## Answer Key-Math 11- Optional Review Homework For Exam 2

1. Compute the derivative for each of the following functions: Please do **not** simplify your derivatives here. (I simlified some, only in the case that you want to check your answers)

(a) $f(x) = \dfrac{x^3 + \dfrac{1}{x^8}}{x^2}$

$f'(x) = \dfrac{x^2[3x^2 - 8x^{-9}] - (x^3 + x^{-8})(2x)}{x^4} = \dfrac{3x^4 - 8x^{-7} - 2x^4 - 2x^{-7}}{x^4} = \dfrac{x^4 - 10x^{-7}}{x^4} = 1 - 10x^{-11}$

or more simply you can rewrite $f(x) = x + x^{-10}$ and then $f'(x) = 1 - 10x^{-11}$.

(b) $f(x) = \sin^2\left(3x - \dfrac{1}{x^2}\right)$

$f'(x) = 2\sin(3x - x^{-2}) \cdot \cos(3x - x^{-2}) \cdot (3 + 2x^{-3})$

(c) $f(x) = x^2 \tan\left(\dfrac{3x}{\sec x}\right)$

$f'(x) = x^2 \sec^2\left(\dfrac{3x}{\sec x}\right) \cdot \left(\dfrac{(\sec x)3 - (3x)\sec x \tan x}{\sec^2 x}\right) + \tan\left(\dfrac{3x}{\sec x}\right) \cdot 2x$

(d) $f(x) = \dfrac{x^3 + (x^5 + 3x)^7}{\sqrt{x^2 - 7x}}$

$$f'(x) = \frac{\sqrt{x^2 - 7x}(3x^2 + 7(x^5 + 3x)^6(5x^4 + 3)) - (x^3 + (x^5 + 3x)^7) \cdot \dfrac{1}{2\sqrt{x^2 - 7x}} \cdot (2x - 7)}{x^2 - 7x}$$

(e) $f(x) = \dfrac{1}{\cos\left(\dfrac{1}{x} - \dfrac{1}{x^7} + x\right)}$

First rewrite $f(x) = \left[\cos\left(\dfrac{1}{x} - \dfrac{1}{x^7} + x\right)\right]^{-1}$

$f'(x) = -1 \cdot \cos^{-2}\left(\dfrac{1}{x} - \dfrac{1}{x^7} + x\right) \cdot \left[-\sin\left(\dfrac{1}{x} - \dfrac{1}{x^7} + x\right)\right] \cdot (-1x^{-2} + 7x^{-8} + 1)$

$= \dfrac{\sin\left(\dfrac{1}{x} - \dfrac{1}{x^7} + x\right) \cdot (-1x^{-2} + 7x^{-8} + 1)}{\cos^2\left(\dfrac{1}{x} - \dfrac{1}{x^7} + x\right)}$

(f) $h(x) = \sqrt{\sqrt{x} + \cos(x^2 + \sqrt{x})}$

$h'(x) = \dfrac{1}{2\sqrt{\sqrt{x} + \cos(x^2 + \sqrt{x})}} \cdot \left[\dfrac{1}{2\sqrt{x}} - \sin(x^2 + \sqrt{x})\left(2x + \dfrac{1}{2\sqrt{x}}\right)\right]$

(g) $g(y) = y^2 \cos\sqrt{y}$

$g'(y) = y^2(-\sin\sqrt{y}) \cdot \dfrac{1}{2\sqrt{y}} + \cos\sqrt{y} \cdot 2y = -\dfrac{y^2 \sin\sqrt{y}}{2\sqrt{y}} + 2y\cos\sqrt{y}$

(h) $f(t) = (3t - 1)^7(4t + 3)^9$

$f'(t) = (3t - 1)^7 \cdot 9(4t + 3)^8(4) + (4t + 3)^9 \cdot 7(3t - 1)^6(3) = 3(3t - 1)^6(4t + 3)^8[12(3t - 1) + 7(4t + 3)] = 3(3t - 1)^6(4t + 3)^8[64t + 9]$

(i) $w(x) = \left(x + \sqrt{x^4+1}\right)^5$

$w'(x) = 5\left(x + \sqrt{x^4+1}\right)^4 \left[1 + \frac{1}{2\sqrt{x^4+1}}(4x^3)\right]$

(j) $f(x) = \frac{\sqrt{\sin(3x^2+17x)}}{\cos^3(4x)}$

$f'(x) =$

$$\frac{\left[\cos^3(4x)\frac{1}{2\sqrt{\sin(3x^2+17x)}} \cdot \cos(3x^2+17x)(6x+17) - \sqrt{\sin(3x^2+17x)}3\cos^2(4x)(-\sin(4x))4\right]}{\cos^6(4x)}$$

2. Compute $y'$ where $xy + y^3 = 4x^2\cos(x^2)$.

Differentiating both sides implicitly yields

$x\frac{dy}{dx} + y + 3y^2\frac{dy}{dx} = 4x^2(-\sin(x^2))(2x) + \cos(x^2)(8x)$ and then

$\frac{dy}{dx}(x+3y^2) = -8x^3\sin(x^2) + 8x\cos(x^2) - y$ so that

$\frac{dy}{dx} = \boxed{\frac{-8x^3\sin(x^2)+8x\cos(x^2)-y}{x+3y^2}}$.

3. Evaluate each of the following limits:

(a) $\lim_{x\to 0}\frac{\sin(6x)}{8x} = \frac{6}{8}\lim_{x\to 0}\frac{\sin(6x)}{6x} = \frac{6}{8}\cdot 1 = \boxed{\frac{6}{8}}$

(b) $\lim_{x\to 0}\frac{\sin(6x)}{\cos x + x} = \frac{0}{1+0} = \boxed{0}$

(c) $\lim_{x\to 0}\frac{3x-8x^2}{\sin(4x)} = \lim_{x\to 0}\frac{x(3-8x)}{\sin(4x)} = \lim_{x\to 0}\frac{x}{\sin(4x)}\cdot\lim_{x\to 0}(3-8x) = \frac{1}{4}\lim_{x\to 0}\frac{4x}{\sin(4x)}\cdot\lim_{x\to 0}(3-8x) = \frac{1}{4}\cdot 3 = \boxed{\frac{3}{4}}$

(d) $\lim_{x\to\infty}\frac{3x-8x^2}{7x^2+5x-9} = \lim_{x\to\infty}\frac{3x-8x^2}{7x^2+5x-9}\cdot\frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty}\frac{\frac{3}{x}-8}{7+\frac{5}{x}-\frac{9}{x^2}} = \boxed{\frac{-8}{7}}$

(e) $\lim_{x\to\infty}\frac{9x^{18}+7x^3-2010}{6x^{31}+2x^7+5} = \lim_{x\to\infty}\frac{9x^{18}+7x^3-2010}{6x^{31}+2x^7+5}\cdot\frac{\left(\frac{1}{x^{31}}\right)}{\left(\frac{1}{x^{31}}\right)} = \lim_{x\to\infty}\frac{\frac{9}{x^{13}}+\frac{7}{x^{28}}-\frac{2010}{x^{31}}}{6+\frac{2}{x^{24}}+\frac{5}{x^{31}}} = \boxed{0}$

(f) $\lim_{x\to\infty}\frac{x^{12}+x^4-2010}{6x^9+x^2} = \lim_{x\to\infty}\frac{x^{12}+x^4-2010}{6x^9+x^2}\cdot\frac{\left(\frac{1}{x^9}\right)}{\left(\frac{1}{x^9}\right)} = \lim_{x\to\infty}\frac{x^3+\frac{1}{x^5}-\frac{2010}{x^9}}{6+\frac{1}{x^7}} = \boxed{\infty}$

4. Consider the function $f(x) = \frac{1}{x^2-4}$. Discuss domain, intervals of increase or decrease, local extreme value(s), concavity, inflection point(s), and any horizontal and vertical asymptotes. Use this information to give a detailed and labelled sketch of the curve.

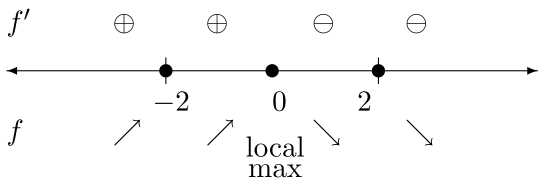
• Domain: $f(x)$ has domain $\{x|x \neq \pm 2\}$

• Vertical asymptotes: VA at $x = \pm 2$.

• Horizontal asymptote: HA at $y = 0$ for this $f$ since $\lim\limits_{x \to \pm\infty} f(x) = 0$.

• First Derivative Information

We compute $f'(x) = \dfrac{-2x}{(x^2-4)^2}$ and set it equal to 0 and solve for $x$ to find critical numbers. The critical points occur where $f'$ is undefined or zero. The latter happens when $x = 0$. The derivative is undefined when $x = \pm 2$, but those values are not in the domain of the original function. As a result, $x = 0$ is the critical number. Using sign testing/analysis for $f'$,



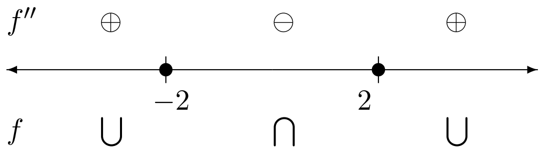

or our $f'$ chart is

| $x$ | $(-\infty, 0)$ | $(0, \infty)$ |
|---|---|---|
| $f'(x)$ | $\oplus$ | $\ominus$ |
| $f(x)$ | $\nearrow$ | $\searrow$ |

So $f$ is increasing on $(-\infty, 0)$; and $f$ is decreasing on $(0, \infty)$. Moreover, $f$ has a local max at $x = 0$ with maximum value $f(0) = -\dfrac{1}{4}$.

• Second Derivative Information

Meanwhile, $f'' = \dfrac{2(3x^2+4)}{(x^2-4)^3}$ is never zero. Using sign testing/analysis for $f''$ around the vertical asymptotes,

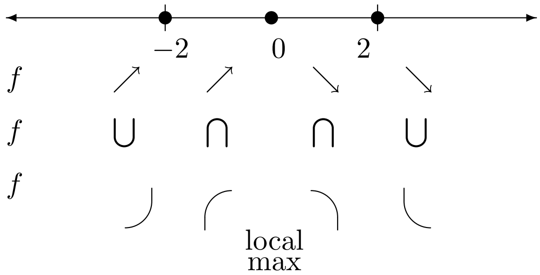



or our $f''$ chart is

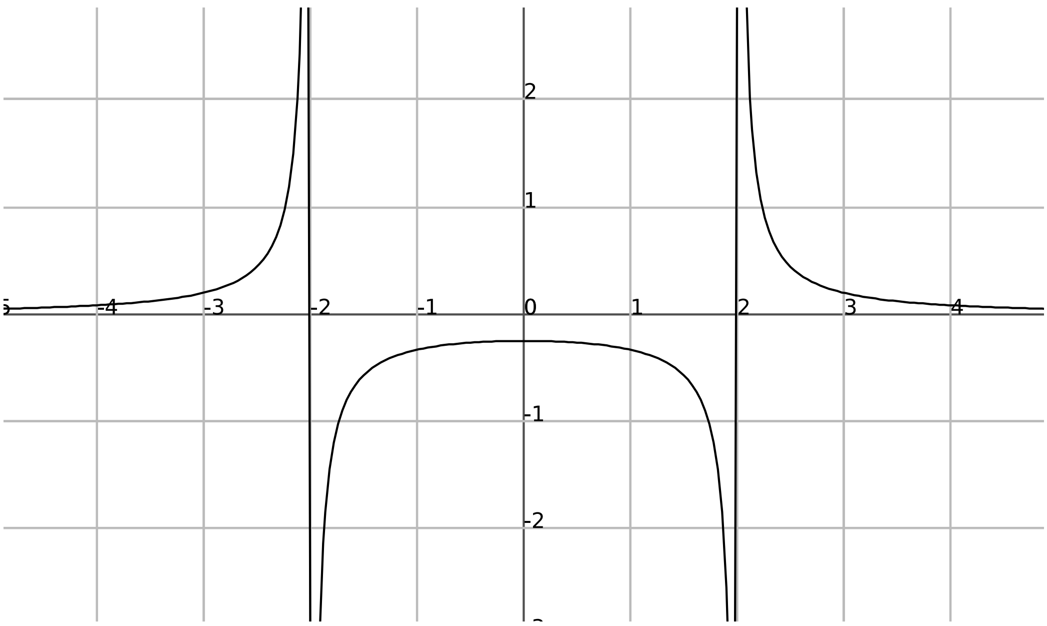
| $x$ | $(-\infty, -2)$ | $(-2, 2)$ | $(2, \infty)$ |
|---|---|---|---|
| $f''(x)$ | $\oplus$ | $\ominus$ | $\oplus$ |
| $f(x)$ | $\cup$ | $\cap$ | $\cup$ |

So $f$ is concave down on $(-2, 2)$ and concave up on $(-\infty, -2)$ and $(2, \infty)$ with no inflection points.

• Piece the first and second derivative information together




• Sketch:




5. Consider the function $f(x) = x^2(x-1)^2$. Discuss domain, intervals of increase or decrease, local extreme value(s), concavity, inflection point(s), and any horizontal and vertical asymptotes. Use this information to give a detailed and labelled sketch of the curve.

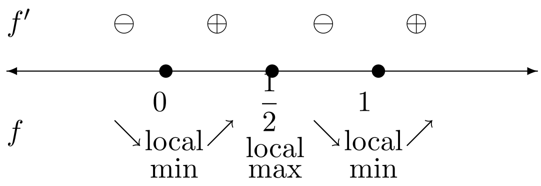
   • Domain: $f(x)$ has domain $(-\infty, \infty)$

   • Vertical Asymptotes: It is a polynomial, continuous everywhere, and so has no vertical asymptotes.

   • Horizontal Asymptotes: There are no horizontal asymptotes for this $f$ since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$.

   • First Derivative Information

   We compute $f'(x) = x^2(2)(x-1) + (x-1)^2(2x) = 2x(x-1)[x+(x-1)] = 2x(x-1)(2x-1)$ and set it equal to 0 and solve for $x$ to find critical numbers. The critical points occur where $f'$ is undefined (never here) or zero. The latter happens when $x = 0, x = 1, x = \frac{1}{2}$ As a result, those numbers are the critical numbers. Using sign testing/analysis for $f'$,



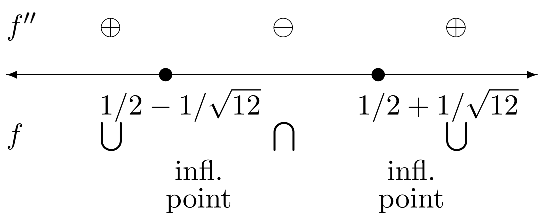

or our $f'$ chart is

| $x$ | $(-\infty, 0)$ | $(0, 1/2)$ | $(1/2, 1)$ | $(1, \infty)$ |
|---|---|---|---|---|
| $f'(x)$ | $\ominus$ | $\oplus$ | $\ominus$ | $\oplus$ |
| $f(x)$ | $\searrow$ | $\nearrow$ | $\searrow$ | $\nearrow$ |

So $f$ is increasing on $(0, 1/2)$ and $(1, \infty)$, and $f$ is dereasing on $-\infty, 0)$ and $(1/2, 1)$. Moreover, $f$ has local mins at $x = 0$ and $x = 1$, but a local max at $x = 1/2$. Here $f(0) = 0$ and $f(1) = 0$, and $f(\frac{1}{2}) = \frac{1}{16}$

- Second Derivative Information

Meanwhile, $f''$ is always defined and continuous, and $f'' = 12x^2 - 12x + 2 = 0$ only at our possible inflection point $x = \dfrac{1}{2} \pm \dfrac{1}{\sqrt{12}}$. Using sign testing/analysis for $f''$,

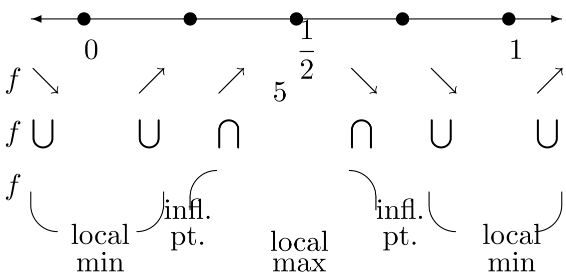



or our $f''$ chart is

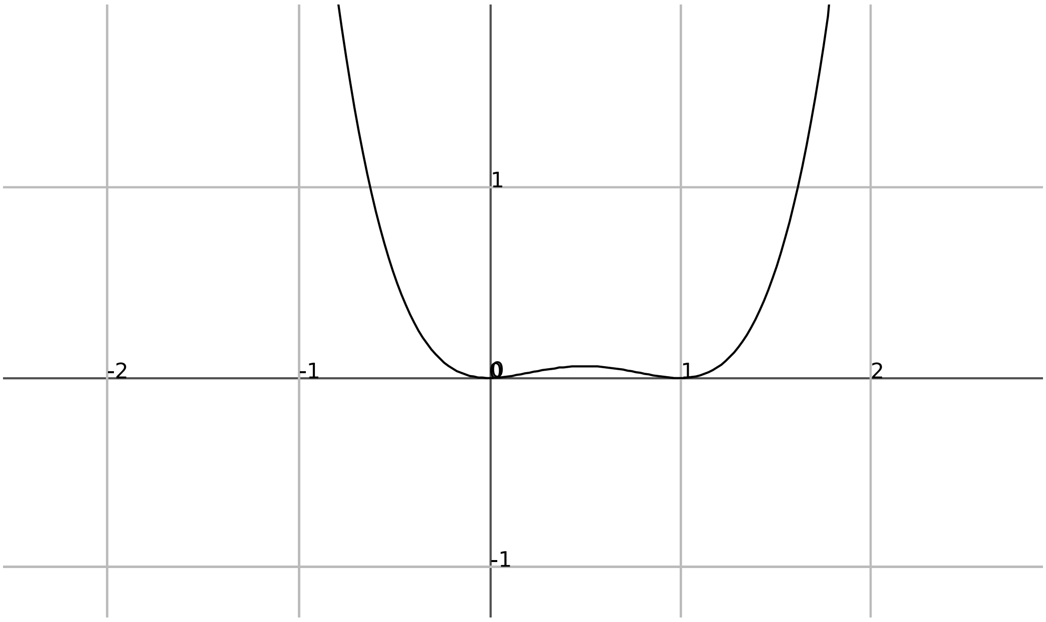
| $x$ | $\left(-\infty, \dfrac{1}{2} - \dfrac{1}{\sqrt{12}}\right)$ | $\left(\dfrac{1}{2} - \dfrac{1}{\sqrt{12}}, \dfrac{1}{2} + \dfrac{1}{\sqrt{12}}\right)$ | $\left(\dfrac{1}{2} + \dfrac{1}{\sqrt{12}}, \infty\right)$ |
|---|---|---|---|
| $f''(x)$ | $\oplus$ | $\ominus$ | $\oplus$ |
| $f(x)$ | $\cup$ | $\cap$ | $\cup$ |

So $f$ is concave down on $\left(\dfrac{1}{2} - \dfrac{1}{\sqrt{12}}, \dfrac{1}{2} + \dfrac{1}{\sqrt{12}}\right)$ and concave up on $\left(-\infty, \dfrac{1}{2} - \dfrac{1}{\sqrt{12}}\right)$ and $\left(\dfrac{1}{2} + \dfrac{1}{\sqrt{12}}, \infty\right)$, with inflection points at $x = \dfrac{1}{2} \pm \dfrac{1}{\sqrt{12}}$. You can check that $f\left(\dfrac{1}{2} \pm \dfrac{1}{\sqrt{12}}\right) = \dfrac{1}{36}$.

- Piece the first and second derivative information together



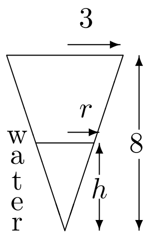
• Sketch:




6. A conical water tank (point facing down) with radius of 3 meters at the top and height of 8 meters is leaking water. At the moment when the water is 2 meters from the top of the tank water is leaking at a rate of 1 cubic meter per minute. How fast is the water level decreasing at that moment?

   The cross section (with water level drawn in) looks like:

   • Diagram




• Variables
Let $r$ = radius of the water level at time $t$
Let $h$ = height of the water level at time $t$
Let $V$ = volume of the water in the tank at time $t$
Find $\frac{dh}{dt}$ =? when $h = 6$ feet (2 from top)
and $\frac{dV}{dt} = -1\frac{\text{m}^3}{\text{min}}$

• Equation relating the variables:

Volume= $V = \frac{1}{3}\pi r^2 h$

• Extra solvable information: Note that $r$ is not mentioned in the problem's info. But there is a relationship, via similar triangles, between $r$ and $h$. We must have

$$\frac{r}{3} = \frac{h}{8} \implies r = \frac{3}{8}h$$

After substituting into our previous equation, we get:

$V = \frac{1}{3}\pi\left(\frac{3}{8}h\right)^2 h = \frac{3}{64}\pi h^3$

• Differentiate both sides w.r.t. time $t$.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{3}{64}\pi h^3\right) \implies \frac{dV}{dt} = \frac{3}{64}\pi \cdot 3h^2 \cdot \frac{dh}{dt} \implies \frac{dV}{dt} = \frac{9}{64}\pi h^2 \frac{dh}{dt}$

• Substitute Key Moment Information (now and not before now!!!):

$-1 = \frac{9}{64}\pi(6)^2\frac{dh}{dt}$

• Solve for the desired quantity:

$\frac{dh}{dt} = \frac{-64}{36\pi \cdot 9} = \frac{-16}{81\pi}\frac{\text{m}}{\text{min}}$

• Answer the question that was asked: The height is decreasing at a rate of $\frac{16}{81\pi}$ meters every second.