{dx}\right|_{(\pi,1)} = \frac{1\,\overset{0}{\cancel{\sin\pi}} + \overset{-1}{\cancel{\cos\pi}}}{-2\pi\underset{0}{\cancel{\sin\pi}} - 3} = \frac{-1}{-3} = \frac{1}{3}$$

Point Slope

$$y - 1 = \tfrac{1}{3}(x - \pi)$$

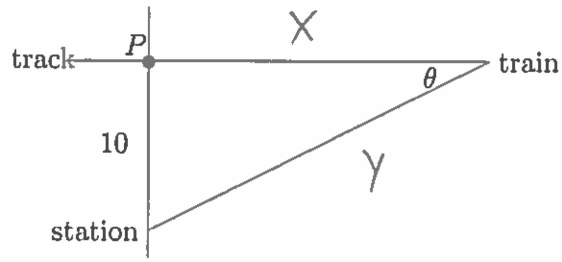
$$\boxed{y = \tfrac{1}{3}x - \tfrac{\pi}{3} + 1}$$

$$\underbrace{\phantom{-\tfrac{\pi}{3}+1}}_{\frac{-\pi+3}{3}}$$

6. [15 Points] Consider a point P on a train track. Suppose a train depot station is 10 feet directly south from this point P. The train is travelling east at 6 feet per second. Consider the angle as shown in the diagram. How fast is this angle changing when 2 seconds has passed since the train passed point P.

- Diagram

The picture at arbitrary time t is:



· Variable.

Let x = distance train travelled at time t

y = distance between train and station at time t.

· Given $\frac{dx}{dt} = 6 f/sec.$ $\frac{d\theta}{dt} = ?$ when $x = 6 ft/sec (2 sec) = 12 ft.$

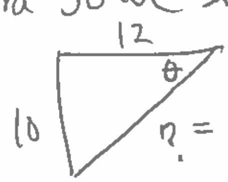
· Equation:

$$\tan\theta = {}^{10}/_{x}$$

· Differentiate.

$$\sec^2\theta \frac{d\theta}{dt} = \frac{-10}{x^2}\frac{dx}{dt}$$

· Extra Solve Info



$? = \sqrt{100+144} = \sqrt{244}$ $\Rightarrow \sec\theta = \frac{H}{A} = \frac{\sqrt{244}}{12}$

· Substitute.

$$\cancel{\left(\frac{\sqrt{244}}{12}\right)^2} \frac{d\theta}{dt} = \frac{-10}{\cancel{(12)^2}} \cdot (6)$$

· Solve $244 \frac{d\theta}{dt} = -60$

$$\frac{d\theta}{dt} = \frac{-60}{244} = \frac{-15}{61} \text{ rad./sec.}$$

· Answer: The angle is decreasing at 15/61 radians every second at that moment.

7. [10 Points]

(a) Let $f(x) = \sin^3(4x) + \sec(4x) - 8\sin(2x)$. Compute $f'\left(\frac{\pi}{12}\right)$. Simplify.

$$f'(x) = 3\sin^2(4x)\cos(4x)\cdot 4 + \sec(4x)\tan(4x)\,4 - 8\cos(2x)\,2$$

$$f'(\pi/12) = 12\sin^2(\pi/3)\cos(\pi/3) + 4\sec(\pi/3)\tan(\pi/3) - 16\cos(\pi/6)$$

$$= 12\left(\frac{\sqrt{3}}{2}\right)^2\cdot\left(\frac{1}{2}\right) + 4\cdot 2\cdot\sqrt{3} - 16\,\sqrt{3}/2.$$

$\cos\pi/3 = 1/2$

$$= 12\cdot\frac{3}{4}\cdot\frac{1}{2} + \cancel{8\sqrt{3}} - \cancel{8\sqrt{3}}$$

$$= \boxed{\frac{9}{2}}$$

(b) Let $f(x) = \cos(2x) + \dfrac{1}{\tan^2 x} + \sin\left(x - \dfrac{\pi}{4}\right)$. Compute $f'\left(\frac{\pi}{4}\right)$. Simplify.

$\tan^{-2}x$

$$f'(x) = -\sin(2x)(2) - 2\tan^{-3}x\cdot\sec^2 x + \cos(x - \pi/4)$$

$$f'(\pi/4) = -2\cancel{\sin}(\pi/2)^{1} - \frac{2}{\cancel{\tan}^3(\pi/4)_{1}}\cdot\sec^2(\pi/4) + \cancel{\cos}(0)^{1}$$

$$= -2 - 2\cdot(\sqrt{2})^2 + 1$$

$\cos(\pi/4) = 1/\sqrt{2}$

$\sec(\pi/4) = \sqrt{2}$

$$= -2 - 4 + 1$$

$$= \boxed{-5}$$