Math 111 Midterm Exam #2 October 26, 2012

1. [12 Points] Compute each of the following **limits**. Justify your answers. Show your work.

(a) $$\lim_{x\to 0}\frac{8x^2-x^3}{\sin^2(3x)}=\lim_{x\to 0}\frac{x^2(8-x)}{\sin^2(3x)}=\lim_{x\to 0}\frac{x}{\sin(3x)}\frac{x}{\sin(3x)}(8-x)$$
$$=\lim_{x\to 0}\frac{3x}{3\sin(3x)}\frac{3x}{3\sin(3x)}(8-x)=\frac{1}{9}\lim_{x\to 0}\frac{3x}{\sin(3x)}\frac{3x}{\sin(3x)}(8-x)$$
$$=\frac{1}{9}\lim_{x\to 0}\frac{3x}{\sin(3x)}\lim_{x\to 0}\frac{3x}{\sin(3x)}\lim_{x\to 0}(8-x)=\frac{1}{9}\cdot 1\cdot 1\cdot 8=\boxed{\frac{8}{9}}$$

(b) $$\lim_{x\to\infty}\frac{3x^7+x^6-2012}{x^2-x+5}=\lim_{x\to\infty}\frac{3x^7+x^6-2012}{x^2-x+5}\cdot\frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{3x^5+x^4-\frac{2012}{x^2}}{1-\frac{1}{x}+\frac{5}{x^2}}=\boxed{\infty}$$

2. [18 Points] **Differentiate** each of the following functions. You **do not** need to simplify your answers. Please do not waste time simplifying your derivative.

(a) $f(x)=\sqrt{x}+\dfrac{1}{\sqrt{x}}+\dfrac{1}{1+\sqrt{x}}+\dfrac{1}{\sqrt{1+x}}$

$$f'(x)=\boxed{\frac{1}{2\sqrt{x}}-\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{(1+\sqrt{x})^2}\left(\frac{1}{2\sqrt{x}}\right)-\frac{1}{2}(1+x)^{-\frac{3}{2}}}$$

(b) $f(x)=\left(\dfrac{\cos(7x)}{x^2+7\sin x}\right)^{\frac{7}{8}}$

$$f'(x)=\boxed{\frac{7}{8}\left(\frac{\cos(7x)}{x^2+7\sin x}\right)^{-\frac{1}{8}}\left(\frac{(x^2+7\sin x)(-7\sin(7x))-\cos(7x)(2x+7\cos x)}{(x^2+7\sin x)^2}\right)}$$

(c) $f(x)=\dfrac{1}{\left(\frac{1}{x^5}+\sqrt{x^2-4}\right)^{\frac{3}{7}}}=\left(\dfrac{1}{x^5}+\sqrt{x^2-4}\right)^{-\frac{3}{7}}$

$$f'(x)=\boxed{-\frac{3}{7}\left(\frac{1}{x^5}+\sqrt{x^2-4}\right)^{-\frac{10}{7}}\left(-\frac{5}{x^6}+\frac{1}{2\sqrt{x^2-4}}(2x)\right)}$$

3. [10 Points] Find the **absolute maximum** and **absolute minimum value(s)** of the function

$$F(x)=(x-1)^2(2x-10)^2 \qquad \text{on the interval} \qquad [0,4].$$

$F'(x)=(x-1)^2\cdot 2(2x-10)(2)+(2x-10)^2\cdot 2(x-1)=2(x-1)(2x-10)[2(x-1)+(2x-10)]=2(x-1)(2x-10)[4x-12]$. On the interval $[0,4]$, F' is always defined. Also, $F'(x)=0$ happens only when $x = 1$, $x = 5$, and $x = 3$ (our critical numbers). Here $x = 5$ is outside of our interval of interest. Applying the closed interval method:

$F(1) = \boxed{0} \longleftarrow$ Absolute Minimum Value

$F(0) = \boxed{100} \longleftarrow$ Absolute Maximum Value

$F(3) = 64$

$F(4) = 36.$

So the absolute maximum value is 100 (attained at $x = 0$), and the absolute minimum value is 0 (attained at $x = 1$).

4. [25 Points] Let $f(x) = \dfrac{x^2 - 9}{x^2 - 4}.$

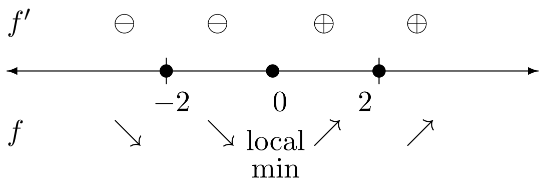
For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve. **Hint:**

Take my word for it that (you do **not** have to compute these)

$$\boldsymbol{f'(x) = \frac{10x}{(x^2-4)^2}} \quad \text{and} \quad \boldsymbol{f''(x) = \frac{-10(3x^2+4)}{(x^2-4)^3}}.$$

- $f(x)$ has domain $\{x | x \neq \pm 2\}$
- Vertical asymptotes at $x = \pm 2$.
- Horizontal asymptote at $y = 1$ for this f since $\lim_{x \to \pm\infty} f(x) = 1$.
- First Derivative Information

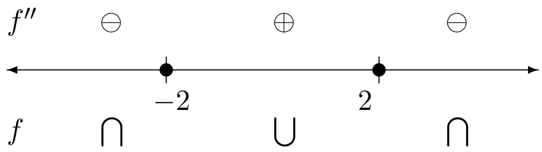
We compute $f'(x) = \dfrac{10x}{(x^2-4)^2}$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined or zero. The latter happens when $x = 0$. The derivative is undefined when $x = \pm 2$, but those values are not in the domain of the original function. As a result, $x = 0$ is the critical number. Using sign testing/analysis for f',



So f is decreasing on $(-\infty, -2) \cup (-2, 0)$; and f is increasing on $(0, 2) \cup (2, \infty)$. Moreover, f has a local min at $(0, f(0)) = \left(0, \frac{9}{4}\right)$.

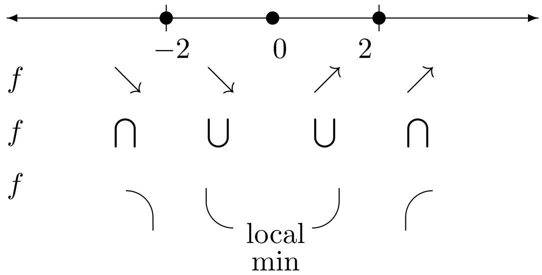
- Second Derivative Information

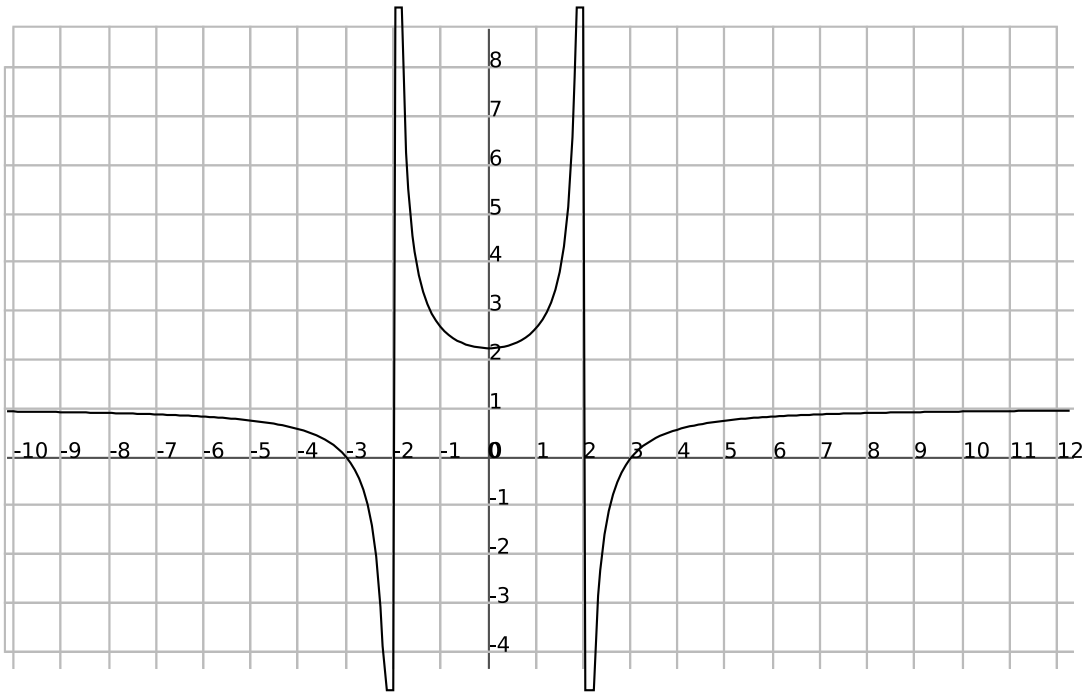
Meanwhile, $f'' = \dfrac{-10(3x^2+4)}{(x^2-4)^3}$ is never zero. No possible inflection points. Using sign testing/analysis for f'' around the vertical asymptotes,



So f is concave up on $(-2, 2)$ and concave down on $(-\infty, -2)$ and $(2, \infty)$. Inflection Points: None.

- Piece the first and second derivative information together





5. [10 Points] Consider the equation $\sin(x^2y) + 6\tan x + 1 = y^3$.

(a) Compute $\dfrac{dy}{dx}$.

$\dfrac{d}{dx}\left(\sin(x^2y) + 6\tan x + 1\right) = \dfrac{d}{dx}\left(y^3\right)$

$\cos(x^2y)\left(x^2\dfrac{dy}{dx} + y(2x)\right) + 6\sec^2 x = 3y^2\dfrac{dy}{dx}$

Distribute:

$x^2\cos(x^2y)\dfrac{dy}{dx} + 2xy\cos(x^2y) + 6\sec^2 x = 3y^2\dfrac{dy}{dx}$

Algebra:

$x^2\cos(x^2y)\dfrac{dy}{dx} - 3y^2\dfrac{dy}{dx} = -2xy\cos(x^2y) - 6\sec^2 x$

Factor:

$\left(x^2\cos(x^2y) - 3y^2\right)\dfrac{dy}{dx} = -2xy\cos(x^2y) - 6\sec^2 x$

Solve:

$\dfrac{dy}{dx} = \boxed{\dfrac{-2xy\cos(x^2y) - 6\sec^2 x}{x^2\cos(x^2y) - 3y^2}}$

(b) Compute the equation of the tangent line to this curve at the point $(0, 1)$.

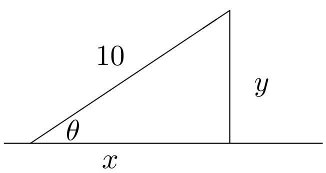
Point=$(0, 1)$

Slope at $(0, 1) = \left.\dfrac{dy}{dx}\right|_{(0,1)} = \dfrac{0 - 6\sec^2 0}{0 - 3} = \dfrac{-6}{-3} = 2$

Point-Slope Form: $y - 1 = 2(x - 0)$ or $\boxed{y = 2x + 1}$

6. [15 Points] The top of a ten foot ladder is sliding down a vertical wall at the rate of one foot every second. Consider the angle formed by the bottom of the ladder and the ground. How fast is this angle changing when the top of the ladder is three feet above the ground?

• Diagram



• Variables

Let x = distance between bottom of ladder and wall at time t

Let y = distance between top of ladder and ground at time t

Let θ = angle formed by the ground and base of ladder at time t

Find $\dfrac{d\theta}{dt}$ =? when $y = 3$ ft

and $\dfrac{dy}{dt} = -1\dfrac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

We have $\sin\theta = \dfrac{y}{10}$.

• Differentiate both sides w.r.t. time t.

$\dfrac{d}{dt}(\sin\theta) = \dfrac{d}{dt}\left(\dfrac{y}{10}\right) \implies \cos\theta\dfrac{d\theta}{dt} = \dfrac{1}{10}\dfrac{dy}{dt}$(Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

We're not given θ for this problem, but we can still compute $\cos\theta$ from trig. relations on the diagram's triangle with $\cos\theta = \frac{\text{adj}}{\text{hyp}}$. When $y = 3$, we can use the Pyth. Theorem to compute $x = \sqrt{(10)^2 - (3)^2} = \sqrt{91}$. Finally, $\cos\theta = \dfrac{\sqrt{91}}{10}$.

$\dfrac{\sqrt{91}}{10}\dfrac{d\theta}{dt} = \dfrac{1}{10}(-1)$

• Solve for the desired quantity:

$$\frac{d\theta}{dt} = -\frac{1}{10}\cdot\frac{10}{\sqrt{91}} = \boxed{-\frac{1}{\sqrt{91}}}\frac{\text{rad}}{\text{sec}}$$

• Answer the question that was asked: The angle is decreasing at a rate of $\dfrac{1}{\sqrt{91}}$ radians every second at that moment.

7. [10 Points] Let $f(x) = \dfrac{1}{2\tan^2 x} + \cos^2 x + \sec(2x)$. Find $f'\left(\dfrac{\pi}{6}\right)$.

Simplify your answer to a single real number.

First, $f(x) = \dfrac{1}{2}\tan^{-2} x + \cos^2 x + \sec(2x)$

Then, $f'(x) = \dfrac{1}{2}(-2)\tan^{-3} x\ (\sec^2 x) - 2\cos x\sin x + 2\sec(2x)\tan(2x)$

$= -1\tan^{-3} x\sec^2 x - 2\cos x\sin x + 2\sec(2x)\tan(2x)$

$$f'\left(\frac{\pi}{6}\right) = -\frac{1}{\tan^3\left(\frac{\pi}{6}\right)}\sec^2\left(\frac{\pi}{6}\right) - 2\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) + 2\sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)$$

$$= -\frac{1}{\left(\frac{1}{\sqrt{3}}\right)^3}\left(\frac{2}{\sqrt{3}}\right)^2 - 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 2(2)\left(\sqrt{3}\right) = -3\sqrt{3}\left(\frac{4}{3}\right) - \frac{\sqrt{3}}{2} + 4\sqrt{3} = \boxed{-\tfrac{\sqrt{3}}{2}}$$

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OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

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OPTIONAL BONUS Two carts, A and B, are connected by a rope 39 feet long that passes over a pulley P on the ceiling. The point Q is on the floor 12 feet directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 feet per second. How fast is cart B moving toward Q at the instant when cart A is 5 feet from Q?