Math 111 Midterm Exam #1 February 17, 2012

1. [30 Points] Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle \lim_{x\to-3} \frac{x^2-2x-15}{x^2+x-6} \overset{\frac{0}{0}}{=} \lim_{x\to-3} \frac{(x+3)(x-5)}{(x+3)(x-2)} = \lim_{x\to-3} \frac{x-5}{x-2} \overset{\text{DSP}}{=} \frac{-8}{-5} = \boxed{\frac{8}{5}}$

(b) $\displaystyle \lim_{x\to5} \frac{x^2-2x-15}{|5-x|} = \boxed{\text{DOES NOT EXIST}}$, RHL$\neq$LHL

RHL: $\displaystyle \lim_{x\to5^+} \frac{x^2-2x-15}{|5-x|} \overset{\frac{0}{0}}{=} \lim_{x\to5^+} \frac{x^2-2x-15}{-(5-x)} = \lim_{x\to5^+} \frac{(x+3)(x-5)}{x-5} = \lim_{x\to5^+} x+3 \overset{\text{DSP}}{=} \boxed{8}$

LHL: $\displaystyle \lim_{x\to5^-} \frac{x^2-2x-15}{|5-x|} \overset{\frac{0}{0}}{=} \lim_{x\to5^-} \frac{x^2-2x-15}{5-x} = \lim_{x\to5^-} \frac{(x+3)(x-5)}{-(x-5)} = \lim_{x\to5^-} -(x+3) \overset{\text{DSP}}{=} \boxed{-8}$

Here, recall that $|5-x| = \begin{cases} 5-x & \text{if} \quad 5-x\geq 0 \\ -(5-x) & \text{if} \quad 5-x<0 \end{cases} = \begin{cases} 5-x & \text{if} \quad x\leq 5 \leftarrow \text{LHL case} \\ x-5 & \text{if} \quad x>5 \leftarrow \text{RHL case} \end{cases}$

Note: $|5-x|$ has a slightly different definition than $|x-5|$.

(c) $\displaystyle \lim_{x\to-3} \frac{x^2+2x-3}{x^2-4x-12} \overset{\text{DSP}}{=} \frac{0}{9} = \boxed{0}$

(d) $\displaystyle \lim_{x\to-6} \frac{\frac{x}{x+2}-\frac{x-3}{x}}{x+6} = \lim_{x\to-6} \frac{\left(\frac{x^2-(x-3)(x+2)}{(x+2)x}\right)}{x+6} = \lim_{x\to-6} \frac{\left(\frac{x^2-(x^2-x-6)}{(x+2)x}\right)}{x+6}$

$\displaystyle = \lim_{x\to-6} \frac{\left(\frac{x^2-x^2+x+6}{(x+2)x}\right)}{x+6} = \lim_{x\to-6} \frac{x+6}{(x+6)x(x+2)}$

$\displaystyle = \lim_{x\to-6} \frac{1}{(x+2)x} \overset{\text{DSP}}{=} \frac{1}{(-6+2)(-6)} = \frac{1}{(-4)(-6)} = \boxed{\frac{1}{24}}$

(e) $\displaystyle \lim_{x\to2} \frac{x+7}{x-2} = \boxed{\text{DNE}}$ since RHL$\neq$LHL.

RHL: $\displaystyle \lim_{x\to2^+} \frac{x+7}{x-2} = \frac{9}{0^+} = \frac{9}{0^+} = \boxed{+\infty}$

LHL: $\displaystyle \lim_{x\to2^-} \frac{x+7}{x-2} = \frac{9}{0^-} = \boxed{-\infty}$

(f) $\displaystyle \lim_{x\to3} \frac{g(x^2)-7}{(g(x))^2-1} =$ where $g(x)=x-2$

$$\lim_{x\to 3} \frac{g(x^2)-7}{(g(x))^2-1} = \lim_{x\to 3} \frac{(x^2-2)-7}{(x-2)^2-1} = \lim_{x\to 3} \frac{x^2-9}{x^2-4x+4-1} = \lim_{x\to 3} \frac{x^2-9}{x^2-4x+3}$$

$$= \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(x-1)} = \lim_{x\to 3} \frac{x+3}{x-1} \overset{\text{DSP}}{=} \lim_{x\to 3} \frac{3+3}{3-1} = \lim_{x\to 3} \frac{6}{2} = \boxed{3}$$

2. [13 Points] Prove that $\lim_{x\to 3} 1-5x = -14$ using the $\varepsilon-\delta$ definition of the limit.

Scratchwork: we want $|f(x)-L| = |(3-4x)-(-5)| < \varepsilon$

$$|f(x)-L| = |(1-5x)-(-14)| = |-5x+15| = |-5(x-3)| = |-5||x-3| = 5|x-3| \text{ (want } < \varepsilon)$$

$$5|x-3| < \varepsilon \text{ means } |x-3| < \frac{\varepsilon}{5}$$

$$\text{So choose } \delta = \frac{\varepsilon}{5} \text{ to restrict } 0 < |x-3| < \delta.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{5}$. Given x such that $0 < |x-3| < \delta$, then

$$|f(x)-L| = |(1-5x)-(-14)| = |-5x+15| = |-5(x-3)| = |-5||x-3| = 5|x-3| < 5\cdot\frac{\varepsilon}{5} = \varepsilon.$$

□

3. [15 Points] Suppose that $f(x) = \sqrt{3-x+x^2}$. Compute $f'(x)$ using the **limit definition of the derivative**.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3-(x+h)+(x+h)^2}-\sqrt{3-x+x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3-(x+h)+(x+h)^2}-\sqrt{3-x+x^2}}{h} \cdot \left(\frac{\sqrt{3-(x+h)+(x+h)^2}+\sqrt{3-x+x^2}}{\sqrt{3-(x+h)+(x+h)^2}+\sqrt{3-x+x^2}}\right)$$

$$= \lim_{h\to 0} \frac{3-(x+h)+(x+h)^2-(3-x+x^2)}{h(\sqrt{3-(x+h)+(x+h)^2}+\sqrt{3-x+x^2})}$$

$$= \lim_{h\to 0} \frac{3-x-h+x^2+2xh+h^2-3+x-x^2}{h(\sqrt{3-(x+h)}+\sqrt{3-x})} = \lim_{h\to 0} \frac{-h-2xh-h^2}{h(\sqrt{3-(x+h)}+\sqrt{3-x})}$$

$$= \lim_{h\to 0} \frac{h(-1+2x+h)}{h(\sqrt{3-(x+h)+(x+h)^2}+\sqrt{3-x+x^2})} = \lim_{h\to 0} \frac{-1+2x+h}{\sqrt{3-(x+h)+(x+h)^2}+\sqrt{3-x+x^2}}$$

$$\overset{\text{cont/L.L.}}{=} \frac{-1+2x}{\sqrt{3-x+x^2}+\sqrt{3-x+x^2}} = \boxed{\frac{-1+2x}{2\sqrt{3-x+x^2}}}$$

4. [10 Points] Suppose that $f(x) = x^3+7x^2-4x+9$. Write the **equation of the tangent line** to the curve $y = f(x)$ when $x = -1$.

Use the limit definition of the derivative when computing the derivative.

First, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{((x+h)^3+7(x+h)^2-4(x+h)+9)-(x^3+7x^2-4x+9)}{h}$

$= \lim_{h\to 0} \dfrac{x^3+3x^2h+3xh^2+h^3+7x^2+14xh+7h^2-4x-4h+9-x^3-7x^2+4x-9}{h}$

$= \lim_{h\to 0} \dfrac{3x^2h+3xh^2+h^3+14xh+7h^2-4h}{h} = \lim_{h\to 0} \dfrac{h(3x^2+3xh+h^2+14x+7h-4)}{h}$

$= \lim_{h\to 0} 3x^2+3xh+h^2+14x+7h-4 = \boxed{3x^2+14x-4}$

Then the slope at $x=-1$ is given by $f'(-1) = 3-14-4 = -15$. The point is given by $(-1, f(-1)) = (1, 19)$. Finally, the equation of the tangent line is given by $y-19 = -15(x-(-1))$ or $\boxed{y=-15x+4}$.

5. [12 Points] Suppose that f and g are functions, **and**

- $\lim_{x\to 7} f(x) = 5$
- $\lim_{x\to 7} g(x) = -3$
- $f(5) = 7$
- $g(x)$ is continuous at $x=7$.
- $f(x)$ is **NOT** continuous at $x=7$.

Evaluate the following quantities and fully **justify** your answers. Do **not** just put down numbers.

(a) $\lim_{x\to 7} \sqrt{3f(x)-7g(x)} = \sqrt{\lim_{x\to 7}(3f(x)-7g(x))} = \sqrt{\lim_{x\to 7}(3f(x)) - \lim_{x\to 7}(7g(x))}$

$= \sqrt{3\lim_{x\to 7} f(x) - 7\lim_{x\to 7} g(x)} = \sqrt{3(5)-7(-3)} = \sqrt{15+21} = \sqrt{36} = \boxed{6}$

These steps are valid by application of the Limit Laws.

(b) $g(7) = \lim_{x\to 7} g(x)$ by definition of continuity assumption for g at $x=7$. We know $\lim_{x\to 7} g(x) = -3$ by assumption, so $\boxed{g(7)=-3}$

(c) $g \circ f(5) = g(f(5)) = g(7) = \boxed{-3}$ using both the assumption $f(5)=7$ and the answer from part(c).

(d) Does $f(7) = 5$? Justify your answer.

No, $f(7) \neq 5$. We know by assumption that f is NOT continuous at $x=7$. By definition of continuity that means, if $f(7)$ even exists, that $\lim_{x\to 7} f(x) \neq f(7)$. So $f(7)$ cannot equal 5 which equals $\lim_{x\to 7} f(x)$.

6. [20 Points] Consider the function defined by

$$f(x) = \begin{cases} 3 & \text{if } x > 12 \\ \sqrt{x-3} & \text{if } 3 < x \le 12 \\ 1 & \text{if } x = 3 \\ 6 - 2x & \text{if } 0 < x < 3 \\ 16 - x^2 & \text{if } -4 < x \le 0 \\ \dfrac{1}{x+4} & \text{if } x < -4 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$. **See me for sketch**.

(b) State the **Domain** of the function $f(x)$. Domain $f(x) = \boxed{\{x | x \neq -4\}}$.

(c) Compute
$$\begin{cases} \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 6 - 2x = \boxed{6} & \leftarrow \text{RHL} \\ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 16 - x^2 = \boxed{16} & \leftarrow \text{LHL} \\ \lim_{x \to 0} f(x) \ \text{ DOES NOT EXIST since RHL} \neq \text{LHL} \end{cases}$$

(d) Compute
$$\begin{cases} \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x-3} = \boxed{0} & \leftarrow \text{RHL} \\ \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 6 - 2x = \boxed{0} & \leftarrow \text{LHL} \\ \lim_{x \to 3} f(x) = \boxed{0} \quad \text{RHL=LHL} \end{cases}$$

(e) Compute
$$\begin{cases} \lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} 16 - x^2 = \boxed{0} & \leftarrow \text{RHL} \\ \lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \dfrac{1}{x+4} = \dfrac{1}{0^-} = \boxed{-\infty} & \leftarrow \text{LHL} \\ \lim_{x \to -4} f(x) = \ \text{ DOES NOT EXIST since RHL} \neq \text{LHL} \end{cases}$$

(f) State the value(s) at which f is discontinuous. Justify your answer(s) using the definition of continuity.

- f is discontinuous at $x = 3$, because despite the fact that $f(3) = 1$ is defined, and $\lim_{x \to 3} f(x) = 0$, those two values are not equal. There is a removable discontinuity at $x = 3$.
- f is discontinuous at $x = 0$, because despite the fact that $f(0) = 16$ is defined, the $\lim_{x \to 0} f(x)$

DOES NOT EXIST. There is a jump discontinuity at $x = 0$.

• f is discontinuous at $x = -4$ for two reasons, $f(-4)$ is undefined, and the $\lim_{x\to -4} f(x)$ DOES NOT EXIST. There is an infinite discontinuity at $x = -4$.

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Let $f(x) = \dfrac{1}{\sqrt{x^3 - 4x^2 + x - 7}}$. Compute $f'(x)$ using the limit definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7}} - \dfrac{1}{\sqrt{x^3 - 4x^2 + x - 7}}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{\sqrt{x^3 - 4x^2 + x - 7} - \sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7}}{\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} \cdot \sqrt{x^3 - 4x^2 + x - 7}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x^3 - 4x^2 + x - 7} - \sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7}}{h\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} \cdot \sqrt{x^3 - 4x^2 + x - 7}}$$ multiply by the conjugate here, short on space...

$$\cdot \left(\frac{\sqrt{x^3 - 4x^2 + x - 7} + \sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7}}{\sqrt{x^3 - 4x^2 + x - 7} + \sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7}} \right)$$

$$= \lim_{h\to 0} \frac{(x^3 - 4x^2 + x - 7) - ((x+h)^3 - 4(x+h)^2 + (x+h) - 7)}{h\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} \cdot \sqrt{x^3 - 4x^2 + x - 7}(****)}$$

where $**** = \sqrt{x^3 - 4x^2 + x - 7} + \sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7}$

$$= \lim_{h\to 0} \frac{(x^3 - 4x^2 + x - 7) - (x^3 + 3x^2h + 3xh^2 + h^3 - 4x^2 - 8xh - 4h^2 + x + h - 7)}{h\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} \cdot \sqrt{x^3 - 4x^2 + x - 7}(****)}$$

$$= \lim_{h\to 0} \frac{x^3 - 4x^2 + x - 7 - x^3 - 3x^2h - 3xh^2 - h^3 + 4x^2 + 8xh + 4h^2 - x - h + 7}{h\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} \cdot \sqrt{x^3 - 4x^2 + x - 7}(****)}$$

$$= \lim_{h\to 0} \frac{-3x^2h - 3xh^2 - h^3 + 8xh + 4h^2 - h}{h\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} \cdot \sqrt{x^3 - 4x^2 + x - 7}(****)}$$

$$= \lim_{h\to 0} \frac{h(-3x^2 - 3xh - h^2 + 8x + 4h - 1)}{h\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} \cdot \sqrt{x^3 - 4x^2 + x - 7}(****)}$$

$$= \lim_{h\to 0} \frac{-3x^2 - 3xh - h^2 + 8x + 4h - 1}{\sqrt{(x+h)^3 - 4(x+h)^2 + (x+h) - 7} \cdot \sqrt{x^3 - 4x^2 + x - 7}(****)}$$

$$= \frac{-3x^2 + 8x - 1}{\sqrt{x^3 - 4x^2 + x - 7} \cdot \sqrt{x^3 - 4x^2 + x - 7} \cdot (\sqrt{x^3 - 4x^2 + x - 7} + \sqrt{x^3 - 4x^2 + x - 7})}$$

$$= \frac{-3x^2+8x-1}{\sqrt{x^3-4x^2+x-7}\cdot\sqrt{x^3-4x^2+x-7}\cdot 2\sqrt{x^3-4x^2+x-7}}$$

$$= \frac{-3x^2+8x-1}{(x^3-4x^2+x-7)\cdot 2\sqrt{x^3-4x^2+x-7}} = \boxed{\frac{-3x^2+8x-1}{2(x^3-4x^2+x-7)^{\frac{3}{2}}}}$$

OPTIONAL BONUS #2 Compute $\lim_{x\to 1} \dfrac{\dfrac{1}{\sqrt{2-x}} - \dfrac{2}{\sqrt{3+x}}}{\dfrac{7}{\sqrt{50-x}} - \dfrac{6}{\sqrt{x+35}}}$

$$= \lim_{x\to 1} \frac{\dfrac{\sqrt{3+x}-2\sqrt{2-x}}{\sqrt{2-x}\sqrt{3+x}}}{\dfrac{7\sqrt{x+35}-6\sqrt{50-x}}{\sqrt{50-x}\sqrt{x+35}}} = \lim_{x\to 1} \frac{\sqrt{50-x}\sqrt{x+35}\left(\sqrt{3+x}-2\sqrt{2-x}\right)}{\sqrt{2-x}\sqrt{3+x}\left(7\sqrt{x+35}-6\sqrt{50-x}\right)} \left(\tfrac{0}{0}\right)$$

$$= \lim_{x\to 1} \frac{\sqrt{50-x}\sqrt{x+35}\left(\sqrt{3+x}-2\sqrt{2-x}\right)}{\sqrt{2-x}\sqrt{3+x}\left(7\sqrt{x+35}-6\sqrt{50-x}\right)} \cdot \left(\frac{\sqrt{3+x}+2\sqrt{2-x}}{\sqrt{3+x}+2\sqrt{2-x}}\right) \qquad \text{one conjugate}$$

$$= \lim_{x\to 1} \frac{\sqrt{50-x}\sqrt{x+35}\left((3+x)-4(2-x)\right)}{\sqrt{2-x}\sqrt{3+x}\left(7\sqrt{x+35}-6\sqrt{50-x}\right)\left(\sqrt{3+x}+2\sqrt{2-x}\right)}$$

$$= \lim_{x\to 1} \frac{\sqrt{50-x}\sqrt{x+35}\,(5x-5)}{\sqrt{2-x}\sqrt{3+x}\left(7\sqrt{x+35}-6\sqrt{50-x}\right)\left(\sqrt{3+x}+2\sqrt{2-x}\right)} \cdot \left(\frac{7\sqrt{x+35}+6\sqrt{50-x}}{7\sqrt{x+35}+6\sqrt{50-x}}\right)$$

other conjugate

$$= \lim_{x\to 1} \frac{\sqrt{50-x}\sqrt{x+35}\,(5x-5)\left(7\sqrt{x+35}+6\sqrt{50-x}\right)}{\sqrt{2-x}\sqrt{3+x}\left(49(x+35)-36(50-x)\right)\left(\sqrt{3+x}+2\sqrt{2-x}\right)}$$

$$= \lim_{x\to 1} \frac{\sqrt{50-x}\sqrt{x+35}\,(5x-5)\left(7\sqrt{x+35}+6\sqrt{50-x}\right)}{\sqrt{2-x}\sqrt{3+x}\left(49x+1715-1800+36x\right)\left(\sqrt{3+x}+2\sqrt{2-x}\right)}$$

$$= \lim_{x\to 1} \frac{\sqrt{50-x}\sqrt{x+35}\,(5(x-1))\left(7\sqrt{x+35}+6\sqrt{50-x}\right)}{\sqrt{2-x}\sqrt{3+x}\,(85x-85)\left(\sqrt{3+x}+2\sqrt{2-x}\right)}$$

$$= \lim_{x\to 1} \frac{\sqrt{50-x}\sqrt{x+35}\,(5(x-1))\left(7\sqrt{x+35}+6\sqrt{50-x}\right)}{\sqrt{2-x}\sqrt{3+x}\,(85(x-1))\left(\sqrt{3+x}+2\sqrt{2-x}\right)}$$

$$= \lim_{x\to 1} \frac{\sqrt{50-x}\sqrt{x+35}\,(5)\left(7\sqrt{x+35}+6\sqrt{50-x}\right)}{\sqrt{2-x}\sqrt{3+x}\,(85)\left(\sqrt{3+x}+2\sqrt{2-x}\right)} \qquad \text{(x-1) factor finally cancels}$$

$$= \frac{7\cdot 6\cdot 5\cdot 84}{1\cdot 2\cdot 85\cdot 4} = \boxed{\frac{441}{17}}$$