Amherst College, Math 11 Final Examination, December 22, 2010

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, or $e^{3\ln 3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [20 Points] Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x^2+4}$

(b) $\displaystyle\lim_{x\to 5^-} \frac{x^2+4x-5}{g(x^2)-49}$, where $g(x) = 2x-1$.

(c) $\displaystyle\lim_{x\to 5} \frac{5-x}{\sqrt{x+4}-3}$

(d) $\displaystyle\lim_{x\to 3^+} \frac{x^2-9}{|3-x|}$

2. [30 Points] Compute each of the following derivatives. Do not simplify your answers.

(a) $f'\left(\frac{\pi}{6}\right)$, where $f(x) = \tan^2 x + \cos(2x)$.

(b) $\displaystyle\frac{d}{dx}\ln\left(\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{e^{\sec x}}\right)$

(c) $\displaystyle\frac{dy}{dx}$, if $e^{xy} = y^3\ln x + e^7$.

(d) $g'(0)$, where $g(x) = \dfrac{f(x)}{e^{3x}}$ with $f(0) = 2$ and $f'(0) = 7$.

(e) $g''(x)$, where $g(x) = \displaystyle\int_x^9 \sqrt{1+\cos t}\,dt$.

(f) $\displaystyle\frac{d}{dx}(\sin x)^x$

3. [25 Points] Compute each of the following integrals.

(a) $\displaystyle\int_0^{\ln 3} \frac{e^{2x}}{1+e^{2x}}\,dx$

(b) $\displaystyle\int_0^3 |4-x^2|\,dx$

(c) $\displaystyle\int \frac{(x+1)(x+2)}{x^3}\,dx$

(d) $\displaystyle\int_0^1 \frac{d}{dx}\left(\frac{\sqrt{1+3x^2}}{x+e^x}\right)dx$

(e) $\displaystyle\int e^{x^2+\ln x}\,dx$

4. [10 Points] Give an ε–δ proof that $\displaystyle\lim_{x\to -2} \frac{1}{2}x + 3 = 2$.

5. [10 Points] Let $f(x) = \dfrac{x+1}{x+2}$. Calculate $f'(x)$, using the **limit definition** of the derivative.

6. [15 Points] Consider the function defined by $f(x) = \begin{cases} |x-1| & \text{if } x \le 7 \\ \dfrac{1}{x-7} & \text{if } x > 7 \end{cases}$

(a) Carefully sketch the graph of $f(x)$.

(b) State the domain of the function $f(x)$.

(c) Compute $\lim_{x\to 7^+} f(x)$, $\lim_{x\to 7^-} f(x)$, and $\lim_{x\to 7} f(x)$.

(d) State the value(s) of x at which f is discontinuous. Justify your answer(s) using the definition of continuity.

(e) State the value(s) of x where $f(x)$ is *not* differentiable. Justify your answer(s).

7. [10 Points] Find the equation of the tangent line to $y = \sin(e^x)$ at the point where the x-coordinate is $\ln \pi$.

8. [20 Points] Let $\boldsymbol{f(x) = \dfrac{x-3}{x^4}}$.

Sketch the graph of $y = f(x)$. State the domain for $f(x)$. Clearly indicate horizontal and vertical asymptotes, local minima/maxima, and inflection points on the graph, as well as where the graph is increasing, decreasing, concave up and concave down. Take my word that

$$\boldsymbol{f'(x) = \frac{3(4-x)}{x^5}} \quad \text{and} \quad \boldsymbol{f''(x) = \frac{12(x-5)}{x^6}}.$$

9. [15 Points] A kite starts flying 300 feet directly above the ground. The kite is being blown horizontally at 10 feet per second. When the kite has blown horizontally for 40 seconds, how fast is the angle between the string and the ground changing?

10. [15 Points] A large box with a square base and top is to be made to hold a fixed volume of 54 cubic feet. The sides cost \$1 per square foot. The top and bottom cost \$2 per square foot. Determine the dimensions that minimize the cost of materials.

(Remember to state the domain of the function you are computing extreme values for.)

11. [15 Points] Consider the region in the first quadrant bounded by $y = e^{2x}$, $y = 4$, and the y–axis.

(a) Draw a picture of the region.

(b) Compute the area of the region.

(c) Compute the volume of the three-dimensional object obtained by rotating the region about the x–axis.

12. [15 Points] Consider an object moving on the number line such that its velocity at time t is $\boldsymbol{v(t) = \sin t}$ feet per second. Also assume that $s(0) = 2$ feet, where as usual $s(t)$ is the position of the object at time t.

(a) Compute the acceleration function $a(t)$ and the position function $s(t)$.

(b) Draw the graph of $v(t)$ for $0 \le t \le 2\pi$, and explain why the object is *not* always moving to the right.

(c) Compute the **displacement** and **total distance** travelled for $0 \le t \le 2\pi$.