

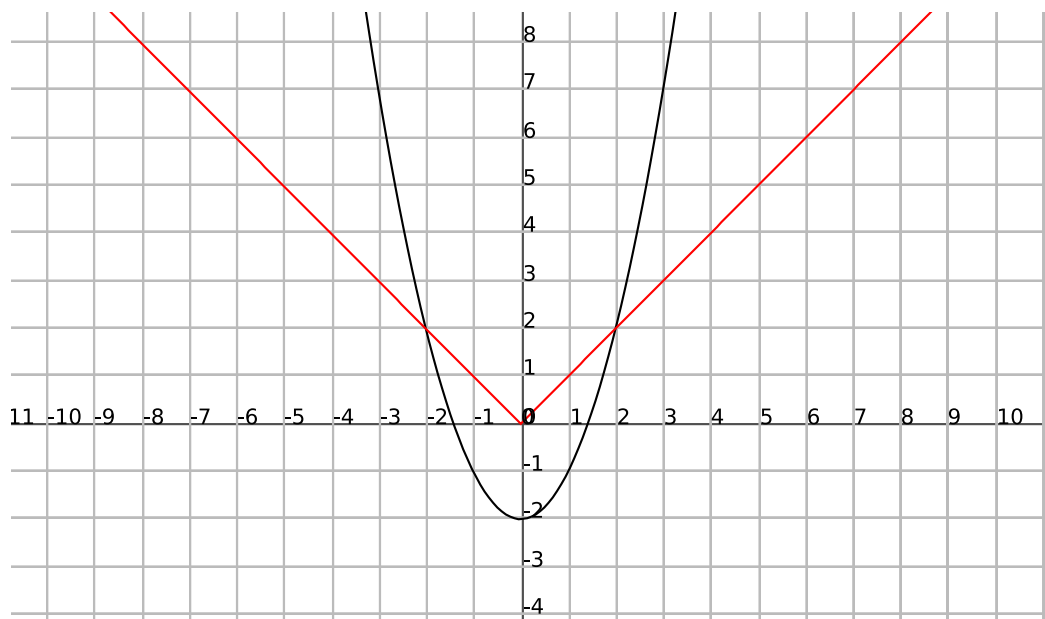
Extra Examples of Areas Between Curves

Professor Danielle Benedetto – Math 11

Find the area of the region bounded by $y = |x|$, $y = x^2 - 2$, $x = -3$ and $x = 3$.

You can solve this problem two different ways. I will detail both. First we will use symmetry to integrate one side and then double the value to capture the entire area. Note that $y = x$ and $y = x^2 - 2$ intersect at $x = 2$ when $x > 0$.

$$\begin{aligned}
 \text{Area} &= 2 \left[\int_0^2 \text{top} - \text{bottom} \, dx + \int_2^3 \text{top} - \text{bottom} \, dx \right] \\
 &= 2 \left[\int_0^2 x - (x^2 - 2) \, dx + \int_2^3 (x^2 - 2) - x \, dx \right] \\
 &= 2 \left[\int_0^2 -x^2 + x + 2 \, dx + \int_2^3 x^2 - x - 2 \, dx \right] \\
 &= 2 \left[\left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x \right) \Big|_0^2 + \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x \right) \Big|_2^3 \right] \\
 &= 2 \left[\left(-\frac{8}{3} + 2 + 4 \right) - 0 + \left(9 - \frac{9}{2} - 6 \right) - \left(\frac{8}{3} - 2 - 4 \right) \right] \\
 &= 2 \left[6 - \frac{8}{3} + 3 - \frac{9}{2} - \frac{8}{3} + 6 \right] \\
 &= 2 \left[15 - \frac{16}{3} - \frac{9}{2} \right] = 2 \left[\frac{90}{6} - \frac{32}{6} - \frac{27}{6} \right] = 2 \left[\frac{31}{6} \right] = \frac{31}{3}
 \end{aligned}$$



Now we will compute the area bounded, without using symmetry. Note that $y = |x|$ and $y = x^2 - 2$ intersect at $x = -2$ and $x = 2$:

$$\begin{aligned}
 & \text{Area} \\
 &= \int_{-3}^{-2} \text{top} - \text{bottom} \, dx + \int_{-2}^0 \text{top} - \text{bottom} \, dx + \int_0^2 \text{top} - \text{bottom} \, dx + \int_2^3 \text{top} - \text{bottom} \, dx \\
 &= \int_{-3}^{-2} (x^2 - 2) - (-x) \, dx + \int_{-2}^0 (-x) - (x^2 - 2) \, dx + \int_0^2 x - (x^2 - 2) \, dx + \int_2^3 (x^2 - 2) - x \, dx \\
 &= \int_{-3}^{-2} x^2 + x - 2 \, dx + \int_{-2}^0 -x^2 - x + 2 \, dx + \int_0^2 -x^2 + x + 2 \, dx + \int_2^3 x^2 - x - 2 \, dx \\
 &= \left(\frac{x^3}{3} + \frac{x^2}{2} - 2x \right) \Big|_{-3}^{-2} + \left(-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right) \Big|_{-2}^0 + \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x \right) \Big|_0^2 + \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x \right) \Big|_2^3 \\
 &= \left(-\frac{8}{3} + 2 + 4 \right) - \left(-9 + \frac{9}{2} + 6 \right) + (0) - \left(-\frac{-8}{3} - 2 - 4 \right) \\
 &\quad + \left(-\frac{8}{3} + 2 + 4 \right) - (0) + \left(9 - \frac{9}{2} - 6 \right) - \left(\frac{8}{3} - 2 - 4 \right) \\
 &= -\frac{8}{3} + 6 + 3 - \frac{9}{2} - \frac{8}{3} + 6 - \frac{8}{3} + 6 + 3 - \frac{9}{2} - \frac{8}{3} + 6 \\
 &= -\frac{32}{3} - \frac{18}{2} + 30 \\
 &= -\frac{32}{3} - 9 + 30 \\
 &= -\frac{32}{3} + 21 \\
 &= -\frac{32}{3} + \frac{63}{3} \\
 &= \frac{31}{3}
 \end{aligned}$$