Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 111

Section 01

Midterm Exam #1

September 26, 2014

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or webpages, or other aids are permitted.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		30
2		15
3		15
4		10
5		10
6		20
Total		100

1. [30 Points] Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim_{x\to -6} \dfrac{x^2+4x-21}{x^2-5x-6} \overset{DSP}{=} \dfrac{36-24-21}{36+30-6} = \dfrac{-9}{60} = \boxed{\dfrac{-3}{20}}$

(b) $\lim_{x\to 6} \dfrac{\overset{(x-6)(x-7)}{x^2-13x+42}}{|6-x|} =$

$|6-x| = \begin{cases} 6-x \text{ if } 6-x \geq 0 & x \leq 6 \quad \text{LHL} \\ -(6-x) \text{ if } 6-x < 0 & x > 6 \quad \text{RHL} \end{cases}$

$x-6$

RHL: $\lim_{x\to 6^+} \dfrac{\cancel{(x-6)}(x-7)}{\cancel{x-6}} = \lim_{x\to 6^+} x-7 = -1$

LHL: $\lim_{x\to 6^-} \dfrac{\overset{(-1)}{\cancel{(x-6)}}(x-7)}{\cancel{6-x}} = \lim_{x\to 6^-} -(x-7) = 1$

DNE RHL $\neq$ LHL.

(c) $\lim_{x\to -3} \dfrac{f(2x)-3f(x)-7}{f(x+2)-2} =$ where $f(x) = x^2+1$

$= \lim_{x\to -3} \dfrac{[(2x)^2+1] - 3(x^2+1) - 7}{[\underbrace{(x+2)^2}_{x^2+4x+4}+1] - 2} = \lim_{x\to -3} \dfrac{4x^2+1-3x^2-3-7}{x^2+4x+5-2}$

$= \lim_{x\to -3} \dfrac{x^2-9}{x^2+4x+3} \overset{0/0}{=} \lim_{x\to -3} \dfrac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}(x+1)} \overset{DSP}{=} \dfrac{-6}{-2} = \boxed{3}$

1. (Continued) Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(d) $\lim_{x\to -5} \dfrac{x^2-x-30}{x^2+10x+25} \overset{0/0}{=} \lim_{x\to -5} \dfrac{(x-6)\cancel{(x+5)}}{(x+5)\cancel{(x+5)}} = \lim_{x\to -5} \dfrac{x-6}{x+5} \quad -11/0$

$$\text{RHL: } \lim_{x\to -5^+} \frac{x-6}{x+5} = \frac{-11}{0^+} = -\infty$$

-4.999

$$\text{LHL: } \lim_{x\to -5^-} \frac{x-6}{x+5} = \frac{-11}{0^-} = +\infty$$

-5.0001

DNE RHL ≠ LHL.

(e) $\lim_{x\to 4} \dfrac{\frac{3-x}{x-5} - \frac{3}{7-x}}{x^2-x-12} = \lim_{x\to 4} \dfrac{\left[\dfrac{(3-x)(7-x)-3(x-5)}{(x-5)(7-x)}\right]}{(x-4)(x+3)}$

$$= \lim_{x\to 4} \frac{\left[\dfrac{21-10x+x^2-3x+15}{(x-5)(7-x)}\right]}{(x-4)(x+3)} = \lim_{x\to 4} \frac{x^2-13x+36}{(x-4)(x+3)(x-5)(7-x)}$$

$$= \lim_{x\to 4} \frac{\cancel{(x-4)}(x-9)}{\cancel{(x-4)}(x+3)(x-5)(7-x)} = \lim_{x\to 4} \frac{x-9}{(x+3)(x-5)(7-x)} = \frac{-5}{7\cdot(-1)(3)} = \frac{-5}{-21} = \boxed{\frac{5}{21}}$$

2. [15 Points] Prove that $\lim_{x \to 2} 5 - 3x = -1$ using the $\varepsilon - \delta$ **definition of the limit.**

Scratchwork:

$$
\begin{aligned}
|f(x) - L| &= |5 - 3x - (-1)| \\
&= |5 - 3x + 1| \\
&= |6 - 3x| \\
&= |-3(x-2)| \\
&= |-3|\,|x-2| \\
&= 3|x-2| \overset{\text{want}}{<} \varepsilon \implies \text{choose } |x-2| < \varepsilon/2 \leftarrow \delta
\end{aligned}
$$

Proof: Let $\varepsilon > 0$ be given.

Choose $\delta = \varepsilon/2$.

Given x such that $0 < |x-2| < \delta$.

Then,

$$
\begin{aligned}
|f(x) - L| &= |5 - 3x - (-1)| = |6 - 3x| = |-2(x-3)| = |-2|\,|x-3| \\
&= 2|x-3| < 2 \cdot \varepsilon/2 = \varepsilon \quad \square
\end{aligned}
$$

3. [15 Points] Suppose that $f(x) = \dfrac{5-2x}{1+6x}$. Compute $f'(x)$ using the **limit definition of the derivative.**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{5-2(x+h)}{1+6(x+h)} - \frac{5-2x}{1+6x}}{h} \quad \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{(5-2x-2h)(1+6x) - (5-2x)(1+6x+6h)}{[1+6(x+h)](1+6x)} \left(\frac{1}{h}\right)$$

$$= \lim_{h\to 0} \frac{5-2x-2h+30x-12x^2-12xh - (5+30x+30h-2x-12x^2-12xh)}{[1+6(x+h)](1+6x)\,h}$$

$$= \lim_{h\to 0} \frac{\cancel{5}\cancel{-2x}-2h\cancel{+30x}\cancel{-12x^2}\cancel{-12xh}\cancel{-5}\cancel{-30x}-30h\cancel{+2x}\cancel{+12x^2}\cancel{+12xh}}{[1+6(x+h)](1+6x)\,h}$$

$$= \lim_{h\to 0} \frac{-32\cancel{h}}{[1+6(x+h)](1+6x)\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-32}{[1+6(x+\underset{0}{h})](1+6x)} = \boxed{\frac{-32}{(1+6x)^2}}$$

Check: Quotient Rule $f'(x) = \dfrac{(1+6x)(-2) - (5-2x)(6)}{(1+6x)^2} = \dfrac{-2\cancel{-12x}-30\cancel{+12x}}{(1+6x)^2}$

$$= \boxed{\frac{-32}{(1+6x)^2}} \text{ Match}$$

4. [10 Points] Suppose that $f(x) = \sqrt{6-x}$. Write the **equation of the tangent line** to the curve $y = f(x)$ where $x = -3$.

Use the limit definition of the derivative when computing the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{0}{0}$$

$$= \lim_{h \to 0} \frac{\sqrt{6-(x+h)} - \sqrt{6-x}}{h} \cdot \frac{\sqrt{6-(x+h)} + \sqrt{6-x}}{\sqrt{6-(x+h)} + \sqrt{6-x}}$$

$$= \lim_{h \to 0} \frac{6-(x+h) - (6-x)}{h\left(\sqrt{6-(x+h)} + \sqrt{6-x}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{6} - \cancel{x} - h - \cancel{6} + \cancel{x}}{h\left(\sqrt{6-(x+h)} + \sqrt{6-x}\right)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\left(\sqrt{6-(x+h)} + \sqrt{6-x}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{6-(x+\underset{0}{h})} + \sqrt{6-x}} = \boxed{\frac{-1}{2\sqrt{6-x}}}$$

Check: Chain Rule $\frac{d}{dx}\sqrt{6-x} = \frac{1}{2\sqrt{6-x}}(-1) = \boxed{\frac{-1}{2\sqrt{6-x}}}$ Match!

5. [10 Points] Suppose that G and H are functions, **and**

- $\lim_{x \to 5} G(x) = 6$
- $\lim_{x \to -9} H(x) = -4$
- $\lim_{x \to 8} G(x) = 7$
- $G(x)$ is continuous at $x = 8$.
- $H(x)$ is continuous at $x = 7$.
- $G(5) = -9$
- $H(7) = -9$

Answer the following questions or evaluate the following quantities and fully **justify** your answers.

(a) Compute $G(8) \underset{\text{by Cont. of } G @ x=8}{=} \lim_{x \to 8} G(x) \underset{\text{Given}}{=} \boxed{7}$

(b) Compute $\lim_{x \to 7} H(x) \underset{\text{by Cont. of } H @ x=7}{=} H(7) \underset{\text{given}}{=} \boxed{-9}$

(c) Compute $H \circ G(8) \underset{\text{by Def.}}{=} H(G(8)) \underset{\text{part a}}{=} H(7) \underset{\text{given}}{=} \boxed{-9}$

(d) Does $H(-9) = -4$? Yes, No, or Not Enough Information? Why or why not?

Note that it was given that $\lim_{x \to -9} H(x) = -4$, so for $H(-9) = -4$ we would require $\lim_{x \to -9} H(x) = H(-9)$ meaning H being continuous @ $x = -9$. We were not given that info. So NOT ENOUGH INFORMATION.

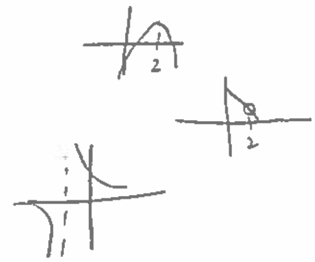
(e) Is $G(x)$ continuous at $x = 5$?

No, $G(x)$ is not continuous @ $x = 5$.

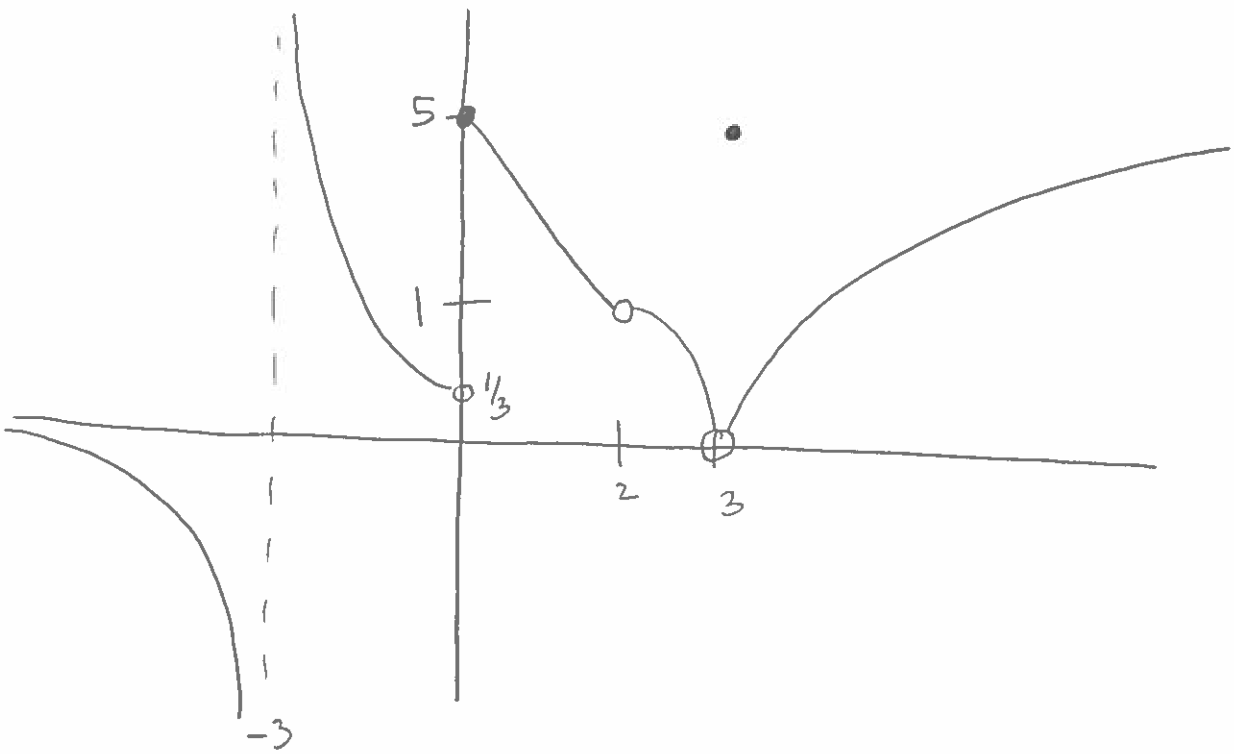
because $\lim_{x \to 5} G(x) = 6 \neq -9 = G(5)$.

6. [20 Points] Consider the function defined by

$$f(x) = \begin{cases} \sqrt{x-3} & \text{if } x > 3 \\ 5 & \text{if } x = 3 \\ 1-(x-2)^2 & \text{if } 2 < x < 3 \\ 5-2x & \text{if } 0 \le x < 2 \\ \dfrac{1}{x+3} & \text{if } x < 0 \end{cases}$$



(a) Carefully sketch the graph of $f(x)$.



(b) State the **Domain** of the function $f(x)$.

Domain $= \{x \mid x \neq -3, 2\}$

6. (Continued) Continue to consider the function defined by

$$f(x) = \begin{cases} \sqrt{x-3} & \text{if } x > 3 \\ 5 & \text{if } x = 3 \\ 1-(x-2)^2 & \text{if } 2 < x < 3 \\ 5-2x & \text{if } 0 \le x < 2 \\ \dfrac{1}{x+3} & \text{if } x < 0 \end{cases}$$

(c) Compute $\lim_{x\to -3} f(x) =$ $\boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to -3^+} \frac{1}{x+3} = \frac{1}{0^+} = \boxed{+\infty}$ LHL: $\lim_{x\to -3^-} \frac{1}{x+3} = \frac{1}{0^-} = \boxed{-\infty}$

(d) Compute $\lim_{x\to 0} f(x) =$ $\boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 5-2x = \boxed{5}$ LHL: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x+3} = \boxed{\frac{1}{3}}$

(e) Compute $\lim_{x\to 2} f(x) =$ $\boxed{1 \text{ b/c RHL} = \text{LHL.}}$

RHL: $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 1-(x-2)^2 = \boxed{1}$ LHL: $\lim_{x\to 2^-} f(x) = 5-2x = \boxed{1}$

(f) Compute $\lim_{x\to 3} f(x) =$ $\boxed{0 \text{ b/c RHL} = \text{LHL}}$

RHL: $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x-3} = \boxed{0}$ LHL: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 1-(x-2)^2 = \boxed{0}$

(g) State the value(s) at which f is discontinuous. Justify your answer(s) using definitions or theorems discussed in class.

Discontinuous @ $x=-3$ b/c $\lim_{x\to -3} f(x)$ DNE (See Above) . or $f(-3)$ undefined

Discontinuous @ $x=0$ b/c $\lim_{x\to 0} f(x)$ DNE (See Above).

Discontinuous @ $x=2$ b/c $f(2)$ undefined.

Discontinuous @ $x=3$ b/c $\lim_{x\to 3} f(x) = 0$ exists and $f(3)=5$ is defined

BUT $\lim_{x\to 3} f(x) \neq f(3)$.

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OPTIONAL BONUS

$\frac{0-2+0+2}{0+\cancel{2}-\cancel{2}}$ %

Do not attempt this unless you are completely done with the rest of the exam.

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DNE b/c RHL≠LHL

OPTIONAL BONUS #1 Compute $\lim_{x\to 1} \frac{2|x-1|-|x+2|+|x|+|x+1|}{|x-1|+|3-x|-|x+1|} =$

$= \boxed{-3}$

RHL: $\lim_{x\to1^+} \frac{2(x-1)-(x+2)+x+x+1}{x-1+3-x-(x+1)} = \lim_{x\to1^+} \frac{2x-2-\cancel{x}-2+\cancel{x}+\cancel{x}+1}{\cancel{x}-1+3-x-x-1} = \lim_{x\to1^+} \frac{3x-3}{-x+1} = \lim_{x\to1^+} \frac{3(\cancel{x-1})}{-(\cancel{x-1})}$

LHL: $\lim_{x\to1^-} \frac{2(1-x)-(x+2)+x+x+1}{1-x+3-x-(x+1)} = \lim_{x\to1^-} \frac{\cancel{2}-2x-x-\cancel{2}+\cancel{x}+\cancel{x}+1}{1-x+3-x-x-\cancel{1}} = \lim_{x\to1^-} \frac{1-x}{-3x+3} = \lim_{x\to1^-} \frac{\cancel{1-x}}{3(\cancel{1-x})}$

$= \boxed{\frac{1}{3}}$

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \quad x>1 \\ -(x-1) & \text{if } x-1<0 \quad x<1 \end{cases}$$

$1-x$

$$|x+2| = \begin{cases} x+2 & \text{if } x+2 \geq 0 \quad x>-2 \text{ this case} \\ -(x+2) & \text{if } x+2<0 \quad x<-2 \end{cases}$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \quad \text{this case} \\ -x & \text{if } x<0 \end{cases}$$

$$|x+1| = \begin{cases} x+1 & \text{if } x+1 \geq 0 \quad x\geq -1 \text{ this case} \\ -(x+1) & \text{if } x+1<0 \quad x<-1 \end{cases}$$

$$|3-x| = \begin{cases} 3-x & \text{if } 3-x \geq 0 \quad x\leq 3 \text{ this case} \\ -(3-x) & \text{if } 3-x<0 \quad x>3 \end{cases}$$