Math 111 Review Packet for Exam #1

Answer Key

Limit Practice Problems

Evaluate the following limits. Be clear if the limit equals a finite value, Does Not Exist, or is $+\infty$ or $-\infty$. Always justify your work:

1. $\displaystyle\lim_{w\to 0}\frac{16}{w} = \boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

   RHL: $\displaystyle\lim_{w\to 0^+}\frac{16}{w} = \frac{16}{0^+} = +\infty$

   LHL: $\displaystyle\lim_{w\to 0^-}\frac{16}{w} = \frac{16}{0^-} = -\infty$

2. $\displaystyle\lim_{t\to 2}\frac{3-t}{t-2} = \boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

   RHL: $\displaystyle\lim_{t\to 2^+}\frac{3-t}{t-2} = \frac{3-2}{0^+} = \frac{1}{0^+} = +\infty$

   LHL: $\displaystyle\lim_{t\to 2^-}\frac{3-t}{t-2} = \frac{3-2}{0^-} = \frac{1}{0^-} = -\infty$

3. $\displaystyle\lim_{t\to 2}\frac{3-t}{(t-2)^2} = \boxed{+\infty}$ since RHL = LHL

   RHL: $\displaystyle\lim_{t\to 2^+}\frac{3-t}{(t-2)^2} = \frac{3-2}{(0^+)^2} = \frac{1}{0^+} = +\infty$

   LHL: $\displaystyle\lim_{t\to 2^-}\frac{3-t}{(t-2)^2} = \frac{3-2}{(0^-)^2} = \frac{1}{0^+} = +\infty$

4. $\displaystyle\lim_{x\to 4}\frac{(x+2)^2}{x^2-3x-4} = \boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

   RHL: $\displaystyle\lim_{x\to 4^+}\frac{(x+2)^2}{x^2-3x-4} = \lim_{x\to 4^+}\frac{(x+2)^2}{(x-4)(x+1)} = \frac{(4+2)^2}{0^+(4+1)} = \frac{36}{0^+(5)} = +\infty$

   LHL: $\displaystyle\lim_{x\to 4^-}\frac{(x+2)^2}{x^2-3x-4} = \lim_{x\to 4^-}\frac{(x+2)^2}{(x-4)(x+1)} = \frac{(4+2)^2}{0^-(4+1)} = \frac{36}{0^-(5)} = -\infty$

5. $\displaystyle\lim_{x\to 4}\frac{x-4}{x^2-3x-4}\ {}^{\frac{0}{0}} = \lim_{x\to 4}\frac{x-4}{(x-4)(x+1)} = \lim_{x\to 4}\frac{1}{x+1} \overset{\text{DSP}}{=} \boxed{\frac{1}{5}}$

6. $\displaystyle\lim_{x\to 4}\frac{x^2-2x-8}{x^2-3x-4}\ {}^{\frac{0}{0}} = \lim_{x\to 4}\frac{(x-4)(x+2)}{(x-4)(x+1)} = \lim_{x\to 4}\frac{x+2}{x+1} \overset{\text{DSP}}{=} \frac{4+2}{4+1} = \boxed{\frac{6}{5}}$

7. $\displaystyle\lim_{x\to 1}\frac{x^2-4x-12}{x^2-3x-18} = \frac{1-4-12}{1-3-18} \overset{\text{DSP}}{=} \frac{-15}{-20} = \boxed{\frac{3}{4}}$

8. $\displaystyle\lim_{x\to 0}\frac{x^2-4x-12}{x^2-3x-18} \overset{\text{DSP}}{=} \frac{-12}{-18} = \boxed{\frac{2}{3}}$

9. $\lim_{x\to-3} \frac{x+2}{x+3} = \boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to-3^+} \frac{x+2}{x+3} = \frac{-1}{0^+} = -\infty$

LHL: $\lim_{x\to-3^-} \frac{x+2}{x+3} = \frac{-1}{0^-} = +\infty$

10. $\lim_{x\to-2} \frac{x^2-4x-12}{x^2-3x-18} \overset{\text{DSP}}{=} \frac{4+8-12}{4+6-18} = \frac{0}{-8} = \boxed{0}$

11. $\lim_{x\to0} \frac{x^2-4x}{x^2-7x}^{\frac{0}{0}} = \lim_{x\to0} \frac{x(x-4)}{x(x-7)} = \lim_{x\to0} \frac{x-4}{x-7} \overset{\text{DSP}}{=} \frac{-4}{-7} = \frac{4}{7}$

12. $\lim_{x\to3} \frac{x^2-9}{|x-3|} = \boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to3^+} \frac{x^2-9}{|x-3|} = \lim_{x\to3^+} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to3^+} x+3 \overset{\text{DSP}}{=} 6$

LHL: $\lim_{x\to3^-} \frac{x^2-9}{|x-3|} = \lim_{x\to3^-} \frac{(x-3)(x+3)}{-(x-3)} = \lim_{x\to3^-} -(x+3) \overset{\text{DSP}}{=} -6$

Recall: $|x-3| = \begin{cases} x-3 & \text{if } x-3 \geq 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \geq 3 \\ -(x-3) & \text{if } x < 3 \end{cases}$

13. $\lim_{x\to-5} \frac{x^2+6x+5}{|x+5|} = \boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to(-5)^+} \frac{x^2+6x+5}{|x+5|} = \lim_{x\to(-5)^+} \frac{(x+5)(x+1)}{x+5} = \lim_{x\to(-5)^+} x+1 \overset{\text{DSP}}{=} -4$

LHL: $\lim_{x\to(-5)^-} \frac{x^2+6x+5}{|x+5|} = \lim_{x\to(-5)^-} \frac{(x+5)(x+1)}{-(x+5)} = \lim_{x\to(-5)^-} -(x+1) \overset{\text{DSP}}{=} 4$

Recall: $|x+5| = \begin{cases} x+5 & \text{if } x+5 \geq 0 \\ -(x+5) & \text{if } x+5 < 0 \end{cases} = \begin{cases} x+5 & \text{if } x \geq -5 \\ -(x+5) & \text{if } x < -5 \end{cases}$

14. $\lim_{t\to1} \frac{t^2-1}{t^2-11t+10}^{\frac{0}{0}} = \lim_{t\to1} \frac{(t-1)(t+1)}{(t-10)(t-1)} = \lim_{t\to1} \frac{t+1}{t-10} \overset{\text{DSP}}{=} \frac{1+1}{1-10} = \boxed{-\frac{2}{9}}$

15. $\lim_{t\to1} \frac{t^2}{t^2+t-1} \overset{\text{DSP}}{=} \frac{1}{1+1-1} = \boxed{1}$

16. $\lim_{t\to-1} \frac{2009(t^2+6t+5)}{t^2+t}^{\frac{0}{0}} = \lim_{t\to-1} \frac{2009(t+5)(t+1)}{t(t+1)} = \lim_{t\to-1} \frac{2009(t+5)}{t} \overset{\text{DSP}}{=} \frac{2009(4)}{-1} = \boxed{-8036}$

17. $\lim_{x\to9} \frac{x^2-10x+9}{x^2+x-90}^{\frac{0}{0}} = \lim_{x\to9} \frac{(x-9)(x-1)}{(x+10)(x-9)} = \lim_{x\to9} \frac{x-1}{x+10} \overset{\text{DSP}}{=} \frac{9-1}{9+10} = \boxed{\frac{8}{19}}$

18. $\lim_{t\to1} t^{500} + t^{400} + t^{300} + t^{200} + t^{100} \overset{\text{DSP}}{=} \boxed{5}$

19. $\lim_{x\to 3} \frac{x^2-x-6}{x^2-2x-3} \overset{\frac{0}{0}}{} = \lim_{x\to 3} \frac{(x-3)(x+2)}{(x-3)(x+1)} = \lim_{x\to 3} \frac{x+2}{x+1} \overset{\text{DSP}}{=} \frac{3+2}{3+1} = \boxed{\frac{5}{4}}$

20. $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} \overset{\frac{0}{0}}{} = \lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4}$

$= \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{x-1} = \lim_{x\to 1} \sqrt{x+3}+2 \overset{\text{L.L.}}{=} \boxed{4}$

21. $\lim_{x\to 9} \frac{9x-x^2}{3-\sqrt{x}} \overset{\frac{0}{0}}{} = \lim_{x\to 9} \frac{9x-x^2}{3-\sqrt{x}} \cdot \frac{3+\sqrt{x}}{3+\sqrt{x}} = \lim_{x\to 9} \frac{x(9-x)(3+\sqrt{x})}{9-x}$

$= \lim_{x\to 9} x(3+\sqrt{x}) \overset{\text{L.L.}}{=} 9(3+3) = \boxed{54}$

22. $\lim_{x\to -1} \frac{5}{1-x} \overset{\text{DSP}}{=} \boxed{\frac{5}{2}}$

23. $\lim_{x\to 5} \frac{6x}{5-x} =$ $\boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 5^+} \frac{6x}{5-x} = \frac{30}{0^-} = -\infty$

LHL: $\lim_{x\to 5^-} \frac{6x}{5-x} = \frac{30}{0^+} = +\infty$

24. $\lim_{x\to 2} \frac{x^2-9x+14}{x^2-4x+4} \overset{\frac{0}{0}}{} = \lim_{x\to 2} \frac{(x-2)(x-7)}{(x-2)(x-2)} = \lim_{x\to 2} \frac{x-7}{x-2}$ $\boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 2^+} \frac{x-7}{x-2} = \lim_{x\to 2^+} \frac{-5}{0^+} = -\infty$

LHL: $\lim_{x\to 2^-} \frac{x-7}{x-2} = \lim_{x\to 2^-} \frac{x-7}{x-2} = \lim_{x\to 2^-} \frac{-5}{0^-} = +\infty$

25. $\lim_{x\to 2} \frac{x^2-4}{|x-2|} =$ $\boxed{\text{DNE b/c RHL} \neq \text{LHL}}$.

RHL: $\lim_{x\to 2^+} \frac{x^2-4}{|x-2|} = \lim_{x\to 2^+} \frac{x^2-4}{x-2} = \lim_{x\to 2^+} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^+} x+2 \overset{\text{DSP}}{=} 4$

LHL: $\lim_{x\to 2^-} \frac{x^2-4}{|x-2|} = \lim_{x\to 2^-} \frac{x^2-4}{-(x-2)} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{-(x-2)} = \lim_{x\to 2^-} -(x+2) \overset{\text{DSP}}{=} -4$

Recall: $|x-2| = \begin{cases} x-2 & \text{if } x-2 \geq 0 \\ -(x-2) & \text{if } x-2 < 0 \end{cases} = \begin{cases} x-2 & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$

26. $\lim_{x\to 3} \frac{\sqrt{x+6}-3}{x^2-x-6} \overset{\frac{0}{0}}{} = \lim_{x\to 3} \frac{\sqrt{x+6}-3}{x^2-x-6} = \lim_{x\to 3} \frac{x+6-9}{(x-3)(x+2)(\sqrt{x+6}+3)}$

$= \lim_{x\to 3} \frac{x-3}{(x-3)(x+2)(\sqrt{x+6}+3)} = \lim_{x\to 3} \frac{1}{(x+2)(\sqrt{x+6}+3)} \overset{\text{L.L.}}{=} \frac{1}{5(3+3)} = \boxed{\frac{1}{30}}$

27. $\lim_{x\to 7} \frac{\frac{1}{7}-\frac{1}{x}}{x-7}^{\frac{0}{0}} = \lim_{x\to 7} \frac{\frac{x}{7x}-\frac{7}{7x}}{x-7} = \lim_{x\to 7} \frac{\frac{x-7}{7x}}{x-7} = \lim_{x\to 7} \frac{x-7}{7x}\cdot\frac{1}{x-7} = \lim_{x\to 7} \frac{x-7}{(7x)(x-7)}$

$= \lim_{x\to 7} \frac{1}{7x} \overset{\text{DSP}}{=} \boxed{\frac{1}{49}}$

28. $\lim_{x\to -6} \frac{\frac{1}{2-x}-\frac{1}{8}}{x+6}^{\frac{0}{0}} = \lim_{x\to -6} \frac{\frac{8-(2-x)}{(2-x)(8)}}{x+6} = \lim_{x\to -6} \frac{\frac{6+x}{(2-x)(8)}}{x+6}$

$= \lim_{x\to -6} \frac{6+x}{(2-x)(8)(x+6)} = \lim_{x\to -6} \frac{1}{(2-x)(8)} \overset{\text{DSP}}{=} \boxed{\frac{1}{64}}$

29. $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{3-x}^{\frac{0}{0}} = \lim_{x\to 3} \frac{\sqrt{x+1}-2}{3-x}\cdot\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right) = \lim_{x\to 3} \frac{x+1-4}{(3-x)(\sqrt{x+1}+2)}$

$= \lim_{x\to 3} \frac{x-3}{(3-x)(\sqrt{x+1}+2)} = \lim_{x\to 3} \frac{x-3}{-(x-3)(\sqrt{x+1}+2)}$

$= \lim_{x\to 3} \frac{-1}{\sqrt{x+1}+2} \overset{\text{L.L.}}{=} \frac{-1}{2+2} = \boxed{-\frac{1}{4}}$

30. $\lim_{x\to 7} \frac{x^2-49}{2-\sqrt{x-3}}^{\frac{0}{0}} = \lim_{x\to 7} \frac{x^2-49}{2-\sqrt{x-3}}\cdot\left(\frac{2+\sqrt{x-3}}{2+\sqrt{x-3}}\right) = \lim_{x\to 7} \frac{(x^2-49)(2+\sqrt{x-3})}{4-(x-3)}$

$= \lim_{x\to 7} \frac{(x-7)(x+7)(2+\sqrt{x-3})}{7-x} = \lim_{x\to 7} \frac{(x-7)(x+7)(2+\sqrt{x-3})}{-(x-7)}$

$= \lim_{x\to 7} -(x+7)(2+\sqrt{x-3}) \overset{\text{L.L.}}{=} -(14)(2+2) = \boxed{-56}$

31. $\lim_{x\to 5} \frac{\frac{1}{\sqrt{x+20}}-\frac{1}{5}}{x-5}^{\frac{0}{0}} = \lim_{x\to 5} \frac{\frac{5-\sqrt{x+20}}{5(\sqrt{x+20})}}{x-5} = \lim_{x\to 5} \frac{5-\sqrt{x+20}}{5(\sqrt{x+20})(x-5)}\cdot\left(\frac{5+\sqrt{x+20}}{5+\sqrt{x+20}}\right)$

$= \lim_{x\to 5} \frac{25-(x+20)}{5(\sqrt{x+20})(x-5)(5+\sqrt{x+20})} = \lim_{x\to 5} \frac{5-x}{5(\sqrt{x+20})(x-5)(5+\sqrt{x+20})}$

$= \lim_{x\to 5} \frac{-(x-5)}{5(\sqrt{x+20})(x-5)(5+\sqrt{x+20})}$

$= \lim_{x\to 5} \frac{-1}{5(\sqrt{x+20})(5+\sqrt{x+20})} \overset{\text{L.L.}}{=} \frac{-1}{5(5)(5+5)} = \boxed{\frac{-1}{250}}$

Challenge!

Functions and Limit Practice Problems Evaluate the following limits:

32. Let $g(x) = 2x + 1$. Compute $\displaystyle\lim_{x\to 1} \frac{x-1}{g(x^2)-3} =$

$$\lim_{x\to 1} \frac{x-1}{(2x^2+1)-3} = \lim_{x\to 1} \frac{x-1}{2x^2-2} = \lim_{x\to 1} \frac{x-1}{2(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{2(x+1)} \overset{\text{DSP}}{=} \boxed{\frac{1}{4}}$$

33. Let $G(u) = u^2 + u$. Compute $\displaystyle\lim_{u\to 2} \frac{u^2-2u}{G(u-3)} = \lim_{u\to 2} \frac{u^2-2u}{(u-3)^2+(u-3)}$

$$= \lim_{u\to 2} \frac{u^2-2u}{u^2-6u+9+u-3} = \lim_{u\to 2} \frac{u(u-2)}{u^2-5u+6} = \lim_{u\to 2} \frac{u(u-2)}{(u-3)(u-2)}$$

$$= \lim_{u\to 2} \frac{u}{u-3} \overset{\text{DSP}}{=} \frac{2}{-1} = \boxed{-2}$$

34. Let $h(y) = y^2 - 3$. Compute $\displaystyle\lim_{x\to -2} \frac{x+2}{h(2x)-h(x+6)} =$

$$\lim_{x\to -2} \frac{x+2}{((2x)^2-3)-((x+6)^2-3)} = \lim_{x\to -2} \frac{x+2}{(4x^2-3)-(x^2+12x+36-3)}$$

$$= \lim_{x\to -2} \frac{x+2}{4x^2-3-x^2-12x-33} = \lim_{x\to -2} \frac{x+2}{3x^2-12x-36} = \lim_{x\to -2} \frac{x+2}{3(x^2-4x-12)}$$

$$= \lim_{x\to -2} \frac{x+2}{3(x-6)(x+2)} = \lim_{x\to -2} \frac{1}{3(x-6)} \overset{\text{DSP}}{=} \boxed{-\frac{1}{24}}$$

35. Let $f(t) = \dfrac{1}{t}$. Compute $\displaystyle\lim_{t\to 4} \frac{f(t-3)-4f(t)}{t-4} =$

$$\lim_{t\to 4} \frac{\left(\dfrac{1}{t-3} - \dfrac{4}{t}\right)}{t-4} = \lim_{t\to 4} \frac{\left(\dfrac{t-4(t-3)}{(t-3)t}\right)}{t-4} = \lim_{t\to 4} \frac{\left(\dfrac{-3t+12}{(t-3)t}\right)}{t-4}$$

$$= \lim_{t\to 4} \left(\frac{-3t+12}{(t-3)t}\right) \cdot \left(\frac{1}{t-4}\right) = \lim_{t\to 4} \frac{-3(t-4)}{(t-3)t(t-4)}$$

$$= \lim_{t\to 4} \frac{-3}{(t-3)t} \overset{\text{DSP}}{=} \frac{-3}{(1)(4)} = \boxed{-\frac{3}{4}}$$

36. Compute $\displaystyle\lim_{x\to -6} \frac{f(x^2)+5x-8}{[f(x)]^2+5x+14} =$ where $f(x) = x + 2$

$$\lim_{x\to -6} \frac{f(x^2)+5x-8}{[f(x)]^2+5x+14} \overset{\frac{0}{0}}{=} \lim_{x\to -6} \frac{x^2+2+5x-8}{[x+2]^2+5x+14} = \lim_{x\to -6} \frac{x^2+5x-6}{x^2+4x+4+5x+14}$$

$$= \lim_{x\to -6} \frac{x^2+5x-6}{x^2+9x+18} = \lim_{x\to -6} \frac{(x+6)(x-1)}{(x+6)(x+3)} = \lim_{x\to -6} \frac{x-1}{x+3} = \frac{-7}{-3} = \boxed{\frac{7}{3}}$$

More Functions

37. Let $f(x) = \sqrt{x}$, $g(x) = x^2 + 4$, and $h(x) = \dfrac{1}{x}$. Compute (and simplify, if possible) the following:

(a) $f \circ g(x) = f(g(x)) = f\left(x^2+4\right) = \boxed{\sqrt{x^2+4}}$

(b) $g \circ f(x) = g(f(x)) = g\left(\sqrt{x}\right) = (\sqrt{x})^2 + 4 = \boxed{x+4}$

(c) $h \circ g \circ f(x) = h(g(f(x)) = h(g(\sqrt{x})) = h(x+4) = \boxed{\dfrac{1}{x+4}}$

(d) $g \circ g(x) = g(g(x)) = g(x^2+4) = (x^2+4)^2 + 4 = x^4 + 8x^2 + 16 + 4 = \boxed{x^4 + 8x^2 + 20}$

$\varepsilon - \delta$ Definition of the Limit

Use the $\varepsilon - \delta$ definition for limits to prove each of the following:

38. $\lim\limits_{x \to 2} 7x - 6 = 8$.

Scratchwork: we want $|f(x) - L| = |(7x-6) - 8| < \varepsilon$

$$|f(x) - L| = |(7x-6) - 8| = |7x - 14| = |7(x-2)| = |7||x-2| = 7|x-2| \text{ (want } < \varepsilon)$$
$$7|x-2| < \varepsilon \text{ means } |x-2| < \frac{\varepsilon}{7}$$
$$\text{So choose } \delta = \frac{\varepsilon}{7} \text{ to restrict } 0 < |x-2| < \delta. \text{ That is } 0 < |x-2| < \frac{\varepsilon}{7}.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \dfrac{\varepsilon}{7}$. Given x such that $0 < |x-2| < \delta$, then

$$|f(x) - L| = |(7x-6) - 8| = |7x-14| = |7(x-2)| = |7||x-2| = 7|x-2| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon.$$

$\square$

39. $\lim\limits_{x \to -7} 2 - \dfrac{3}{7}x = 5$.

Scratchwork: we want $|f(x) - L| = \left|\left(2 - \dfrac{3}{7}x\right) - 5\right| < \varepsilon$

$$|f(x) - L| = \left|\left(2 - \frac{3}{7}x\right) - 5\right| = \left|-\frac{3}{7}x - 3\right| = \left|-\frac{3}{7}(x+7)\right| = \left|-\frac{3}{7}\right| |x-(-7)| = \frac{3}{7}|x-(-7)|$$
$$\text{(want } < \varepsilon)$$
$$\frac{3}{7}|x-(-7)| < \varepsilon \text{ means } |x-(-7)| < \frac{7}{3}\varepsilon$$
$$\text{So choose } \delta = \frac{7}{3}\varepsilon \text{ to restrict } 0 < |x-(-7)| < \delta. \text{ That is } 0 < |x-(-7)| < \frac{7}{3}\varepsilon.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{7}{3}\varepsilon$. Given x such that $0 < |x-(-7)| < \delta$, then

$$|f(x)-L| = \left|\left(2-\frac{3}{7}x\right)-5\right| = \left|-\frac{3}{7}x-3\right| = \left|-\frac{3}{7}(x+7)\right| = \left|-\frac{3}{7}\right||x-(-7)| = \frac{3}{7}|x-(-7)|$$
$$< \frac{3}{7}\cdot\frac{7}{3}\varepsilon = \varepsilon.$$

□

40. $\lim_{x\to -2} 2x+1 = -3$.

Scratchwork: we want $|f(x)-L| = |(2x+1)-(-3)| < \varepsilon$

$$|f(x)-L| = |(2x+1)-(-3)| = |2x+4| = |2(x+2)| = |2||x+2| = 2|x+2| = 2|x-(-2)|$$
$$(\text{want} < \varepsilon)$$
$$2|x-(-2)| < \varepsilon \text{ means } |x-(-2)| < \frac{\varepsilon}{2}$$
$$\text{So choose } \delta = \frac{\varepsilon}{2} \text{ to restrict } 0 < |x-(-2)| < \delta. \text{ That is } 0 < |x-(-2)| < \frac{\varepsilon}{2}.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{2}$. Given x such that $0 < |x-(-2)| < \delta$, then

$$|f(x)-L| = |(2x+1)-(-3)| = |2x+4| = |2(x+2)| = |2|\,|x-(-2)| = 2|x-(-2)|$$
$$< 2\cdot\frac{1}{2}\varepsilon = \varepsilon.$$

□

41. $\lim_{x\to 3} 1-4x = -11$.

Scratchwork: we want $|f(x)-L| = |(1-4x)-(-11)| < \varepsilon$

$$|f(x)-L| = |(1-4x)-(-11)| = |-4x+12| = |-4(x-3)| = |-4||x-3| = 4|x-3|$$
$$(\text{want} < \varepsilon)$$
$$4|x-3| < \varepsilon \text{ means } |x-3| < \frac{\varepsilon}{4}$$
$$\text{So choose } \delta = \frac{\varepsilon}{4} \text{ to restrict } 0 < |x-3| < \delta. \text{ That is } 0 < |x-3| < \frac{\varepsilon}{4}.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{4}$. Given x such that $0 < |x-3| < \delta$, then

$$|f(x)-L| = |1-4x-(-11)| = |12-4x| = |-4(x-3)| = |-4|\,|x-3| = 4|x-3|$$
$$< 4\cdot\frac{\varepsilon}{4} = \varepsilon.$$

□

42. $\lim_{x \to -3} 1 - 5x = 16$.

Scratchwork: we want $|f(x) - L| = |(1 - 5x) - 16| < \varepsilon$

$$|f(x) - L| = |(1 - 5x) - 16| = |-5x - 15| = |-5(x + 3)| = |-5||x - (-3)| = 5|x - (-3)|$$

(want $< \varepsilon$)

$$5|x - (-3)| < \varepsilon \text{ means } |x - (-3)| < \frac{\varepsilon}{5}$$

So choose $\delta = \frac{\varepsilon}{5}$ to restrict $0 < |x - (-3)| < \delta$. That is $0 < |x - (-3)| < \frac{\varepsilon}{5}$.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{5}$. Given x such that $0 < |x - (-3)| < \delta$, then

$$\begin{aligned} |f(x) - L| = |1 - 5x - 16| = |-5x - 15| = |-5(x + 3)| = |-5|\,|x - (-3)| &= 5|x - (-3)| \\ &< 5 \cdot \frac{\varepsilon}{5} = \varepsilon. \end{aligned}$$

□

43. $\lim_{x \to -14} \frac{4}{7}x + 3 = -5$.

Scratchwork: we want $|f(x) - L| = \left|\left(\frac{4}{7}x + 3\right) - (-5)\right| < \varepsilon$

$$|f(x) - L| = \left|\left(\frac{4}{7}x + 3\right) - (-5)\right| = \left|\frac{4}{7}x + 8\right| = \left|\frac{4}{7}(x + 14)\right| = \left|\frac{4}{7}\right| |x - (-14)| = \frac{4}{7}|x - (-14)|$$

(want $< \varepsilon$)

$$\frac{4}{7}|x - (-14)| < \varepsilon \text{ means } |x - (-14)| < \frac{7}{4}\varepsilon$$

So choose $\delta = \frac{7}{4}\varepsilon$ to restrict $0 < |x - (-14)| < \delta$. That is $0 < |x - (-14)| < \frac{7}{4}\varepsilon$.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{7}{4}\varepsilon$. Given x such that $0 < |x - (-14)| < \delta$, then

$$\begin{aligned} |f(x) - L| = \left|\left(\frac{4}{7}x + 3\right) - (-5)\right| = \left|\frac{4}{7}x + 8\right| = \left|\frac{4}{7}(x + 14)\right| = \left|\frac{4}{7}\right| |x - (-14)| &= \frac{4}{7}|x - (-14)| \\ &< \frac{4}{7} \cdot \frac{7}{4}\varepsilon = \varepsilon. \end{aligned}$$

□

Derivatives Use the limit definition of the derivative to calculate the derivative for each of the following functions:

44. $f(x) = 3 - 9x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(3-9(x+h)^2)-(3-9x^2)}{h}$$
$$= \lim_{h\to 0} \frac{3-9x^2-18xh-9h^2-3+9x^2}{h} = \lim_{h\to 0} \frac{-18xh-9h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(-18x-9h)}{h} = \lim_{h\to 0} -18x-9h = \boxed{-18x}$$

45. $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3-x^3}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} = \lim_{h\to 0} 3x^2+3xh+h^2 = \boxed{3x^2}$$

46. $f(x) = \dfrac{1}{x^2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\left(\frac{x^2-(x+h)^2}{(x+h)^2x^2}\right)}{h}$$
$$= \lim_{h\to 0} \frac{x^2-x^2-2xh-h^2}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{-2xh-h^2}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{h(-2x-h)}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{-2x-h}{(x+h)^2x^2} = \frac{-2x}{x^4}$$
$$= \boxed{\frac{-2}{x^3}}$$

47. $f(x) = \sqrt{x-7}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)-7}-\sqrt{x-7}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{(x+h)-7}-\sqrt{x-7}}{h} \cdot \frac{\sqrt{(x+h)-7}+\sqrt{x-7}}{\sqrt{(x+h)-7}+\sqrt{x-7}}$$
$$= \lim_{h\to 0} \frac{(x+h-7)-(x-7)}{h(\sqrt{(x+h)-7}+\sqrt{x-7})} = \lim_{h\to 0} \frac{x+h-7-x+7}{h(\sqrt{(x+h)-7}+\sqrt{x-7})} = \lim_{h\to 0} \frac{h}{h(\sqrt{(x+h)-7}+\sqrt{x-7})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{(x+h)-7}+\sqrt{x-7}} = \boxed{\frac{1}{2\sqrt{x-7}}}$$

48. $f(x) = \dfrac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}\right)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} = \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \cdot \left(\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right)$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0} \frac{-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{(\sqrt{x})^2 2\sqrt{x}} = \boxed{\frac{-1}{2x^{\frac{3}{2}}}}$$

49. $f(x) = \dfrac{3-x}{x-4}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{3-(x+h)}{(x+h)-4} - \frac{3-x}{x-4}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{[3-x-h](x-4)-(3-x)[x+h-4]}{(x+h-4)(x-4)}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{3x-x^2-xh-12+4x+4h-3x-3h+12+x^2+xh-4x}{(x+h-4)(x-4)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{h}{(x+h-4)(x-4)}\right)}{h} = \lim_{h\to 0} \frac{h}{(x+h-4)(x-4)}\left(\frac{1}{h}\right)$$

$$= \lim_{h\to 0} \frac{1}{(x+h-4)(x-4)} = \boxed{\frac{1}{(x-4)^2}}$$

50. $f(x) = \dfrac{3x-1}{2-5x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{3(x+h)-1}{2-5(x+h)} - \frac{3x-1}{2-5x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{[3x+3h-1](2-5x)-(3x-1)[2-5x-5h]}{(2-5(x+h))(2-5x)}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{6x+6h-2-15x^2-15xh+5x-6x+15x^2+15xh+2-5x-5h}{(2-5(x+h))(2-5x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{h}{(2-5(x+h))(2-5x)}\right)}{h} = \lim_{h\to 0} \frac{h}{(2-5(x+h))(2-5x)}\left(\frac{1}{h}\right)$$

$$= \lim_{h\to 0} \frac{1}{(2-5(x+h))(2-5x)} = \boxed{\frac{1}{(2-5x)^2}}$$

Tangent Lines Please use the limit definition for the derivative when computing the derivatives in this section.

51. Find an equation for the tangent line to the graph of $f(x) = x - 2x^2$ at the point $(1, -1)$

First compute the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{((x+h) - 2(x+h)^2) - (x - 2x^2)}{h}$$

$$= \lim_{h\to 0} \frac{x + h - 2x^2 - 4xh - 2h^2 - x + 2x^2}{h} = \lim_{h\to 0} \frac{h - 4xh - 2h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(1 - 4x - 2h)}{h} = \lim_{h\to 0} 1 - 4x - 2h = 1 - 4x$$

Note: $f'(1) = 1 - 4(1) = -3$, so using *point slope form*, the equation of the tangent line through the point $(1, -1)$ with slope -3 is given by

$y - (-1) = -3(x - 1)$ or $\boxed{y = -3x + 2}$.

52. Find an equation for the tangent line to the graph of $f(x) = \sqrt{x}$ at $x = 4$

First compute the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Note: $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. The point is $(4, f(4)) = (4, \sqrt{4}) = (4, 2)$. Therefore, using *point slope form*, the equation of the tangent line throught the point $(\ 4, 2)$ with slope $\frac{1}{4}$ is given by

$y - 2 = \frac{1}{4}(x - 4)$ or $\boxed{y = \frac{1}{4}x + 1}$.

53. At which point(s) does the graph of $f(x) = -x^2 + 13$ have a horizontal tangent line?

First compute the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(-(x+h)^2 + 13) - (-x^2 + 13)}{h}$$

$$= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 + 13 + x^2 - 13}{h} = \lim_{h\to 0} \frac{-2xh - h^2}{h} = \lim_{h\to 0} \frac{h(-2x - h)}{h} = \lim_{h\to 0} -2x - h$$

$$= -2x$$

Note: Set $f'(x) = 0$ and solve $f'(x) = -2x = 0 \Rightarrow x = 0$ so the point is $(0, f(0)) = \boxed{(0, 13)}$.

54. At which point(s) of the graph of $f(x) = -x^3 + 13$ is the slope of the tangent line equal to -27? What's the picture representing this problem?

First compute the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(-(x+h)^3 + 13) - (-x^3 + 13)}{h}$$

$= \lim_{h\to 0} \dfrac{-x^3 - 3x^2h - 3xh^2 - h^3 + 13 + x^3 - 13}{h} = \lim_{h\to 0} \dfrac{-3x^2h - 3xh^2 - h^3}{h}$

$= \lim_{h\to 0} \dfrac{h(-3x^2 - 3xh - h^2)}{h} = \lim_{h\to 0} -3x^2 - 3xh - h^2 = -3x^2$

Note: Set $f'(x) = -27$ and solve $f'(x) = -3x^2 = -27 \Rightarrow x^{\ 2} = 9 \Rightarrow x = \pm 3$ so the points are $(3, f(3)) = \boxed{(3, -14)}$ and $(-3, f(-3)) = \boxed{(-3, 40)}$.

55. There are two points on the graph of the curve $y = -x^2 + 7$ whose tangent line to the graph at those points passes through the point $(0, 11)$. Find those two points.

CHALLENGE!!

First compute the derivative $f'(x)$.

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{(-(x+h)^2 + 7) - (-x^2 + 7)}{h}$

$= \lim_{h\to 0} \dfrac{-x^2 - 2xh - h^2 + 7 + x^2 - 7}{h} = \lim_{h\to 0} \dfrac{-2xh - h^2}{h} = \lim_{h\to 0} \dfrac{h(-2x - h)}{h} = \lim_{h\to 0} -2x - h$

$= -2x$

Let a point on the graph be given by $(a, f(a)) = (a, -a^2 + 7)$. The slope of the tangent line at this point $(a, -a^2+7)$ is given by $f'(a) = -2a$. The tangent line to this curve through the point $(a, -a^2+7)$ with slope $-2a$ is given by $y - (-a^2+7) = -2a(x - a)$ or $y + a^2 - 7 = -2ax + 2a^2$. For this tangent line to pass through the exterior point $(0, 11)$, that means the point $(0, 11)$ satisfies the equation of the tangent line. Then, $11 + a^2 - 7 = 0 + 2a^2$ or $a^2 = 4 \Rightarrow a = \pm 2$. So the two points of interest here are $(2, f(2)) = \boxed{(2, 3)}$ and $(-2, f(-2)) = \boxed{(-2, 3)}$.

56. Find the equation of the line passing through $(2, 3)$ which is perpendicular to the tangent to the curve $y = x^3 - 3x + 1$ at the point $(2, 3)$.

First we will find the slope of the tangent line to this curve when $x = 2$. Then we will take minus the reciprical of that slope to finish the problem.

First compute the derivative $f'(x)$.

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{((x+h)^3 - 3(x+h) + 1) - (x^3 - 3x + 1)}{h}$

$= \lim_{h\to 0} \dfrac{x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h + 1 - x^3 + 3x - 1}{h}$

$= \lim_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3 - 3h}{h} = \lim_{h\to 0} \dfrac{h(3x^2 + 3xh + h^2 - 3)}{h}$

$= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 3 = 3x^2 - 3$

Thus, $f'(2) = 9$, so the line perpendicular to that would have slope equal to $-\dfrac{1}{9}$. The equation of the line through the point $(2, 3)$ with slope $-\dfrac{1}{9}$ is given by *point slope form* as $y - 3 = -\dfrac{1}{9}(x - 2)$. So, $\boxed{y = -\dfrac{1}{9}x + \dfrac{29}{9}}$.

57. Find the equation of the tangent line to the curve $y = x^3 + x$ at the point(s) where the slope equals 4.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^3 + (x+h)) - (x^3 + x)}{h}$$
$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + x + h - x^3 - x}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + h}{h}$$
$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 + 1)}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 + 1 = 3x^2 + 1$$

Set $f'(x) = 3x^2 + 1 = 4$ and solve for $x = \pm 1$. Therefore, the points where slope is equal to 4 are $(1, (f(1)) = (1, 2)$ and $(-1, f(-1)) = (-1, -2)$.

The equation of the tangent line to the curve, at the point $(1, 2)$ with slope equaling 4, is given by $y - 2 = 4(x - 1)$ or $\boxed{y = 4x - 2}$.

Finally, the equation of the tangent line to the curve, at the point $(-1, -2)$ with slope equaling 4, is given by $y - (-2) = 4(x - (-1))$ or $\boxed{y = 4x + 2}$.

58. Find an equation for the tangent line to the graph of $f(x) = \dfrac{1}{x-1}$ at the point $(0, -1)$.

First we compute the slope $f'(x)$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left(\dfrac{1}{(x+h)-1} - \dfrac{1}{x-1}\right)}{h} = \lim_{h\to 0} \frac{\left(\dfrac{(x-1) - (x+h-1)}{((x+h)-1)(x-1)}\right)}{h}$$
$$= \lim_{h\to 0} \frac{\left(\dfrac{x-1-x-h+1)}{((x+h)-1)(x-1)}\right)}{h} = \lim_{h\to 0} \frac{-h}{h(x+h-1)(x-1)}$$
$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = -\frac{1}{(x-1)^2}$$

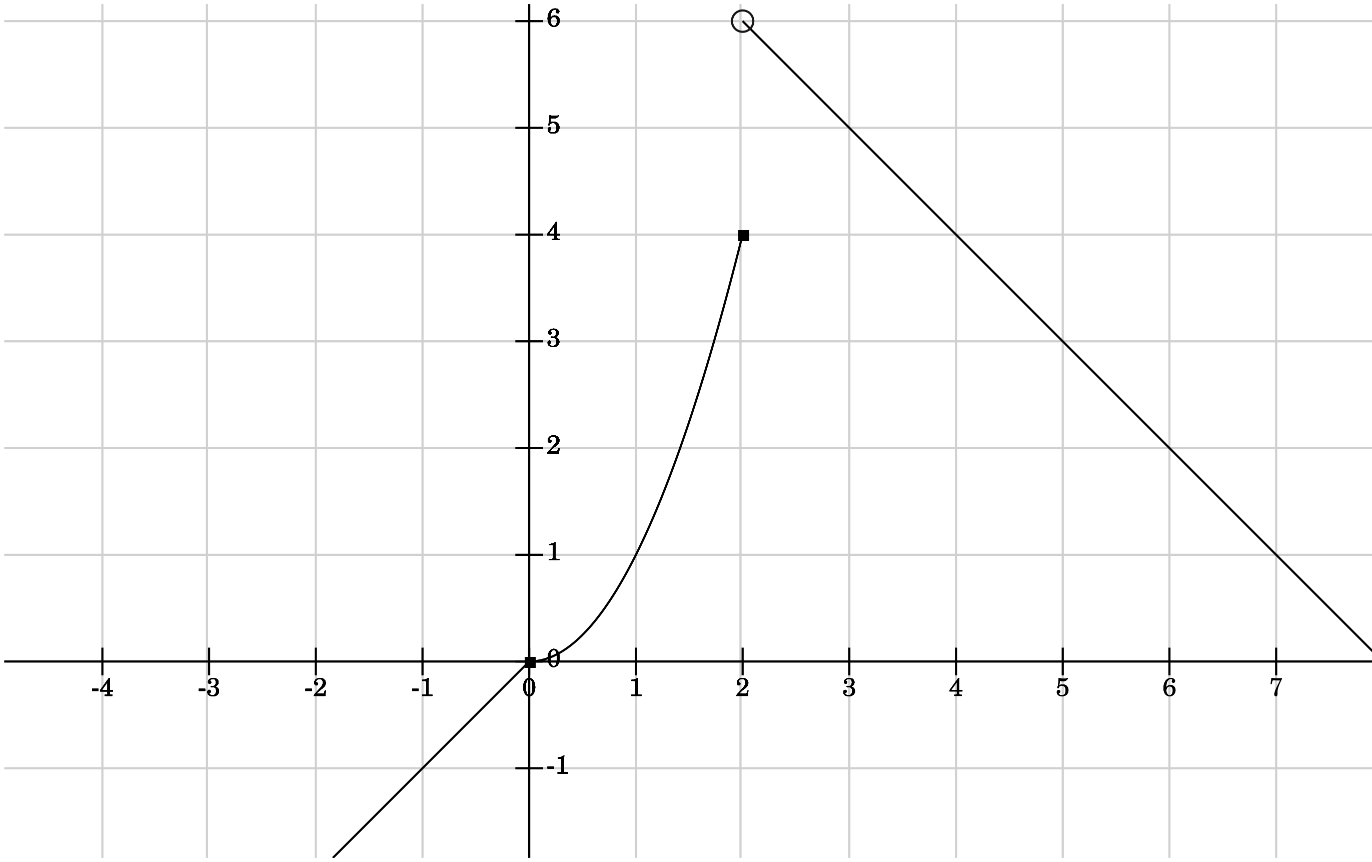
Note: $f'(0) = -1$. Therefore, using *point slope form*, the equation of the tangent line through the point $(0, -1)$ with slope equal to -1 is given by $y - (-1) = -1(x - 0)$ or $\boxed{y = -x - 1}$.

Piece-wise defined functions

Consider each of the following piecewise defined functions. Answer the related questions. *Justify* your answers please.

59. Let $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 2 \\ 8 - x & \text{if } x > 2 \end{cases}$

Sketch the graph.



Find the numbers at which f is discontinuous.

Evaluate:

$\lim_{x\to 2} f(x) =$ $\boxed{\text{DNE b/c LHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 2^+} f(x) = 6$

LHL: $\lim_{x\to 2^-} f(x) = 4$

$\lim_{x\to 0} f(x) = \boxed{0}$

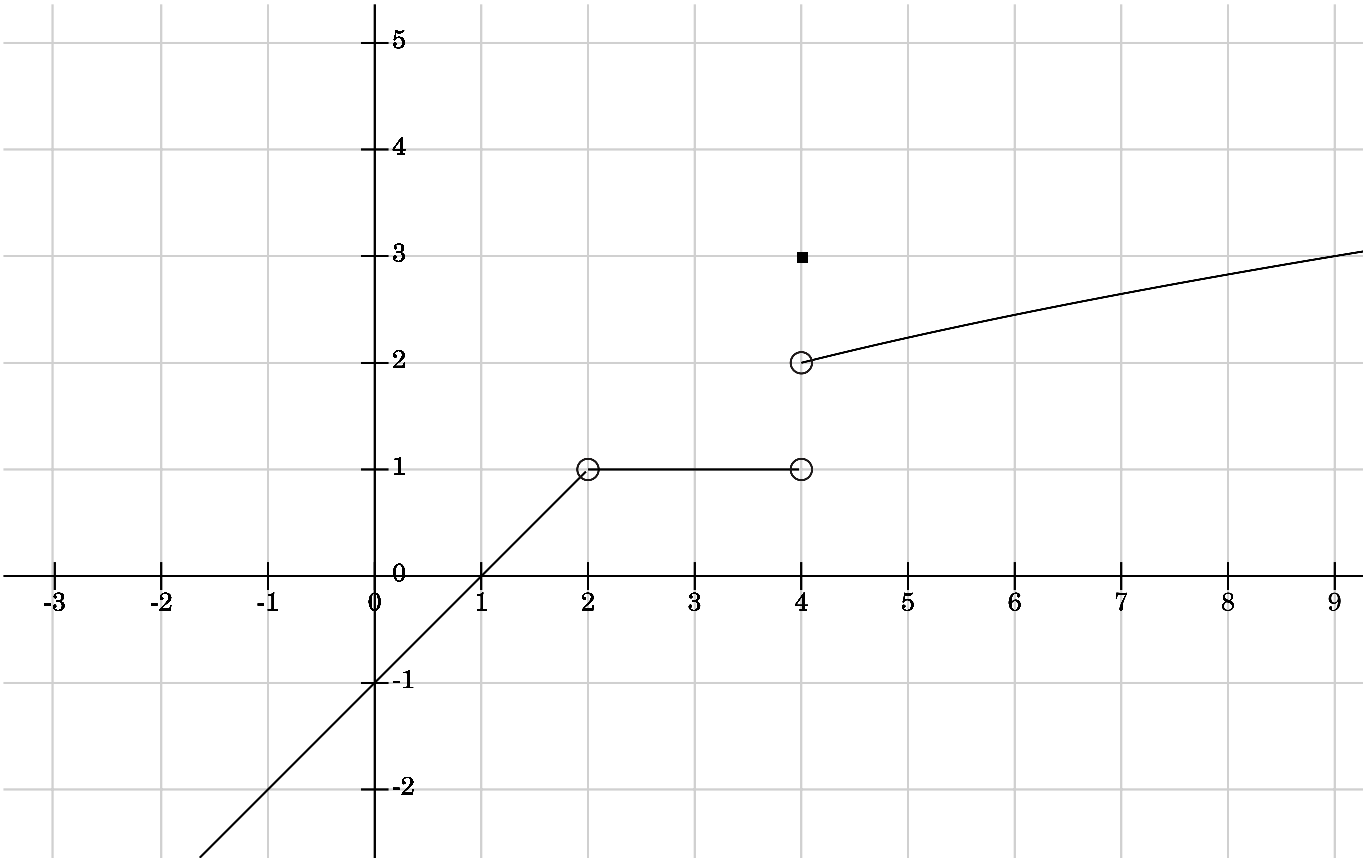
RHL: $\lim_{x\to 0^+} f(x) = 0$

LHL: $\lim_{x\to 0^-} f(x) = 0$

$\boxed{f \text{ is discontinuous at } x = 2 \text{ since } \lim_{x\to 2} f(x) \text{ DNE}}$

60. Let $f(x) = \begin{cases} x-1 & \text{if } x < 2 \\ 1 & \text{if } 2 < x < 4 \\ 3 & \text{if } x = 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

Sketch the graph.



Find the numbers at which f is discontinuous. Evaluate:

$\lim_{x\to 0} f(x) = \boxed{-1}$ b/c RHL=LHL

RHL: $\lim_{x\to 0^+} f(x) = -1$

LHL: $\lim_{x\to 0^-} f(x) = -1$

$\lim_{x\to 2} f(x) = \boxed{1}$ b/c RHL=LHL

RHL: $\lim_{x\to 2^+} f(x) = 1$

LHL: $\lim_{x\to 2^-} f(x) = 1$

$\lim_{x\to 4} f(x) = \boxed{\text{DNE b/c LHL} \neq \text{LHL}}$

RHL: $\lim_{x \to 4^+} f(x) = 2$

LHL: $\lim_{x \to 4^-} f(x) = 1$

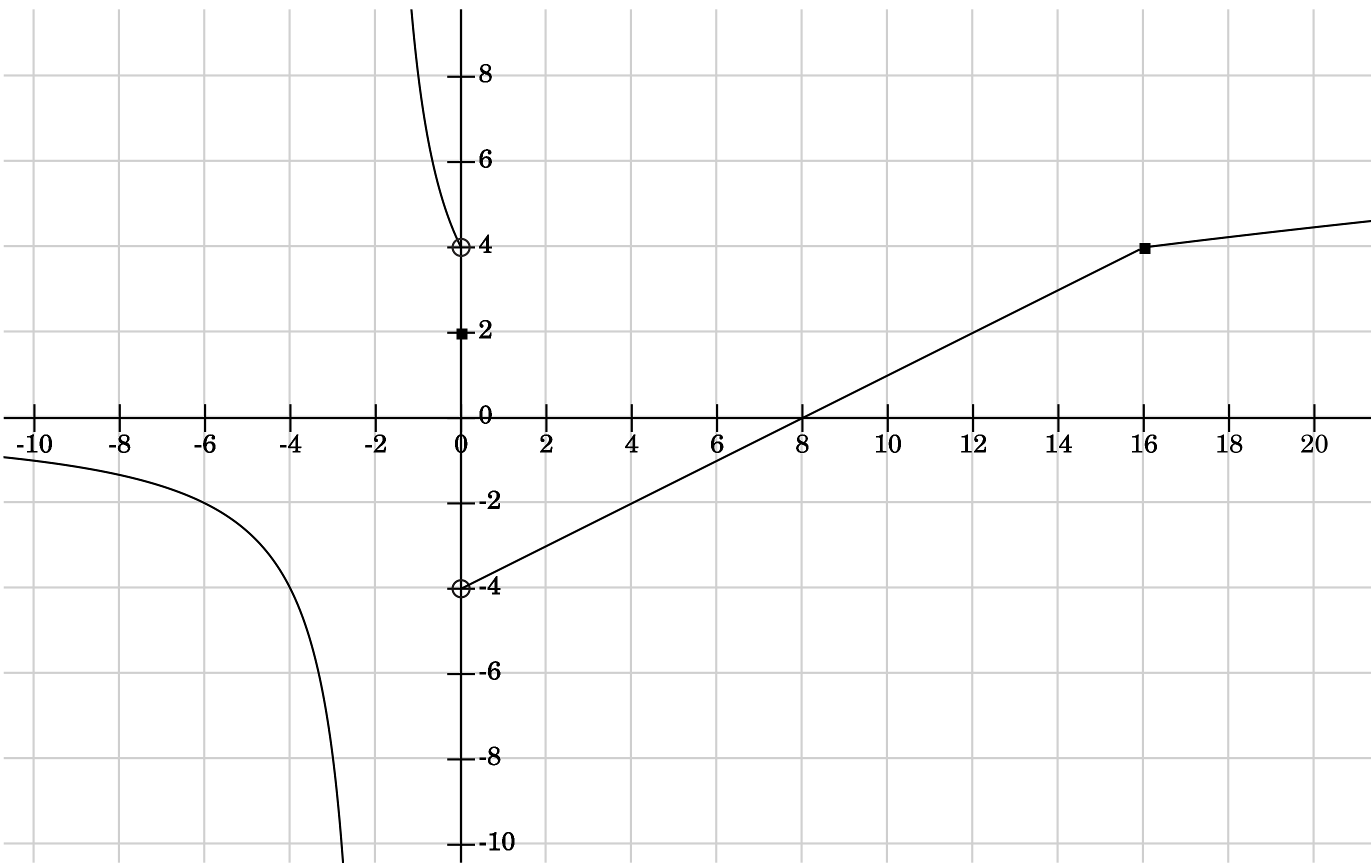
$f(4) = \boxed{3}$

$\boxed{f \text{ is discontinuous at } x = 2 \text{ because } f(2) \text{ is undefined}}$

$\boxed{f \text{ is discontinuous at } x = 4 \text{ because } \lim_{x \to 4} f(x) \ \boxed{\text{DNE b/c LHL} \neq \text{LHL}}}$

61. Let $h(x) = \begin{cases} \dfrac{8}{x+2} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ \dfrac{1}{2}x - 4 & \text{if } 0 < x \leq 16 \\ \sqrt{x} & \text{if } x > 16 \end{cases}$

Sketch the graph.



Find the numbers at which h is discontinuous. Evaluate:

$\lim_{x \to -2} h(x) = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x \to -2^+} h(x) = +\infty$

LHL: $\lim_{x \to -2^-} h(x) = -\infty$

$\lim_{x \to 0} h(x) = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x \to 0^+} h(x) = -4$

LHL: $\lim_{x \to 0^-} h(x) = 4$

$\lim_{x \to 16} h(x) = \boxed{4}$ b/c RHL=LHL

RHL: $\lim_{x \to 16^+} h(x) = 4$

LHL: $\lim_{x \to 16^-} h(x) = 4$

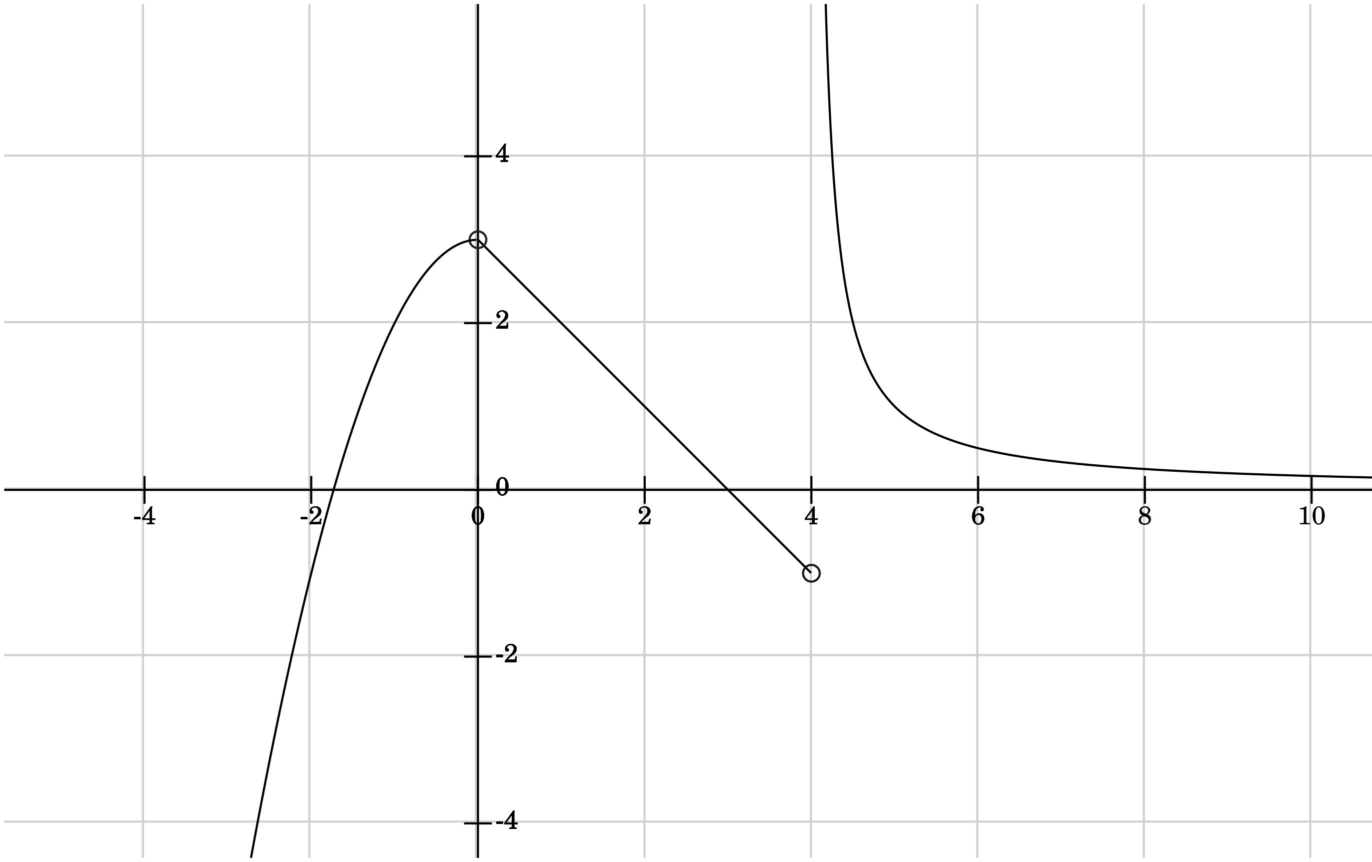
$\boxed{h \text{ is discontinuous at } x = -2 \text{ because } \lim_{x \to -2} h(x) \text{ DNE or because } f(-2) \text{ is undefined}}$

$\boxed{h \text{ is discontinuous at } x = 0 \text{ because } \lim_{x \to 0} h(x) \text{ DNE}}$

Note h is continuous at $x = 16$ because $\lim_{x \to 16} h(x) = h(16)$

62. Let $F(x) = \begin{cases} \dfrac{1}{x-4} & \text{if } x > 4 \\ 3 - x & \text{if } 0 < x < 4 \\ 3 - x^2 & \text{if } x < 0 \end{cases}$

Sketch the graph.



Find the numbers at which F is discontinuous. Evaluate:

$\lim_{x\to 0} F(x) = \boxed{3}$ b/c RHL=LHL

RHL: $\lim_{x\to 0^+} F(x) = 3$

LHL: $\lim_{x\to 0^-} F(x) = 3$

$\lim_{x\to 4} F(x) = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 4^+} F(x) = +\infty$

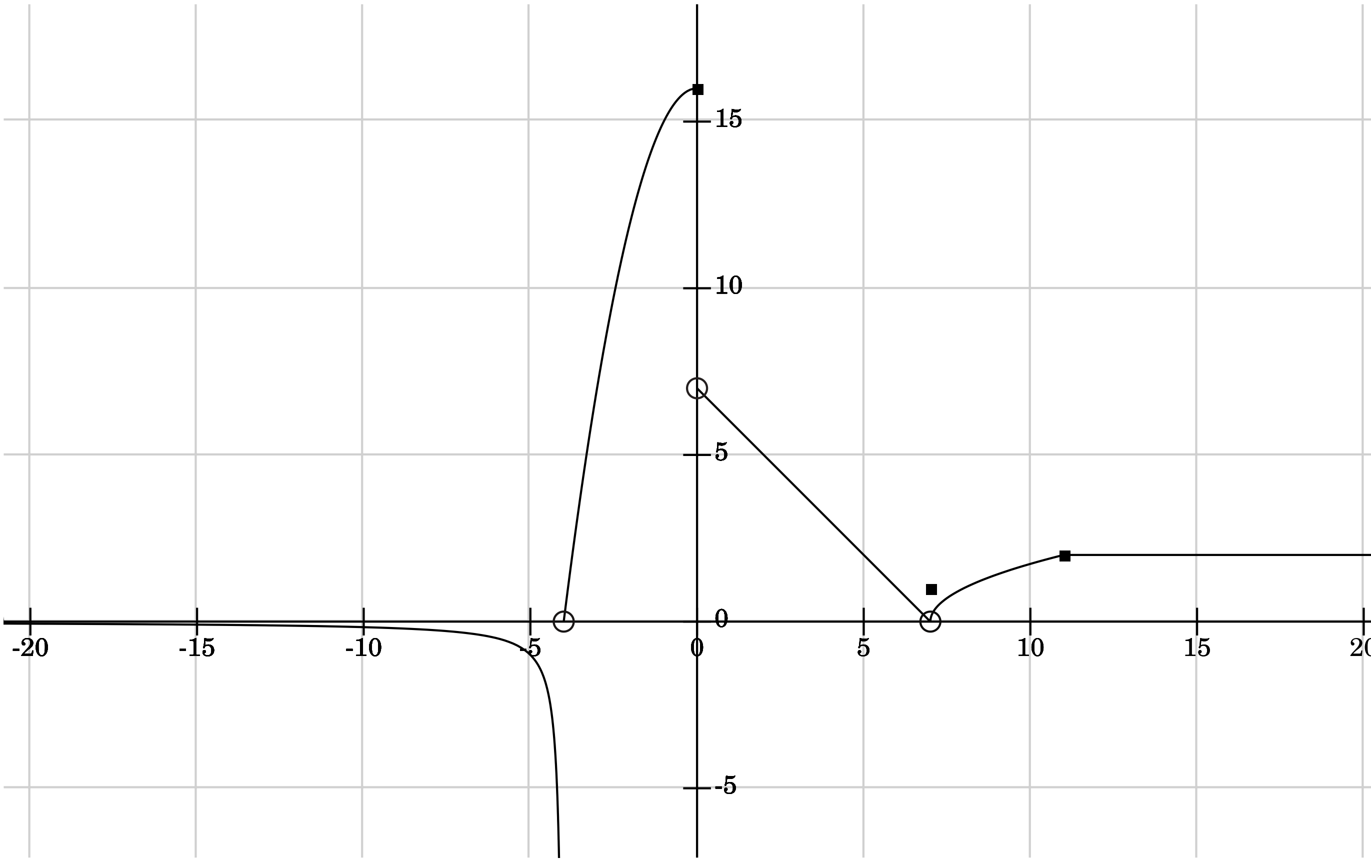
LHL: $\lim_{x\to 4^-} F(x) = -1$

$\boxed{f \text{ is discontinuous at } x = 0 \text{ because } f(0) \text{ is undefined}}$

$\boxed{f \text{ is discontinuous at } x = 4 \text{ because } f(4) \text{ is undefined or because } \lim_{x\to 4} F(x) \text{ DNE}}$

63. Let $f(x) = \begin{cases} 2 & \text{if } x \geq 11 \\ \sqrt{x-7} & \text{if } 7 < x < 11 \\ 1 & \text{if } x = 7 \\ 7 - x & \text{if } 0 < x < 7 \\ 16 - x^2 & \text{if } -4 < x \leq 0 \\ \dfrac{1}{x+4} & \text{if } x < -4 \end{cases}$

Sketch the graph.



Find the numbers at which f is discontinuous. Justify your answer(s) using the definition of continuity.

• f is discontinuous at $x = 7$, because despite the fact that $f(7) = 1$ is defined, and $\lim_{x \to 7} f(x) = 0$, those two values are not equal.

• f is discontinuous at $x = 0$, because despite the fact that $f(0) = 16$ is defined, the $\lim_{x \to 0} f(x)$ DOES NOT EXIST.

• f is discontinuous at $x = -4$ for two reasons, $f(-4)$ is undefined, and the $\lim_{x \to -4} f(x)$ DOES NOT EXIST.

Note that f is continuous at $x = 11$ because $\lim_{x \to 11} f(x) = 2 = f(11)$

Evaluate:

$\lim_{x \to -4} f(x) =$ $\boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

RHL: $\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} 16 - x^2 = 0$

LHL: $\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \dfrac{1}{x+4} = \dfrac{1}{0^-} = -\infty$

$\lim_{x \to 0} f(x) =$ $\boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

RHL: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 7 - x = 7$ LHL: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 16 - x^2 = 16$

$\lim_{x \to 7} f(x) = \boxed{0}$ since RHL=LHL

RHL: $\lim_{x \to 7^+} f(x) = \lim_{x \to 7^+} \sqrt{x-7} = 0$ LHL: $\lim_{x \to 7^-} f(x) = \lim_{x \to 7^-} 7 - x = 0$

$\lim_{x \to 11} f(x) == \boxed{2}$

RHL: $\lim_{x \to 11^+} f(x) = \lim_{x \to 11^+} 2 = 2$

LHL: $\lim_{x \to 11^-} f(x) = \lim_{x \to 11^-} \sqrt{x-7} = \sqrt{4} = 2$