Math 111 **Answer Key** **Exam #1** **September 28, 2012**

1. [30 Points] Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim_{x\to-7} \dfrac{x^2+5x-14}{x^2-4x+4} \overset{\text{DSP}}{=} \dfrac{0}{81} = \boxed{0}$

(b) $\lim_{x\to 4} \dfrac{x^2-9x+20}{|4-x|} = \boxed{\text{DOES NOT EXIST}}$, RHL$\neq$LHL

RHL: $\lim_{x\to4^+} \dfrac{x^2-9x+20}{|4-x|} \overset{\frac{0}{0}}{=} \lim_{x\to4^+} \dfrac{x^2-9x+20}{-(4-x)} = \lim_{x\to4^+} \dfrac{(x-4)(x-5)}{x-4} = \lim_{x\to4^+} x-5 \overset{\text{DSP}}{=} \boxed{-1}$

LHL: $\lim_{x\to4^-} \dfrac{x^2-9x+20}{|4-x|} \overset{\frac{0}{0}}{=} \lim_{x\to4^-} \dfrac{x^2-9x+20}{4-x} = \lim_{x\to4^-} \dfrac{(x-4)(x-5)}{-(x-4)} = \lim_{x\to4^-} -(x-5) \overset{\text{DSP}}{=} \boxed{1}$

Here, recall that $|4-x| = \begin{cases} 4-x & \text{if } 4-x\geq 0 \\ -(4-x) & \text{if } 4-x<0 \end{cases} = \begin{cases} 4-x & \text{if } x\leq 4 \leftarrow \text{LHL case} \\ x-4 & \text{if } x>4 \leftarrow \text{RHL case} \end{cases}$

(c) $\lim_{x\to-6} \dfrac{f(x^2)+5x-8}{[f(x)]^2+5x+14} =$ where $f(x)=x+2$

$$\lim_{x\to-6} \frac{f(x^2)+5x-8}{[f(x)]^2+5x+14} = \lim_{x\to-6} \frac{x^2+2+5x-8}{[x+2]^2+5x+14} = \lim_{x\to-6} \frac{x^2+5x-6}{x^2+4x+4+5x+14}$$

$$= \lim_{x\to-6} \frac{x^2+5x-6}{x^2+9x+18} = \lim_{x\to-6} \frac{(x+6)(x-1)}{(x+6)(x+3)} = \lim_{x\to-6} \frac{x-1}{x+3} = \frac{-7}{-3} = \boxed{\frac{7}{3}}$$

(d) $\lim_{x\to2} \dfrac{x^2+5x-14}{x^2-4x+4} = \lim_{x\to2} \dfrac{(x+7)(x-2)}{(x-2)(x-2)} = \lim_{x\to2} \dfrac{x+7}{x-2} \boxed{\text{DOES NOT EXIST}}$, RHL$\neq$LHL

RHL: $\lim_{x\to2^+} \dfrac{x+7}{x-2} = \dfrac{9}{0^+} = \boxed{+\infty}$

LHL: $\lim_{x\to2^-} \dfrac{x+7}{x-2} = \dfrac{9}{0^-} = \boxed{-\infty}$

(e) $\lim_{x\to8} \dfrac{3-\sqrt{x+1}}{x^2-7x-8} = \lim_{x\to8} \dfrac{3-\sqrt{x+1}}{x^2-7x-8} \cdot \left(\dfrac{3+\sqrt{x+1}}{3+\sqrt{x+1}}\right) = \lim_{x\to8} \dfrac{9-(x+1)}{(x^2-7x-8)(3+\sqrt{x+1})}$

$$= \lim_{x\to8} \frac{8-x}{(x-8)(x+1)(3+\sqrt{x+1})} = \lim_{x\to8} \frac{-(x-8)}{(x-8)(x+1)(3+\sqrt{x+1})}$$

$$= \lim_{x\to8} \frac{-1}{(x+1)(3+\sqrt{x+1})} = \frac{-1}{(9)(3+\sqrt{9})} = \frac{-1}{(9)(6)} = \boxed{\frac{-1}{54}}$$

2. [13 Points] Prove that $\lim_{x\to 5} 7-2x=-3$ using the $\varepsilon-\delta$ definition of the limit.

Scratchwork: we want $|f(x)-L|=|(7-2x)-(-3)|<\varepsilon$

$$|f(x)-L|=|(7-2x)-(-3)|=|-2x+10|=|-2(x-5)|=|-2||x-5|=2|x-5| \text{ (want } < \varepsilon)$$

$$2|x-5|<\varepsilon \text{ would require } |x-5|<\frac{\varepsilon}{2}$$

$$\text{So choose } \delta=\frac{\varepsilon}{2} \text{ to restrict } 0<|x-5|<\delta. \text{ That is } 0<|x-5|<\frac{\varepsilon}{2}.$$

Proof: Let $\varepsilon>0$ be given. Choose $\delta=\frac{\varepsilon}{2}$. Given x such that $0<|x-5|<\delta$, then

$$|f(x)-L|=|(7-2x)-(-3)|=|-2x+10|=|-2(x-5)|=|-2||x-5|=2|x-5|<2\cdot\frac{\varepsilon}{2}=\varepsilon.$$

□

3. [15 Points] Suppose that $f(x)=\frac{x+7}{x-3}$. Compute $f'(x)$ using the **limit definition of the derivative**.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{(x+h)+7}{(x+h)-3}-\frac{x+7}{x-3}}{h}=\lim_{h\to 0}\frac{\left(\frac{(x+h+7)(x-3)-(x+7)(x+h-3)}{(x+h-3)(x-3)}\right)}{h}$$

$$=\lim_{h\to 0}\frac{x^2+xh+7x-3x-3h-21-(x^2+xh-3x+7x+7h-21)}{h(x+h-3)(x-3)}$$

$$=\lim_{h\to 0}\frac{x^2+xh+7x-3x-3h-21-x^2-xh+3x-7x-7h+21}{h(x+h-3)(x-3)}$$

$$=\lim_{h\to 0}\frac{-3h-7h}{h(x+h-3)(x-3)}=\lim_{h\to 0}\frac{-10h}{h(x+h-3)(x-3)}=\lim_{h\to 0}\frac{-10}{(x+h-3)(x-3)}$$

$$=\boxed{\frac{-10}{(x-3)^2}}$$

4. [10 Points] Suppose that $f(x)=5-7x+4x^2-x^3$. Write the **equation of the tangent line** to the curve $y=f(x)$ when $x=1$.

Use the limit definition of the derivative when computing the derivative.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{5-7(x+h)+4(x+h)^2-(x+h)^3-(5-7x+4x^2-x^3)}{h}$$

$$=\lim_{h\to 0}\frac{5-7x-7h+4(x^2+2xh+h^2)-(x^3+3x^2h+3xh^2+h^3)-(5-7x+4x^2-x^3)}{h}$$

$$=\lim_{h\to 0}\frac{5-7x-7h+4x^2+8xh+4h^2-x^3-3x^2h-3xh^2-h^3-5+7x-4x^2+x^3}{h}$$

$= \lim_{h \to 0} \dfrac{-7h + 8xh + 4h^2 - 3x^2h - 3xh^2 - h^3}{h} = \lim_{h \to 0} \dfrac{h(-7 + 8x + 4h - 3x^2 - 3xh - h^2)}{h}$

$= \lim_{h \to 0} -7 + 8x + 4h - 3x^2 - 3xh - h^2 = -7 + 8x - 3x^2$

Then the slope at $x = 1$ is given by $f'(1) = -7+8-3 = -2$. The point is given by $(1, f(1)) = (1, 1)$. Finally, the equation of the tangent line is given by $y - 1 = -2(x - 1)$ or $\boxed{y = -2x + 3}$.

5. [6 Points] Suppose that f and g are functions, **and**

- $\lim_{x \to 3} f(x) = 9$
- $\lim_{x \to 7} g(x) = -6$
- $\lim_{x \to 4} f(x) = 7$
- $g(x)$ is continuous at $x = 7$.
- $f(x)$ is continuous at $x = 4$.

(a) Compute $g \circ f(4) =$ (Do **not** just put down a value. Justify your answer.)

First, note that because f was assumed to be continuous at $x = 4$ that means that $\lim_{x \to 4} f(x) = f(4)$. Therefore, $f(4) = \lim_{x \to 4} f(x) = 7$ by assumption of the limit.

Second, note that because g was assumed to be continuous at $x = 7$ that means that $\lim_{x \to 7} g(x) = g(7)$. Therefore, $g(7) = \lim_{x \to 7} g(x) = -6$ by assumption of the limit.

Finally, $g \circ f(4) = g(f(4)) = g(7) = \boxed{-6}$

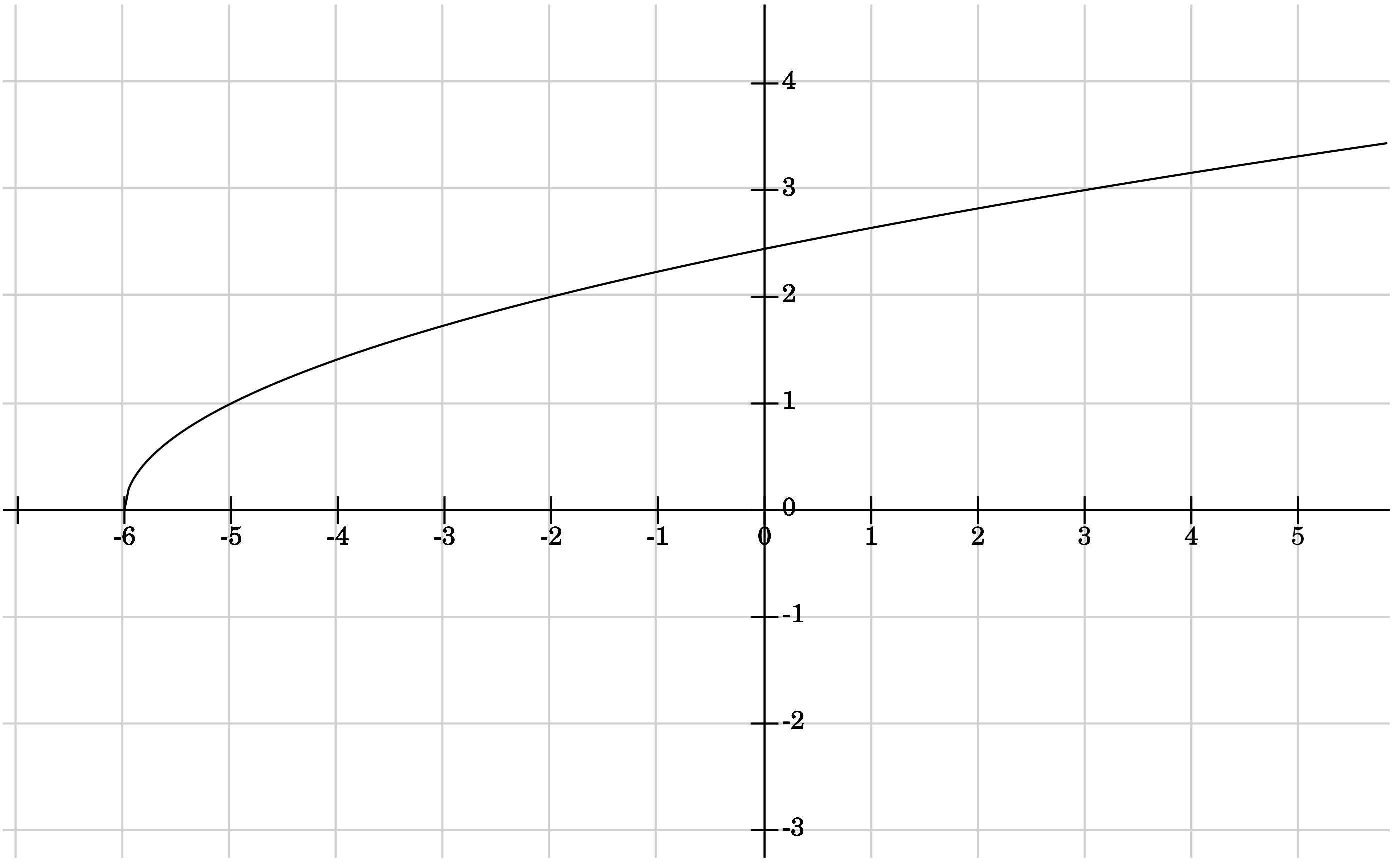
(b) Does $f(3) = 9$? Why or why not?

$\boxed{\text{No}}$ not necessarily, since we did not assume that f was continuous at $x = 3$. If f was continuous at $x = 3$ we would know that $f(3) = \lim_{x \to 3} f(x) = 9$.

6. [6 Points] Suppose that $f(x) = \sqrt{x + 4}$ and $g(x) = x + 2$.

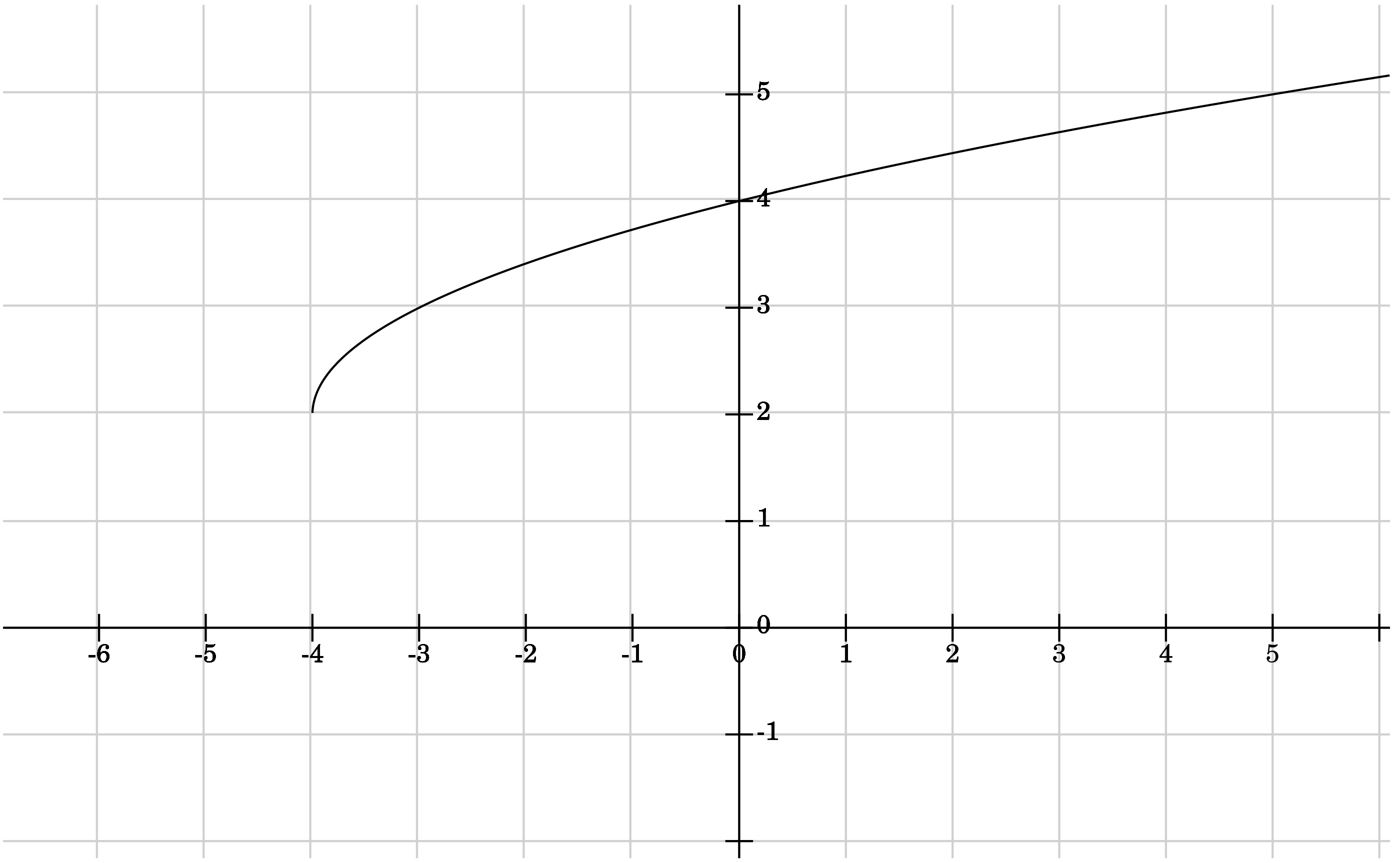
(a) Compute **and** graph $f \circ g(x)$.

First, $f \circ g(x) = f(g(x)) = f(x + 2) = \sqrt{(x + 2) + 4} = \boxed{\sqrt{x + 6}}$.



(b) Compute **and** graph $g \circ f(x)$.

Second, $g \circ f(x) = g(f(x)) = g(\sqrt{x+4}) = \boxed{\sqrt{x+4}+2}$.



7. [20 Points] Consider the function defined by

$$f(x) = \begin{cases} \sqrt{x-7} & \text{if } x > 7 \\ 1 & \text{if } x = 7 \\ 7 - x & \text{if } 0 < x < 7 \\ 16 - x^2 & \text{if } -4 < x \leq 0 \\ \dfrac{1}{x+4} & \text{if } x < -4 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$.

See me for sketch.

(b) State the **Domain** of the function $f(x)$.

Domain $f(x) = \boxed{\{x | x \neq -4\}}$.

(c) Compute
$$\begin{cases} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 7 - x = \boxed{7} & \leftarrow \text{RHL} \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 16 - x^2 = \boxed{16} & \leftarrow \text{LHL} \\ \lim_{x\to 0} f(x) = \text{ DOES NOT EXIST since RHL} \neq \text{LHL} \end{cases}$$

(d) Compute
$$\begin{cases} \lim_{x\to 7^+} f(x) = \lim_{x\to 7^+} \sqrt{x-7} = \boxed{0} & \leftarrow \text{RHL} \\ \lim_{x\to 7^-} f(x) = \lim_{x\to 7^-} 7 - x = \boxed{0} & \leftarrow \text{LHL} \\ \lim_{x\to 7} f(x) = \boxed{0} & \text{RHL=LHL} \end{cases}$$

(e) Compute
$$\begin{cases} \lim_{x\to -4^+} f(x) = \lim_{x\to -4^+} 16 - x^2 = \boxed{0} & \leftarrow \text{RHL} \\ \lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} \dfrac{1}{x+4} = \dfrac{1}{0^-} = \boxed{-\infty} & \leftarrow \text{LHL} \\ \lim_{x\to -4} f(x) = \text{ DOES NOT EXIST since RHL} \neq \text{LHL} \end{cases}$$

(f) State the value(s) at which f is discontinuous. Justify your answer(s) using definitions or theorems discussed in class.

• f is discontinuous at $x = 7$, because despite the fact that $f(7) = 1$ is defined, and $\lim_{x\to 7} f(x) = 0$, those two values are not equal. (There is a removable discontinuity at $x = 7$.)

• f is discontinuous at $x = 0$, because despite the fact that $f(0) = 16$ is defined, the $\lim_{x\to 0} f(x)$ DOES NOT EXIST. (There is a jump discontinuity at $x = 0$.)

• f is discontinuous at $x = -4$ for two reasons, $f(-4)$ is undefined, and the $\lim_{x\to -4} f(x)$ DOES NOT EXIST. (There is an infinite discontinuity at $x = -4$.)

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute $\displaystyle\lim_{x\to 2} \frac{(4-\sqrt{x+14})(\sqrt{13-x^2}-3)}{(6-\sqrt{40-2x})(\sqrt{x^2+21}-5)} =$

$\displaystyle= \lim_{x\to 2} \frac{(4-\sqrt{x+14})(\sqrt{13-x^2}-3)}{(6-\sqrt{40-2x})(\sqrt{x^2+21}-5)}$ why not throw in all the conjugates at once...

$$\cdot \left(\frac{4+\sqrt{x+14}}{4+\sqrt{x+14}}\right) \cdot \left(\frac{\sqrt{13-x^2}+3}{\sqrt{13-x^2}+3}\right) \cdot \left(\frac{6+\sqrt{40-2x}}{6+\sqrt{40-2x}}\right) \cdot \left(\frac{\sqrt{x^2+21}+5}{\sqrt{x^2+21}+5}\right)$$

$$= \lim_{x\to 2} \frac{(16-(x+14))((13-x^2)-9)(6+\sqrt{40-2x})(\sqrt{x^2+21}+5)}{(36-(40-2x))((x^2+21)-25)(4+\sqrt{x+14})(\sqrt{13-x^2}+3)}$$

$$= \lim_{x\to 2} \frac{(2-x)(4-x^2)(6+\sqrt{40-2x})(\sqrt{x^2+21}+5)}{(2x-4)(x^2-4)(4+\sqrt{x+14})(\sqrt{13-x^2}+3)}$$

$$= \lim_{x\to 2} \frac{-(x-2)(-1)(x^2-4)(6+\sqrt{40-2x})(\sqrt{x^2+21}+5)}{2(x-2)(x^2-4)(4+\sqrt{x+14})(\sqrt{13-x^2}+3)}$$

$$= \lim_{x\to 2} \frac{(6+\sqrt{40-2x})(\sqrt{x^2+21}+5)}{2(4+\sqrt{x+14})(\sqrt{13-x^2}+3)} = \frac{(6+\sqrt{36})(\sqrt{25}+5)}{2(4+\sqrt{16})(\sqrt{9}+3)} = \frac{(6+6)(5+5)}{2(4+4)(3+3)} = \boxed{\frac{5}{4}}$$

OPTIONAL BONUS #2 Let $f(x) = \sqrt{\dfrac{x^2+1}{7-x^3}}$. Compute $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{\frac{(x+h)^2+1}{7-(x+h)^3}} - \sqrt{\frac{x^2+1}{7-x^3}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{\frac{(x+h)^2+1}{7-(x+h)^3}} - \sqrt{\frac{x^2+1}{7-x^3}}}{h} \cdot \left(\frac{\sqrt{\frac{(x+h)^2+1}{7-(x+h)^3}} + \sqrt{\frac{x^2+1}{7-x^3}}}{\sqrt{\frac{(x+h)^2+1}{7-(x+h)^3}} + \sqrt{\frac{x^2+1}{7-x^3}}}\right)$$

$$= \lim_{h\to 0} \frac{\frac{(x+h)^2+1}{7-(x+h)^3} - \frac{x^2+1}{7-x^3}}{h\left(\sqrt{\frac{(x+h)^2+1}{7-(x+h)^3}} + \sqrt{\frac{x^2+1}{7-x^3}}\right)} = \lim_{h\to 0} \frac{\left(\frac{((x+h)^2+1)(7-x^3)-(x^2+1)(7-(x+h)^3)}{(7-(x+h)^3)(7-x^3)}\right)}{h\left(\sqrt{\frac{(x+h)^2+1}{7-(x+h)^3}} + \sqrt{\frac{x^2+1}{7-x^3}}\right)}$$

$$= \lim_{h\to 0} \frac{\left(\frac{(x^2+2xh+h^2+1)(7-x^3)-(x^2+1)(7-(x^3+3x^2h+3xh^2+h^3))}{(7-(x+h)^3)(7-x^3)}\right)}{h\left(\sqrt{\frac{(x+h)^2+1}{7-(x+h)^3}} + \sqrt{\frac{x^2+1}{7-x^3}}\right)}$$

$$= \lim_{h\to 0} \frac{\left(\frac{(x^2+2xh+h^2+1)(7-x^3)-(x^2+1)(7-x^3-3x^2h-3xh^2-h^3)}{(7-(x+h)^3)(7-x^3)}\right)}{h\left(\sqrt{\frac{(x+h)^2+1}{7-(x+h)^3}} + \sqrt{\frac{x^2+1}{7-x^3}}\right)}$$

$= \lim_{h\to 0}$ ...(continue to fit space here)

$$\frac{\dfrac{7x^2+14xh+7h^2+7-x^5-2x^4h-x^3h^2-x^3-(7x^2-x^5-3x^4h-3x^3h^2-x^2h^3+7-x^3-3x^2h-3xh^2-}{(7-(x+h)^3)(7-x^3)}}{h\left(\sqrt{\dfrac{(x+h)^2+1}{7-(x+h)^3}}+\sqrt{\dfrac{x^2+1}{7-x^3}}\right)}$$

$= \lim\limits_{h\to 0}$ ...(continue to fit space here)

$$\frac{\dfrac{7x^2+14xh+7h^2+7-x^5-2x^4h-x^3h^2-x^3-7x^2+x^5+3x^4h+3x^3h^2+x^2h^3-7+x^3+3x^2h+3xh^2+h}{(7-(x+h)^3)(7-x^3)}}{h\left(\sqrt{\dfrac{(x+h)^2+1}{7-(x+h)^3}}+\sqrt{\dfrac{x^2+1}{7-x^3}}\right)}$$

$$=\lim_{h\to 0}\frac{\dfrac{14xh+7h^2-2x^4h-x^3h^2+3x^4h+3x^3h^2+x^2h^3+3x^2h+3xh^2+h^3}{(7-(x+h)^3)(7-x^3)}}{h\left(\sqrt{\dfrac{(x+h)^2+1}{7-(x+h)^3}}+\sqrt{\dfrac{x^2+1}{7-x^3}}\right)}$$

$$=\lim_{h\to 0}\frac{14xh+7h^2-2x^4h-x^3h^2+3x^4h+3x^3h^2+x^2h^3+3x^2h+3xh^2+h^3}{(7-(x+h)^3)(7-x^3)h\left(\sqrt{\dfrac{(x+h)^2+1}{7-(x+h)^3}}+\sqrt{\dfrac{x^2+1}{7-x^3}}\right)}$$

$$=\lim_{h\to 0}\frac{h(14x+7h-2x^4-x^3h+3x^4+3x^3h+x^2h^2+3x^2+3xh+h^2)}{(7-(x+h)^3)(7-x^3)h\left(\sqrt{\dfrac{(x+h)^2+1}{7-(x+h)^3}}+\sqrt{\dfrac{x^2+1}{7-x^3}}\right)}$$

$$=\lim_{h\to 0}\frac{14x+7h-2x^4-x^3h+3x^4+3x^3h+x^2h^2+3x^2+3xh+h^2}{(7-(x+h)^3)(7-x^3)\left(\sqrt{\dfrac{(x+h)^2+1}{7-(x+h)^3}}+\sqrt{\dfrac{x^2+1}{7-x^3}}\right)}$$

$$=\frac{14x-2x^4+3x^4+3x^2}{(7-x^3)(7-x^3)\left(\sqrt{\dfrac{x^2+1}{7-x^3}}+\sqrt{\dfrac{x^2+1}{7-x^3}}\right)}$$

$$=\boxed{\frac{x^4+3x^2+14x}{(7-x^3)^2 2\sqrt{\dfrac{x^2+1}{7-x^3}}}}$$

OPTIONAL BONUS #3

Compute $\lim\limits_{x\to 0}\dfrac{|x-1|-|x+1|-|x|}{|x|+|2-x|-|x+2|}=$

$\boxed{\text{DOES NOT EXIST, RHL} \neq \text{LHL}}$

RHL: $\lim\limits_{x\to 0}\dfrac{|x-1|-|x+1|-|x|}{|x|+|2-x|-|x+2|}=\lim\limits_{x\to 0^+}\dfrac{-(x-1)-(x+1)-x}{x+(2-x)-(x+2)}$

$=\lim\limits_{x\to 0^+}\dfrac{-x+1-x-1-x}{x+2-x-x-2}=\lim\limits_{x\to 0^+}\dfrac{-3x}{-x}=\boxed{3}$

LHL: $\lim_{x\to 0} \dfrac{|x-1|-|x+1|-|x|}{|x|+|2-x|-|x+2|} = \lim_{x\to 0^-} \dfrac{-(x-1)-(x+1)-(-x)}{-x+(2-x)-(x+2)} =$

$= \lim_{x\to 0^-} \dfrac{-x+1-x-1+x}{-x+2-x-x-2} = \lim_{x\to 0^-} \dfrac{-x}{-3x} = \boxed{\dfrac{1}{3}}$

Here, recall that $|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ \\ -x & \text{if} \quad x < 0 \end{cases}$

$$|x-1| = \begin{cases} x-1 & \text{if} \quad x-1 \geq 0 \\ \\ -(x-1) & \text{if} \quad x-1 < 0 \end{cases} = \begin{cases} x-1 & \text{if} \quad x \geq 1 \\ \\ 1-x & \text{if} \quad x < 1 \end{cases}$$

$$|x+1| = \begin{cases} x+1 & \text{if} \quad x+1 \geq 0 \\ \\ -(x+1) & \text{if} \quad x+1 < 0 \end{cases} = \begin{cases} x+1 & \text{if} \quad x \geq -1 \\ \\ -x-1 & \text{if} \quad x < -1 \end{cases}$$

$$|2-x| = \begin{cases} 2-x & \text{if} \quad 2-x \geq 0 \\ \\ -(2-x) & \text{if} \quad 2-x < 0 \end{cases} = \begin{cases} 2-x & \text{if} \quad x \leq 2 \\ \\ x-2 & \text{if} \quad x > 2 \end{cases}$$

$$|x+2| = \begin{cases} x+2 & \text{if} \quad x+2 \geq 0 \\ \\ -(x+2) & \text{if} \quad x+2 < 0 \end{cases} = \begin{cases} x+2 & \text{if} \quad x \geq -2 \\ \\ -x-2 & \text{if} \quad x < -2 \end{cases}$$