Math 11 Midterm Exam #3 December 3, 2010

1. [15 Points]

(a) Differentiate $f(x) = \int_{\sin x}^{7} \sqrt{1-e^t}\ dt$

$$f'(x) = \frac{d}{dx}\int_{\sin x}^{7} \sqrt{1-e^t}\ dt = -\frac{d}{dx}\int_{7}^{\sin x} \sqrt{1-e^t}\ dt = \boxed{-\sqrt{1-e^{\sin x}}\cdot\cos x}$$

(b) Differentiate $f(x) = e^{\left(e^x\right)} + (x^e)^e$

Rewrite $f(x) = e^{\left(e^x\right)} + x^{\left(e^2\right)}$

$f'(x) = \boxed{e^{(e^x)}e^x + e^2x^{e^2-1}}$

(c) Find $f(x)$ if $\int_1^x f(t)\ dt = e^{\frac{1}{x}} - e.$

Differentiate both sides

$$\frac{d}{dx}\int_1^x f(t)\ dt = \frac{d}{dx}\left(e^{\frac{1}{x}} - e\right).$$

By the FTC (Part 1), $f(x) = \boxed{e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)}$

2. [20 Points] Compute the following integrals:

(a) $$\int \left(e^{3x} + \frac{1}{e^x}\right)^2 dx = \int e^{6x} + 2e^{2x} + \frac{1}{e^{2x}}\ dx = \int e^{6x} + 2e^{2x} + e^{-2x}\ dx$$

$$= \boxed{\frac{1}{6}e^{6x} + e^{2x} - \frac{1}{2}e^{-2x} + C}$$

(b) $$\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \sec^2(3x)\ dx = \frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2(u)\ du = \frac{1}{3}\tan(u)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{3}\left(\tan\frac{\pi}{3} - \tan\frac{\pi}{6}\right) = \boxed{\frac{1}{3}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)}$$

Here $\begin{cases} u &= 3x \\ du &= 3dx \\ \frac{1}{3}du &= dx \end{cases}$ and $\begin{cases} x = \frac{\pi}{18} & \Longrightarrow u = \frac{\pi}{6} \\ x = \frac{\pi}{9} & \Longrightarrow u = \frac{\pi}{3} \end{cases}$

(c) $$\int x(1-x)^{\frac{1}{3}}\ dx = -\int (1-u)u^{\frac{1}{3}}\ du = -\int u^{\frac{1}{3}} - u^{\frac{4}{3}}\ du = -\left(\frac{3}{4}u^{\frac{4}{3}} - \frac{3}{7}u^{\frac{7}{3}}\right) + C$$

$$= \boxed{-\frac{3}{4}(1-x)^{\frac{4}{3}} + \frac{3}{7}(1-x)^{\frac{7}{3}} + C}$$

Here $\begin{cases} u &= 1-x \quad \Rightarrow x = 1-u \\ du &= -dx \\ -du &= dx \end{cases}$

3. [8 Points] Find the function $f(x)$ that satisfies $f'(x) = \dfrac{e^x + \cos x}{\sqrt{e^x + \sin x}}$ and $f(0) = 5$.

$$f(x) = \int f'(x)\, dx = \int \frac{e^x + \cos x}{\sqrt{e^x + \sin x}}\, dx = \int \frac{1}{\sqrt{u}}\, du = 2\sqrt{u} + C = 2\sqrt{e^x + \sin x} + C$$

Here $\begin{cases} u &= e^x + \sin x \\ du &= e^x + \cos x dx \end{cases}$

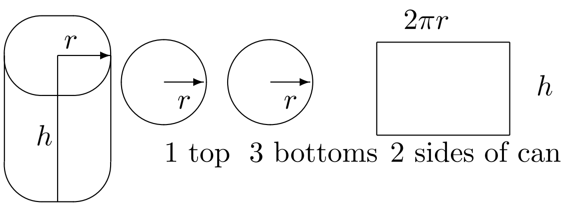
Now we use the initial valued information.

$f(0) = 2\sqrt{e^0 + \sin 0} + C = 2 + C \overset{\text{set}}{=} 5 \Rightarrow C = 3$. Finally, $f(x) = \boxed{2\sqrt{e^x + \sin x} + 3}$.

4. [15 Points] You need to construct a soup can in the shape of a cylinder. The bottom of the can needs to be covered with three layers of material. The sides of the can need to be covered with two layers of material. The top only needs one layer. You are to use a fixed 1200π square cm of material. What is the maximum volume of your soup can?

(Don't forget to state the common sense bounds.)

• Diagram:



• Variables:

Let r =radius of can.
Let h =height of can.
Let M =amount of material (surface area).
Let V =volume of can.

• Equations:

We know that the amount of material used $M = 4\pi r^2 + 4\pi r h = 1200\pi$ is fixed so that

$$h = \frac{1200\pi - 4\pi r^2}{4\pi r} = \frac{300 - r^2}{r}.$$

Then the volume $V = \pi r^2 h = \pi r^2 \left(\dfrac{300 - r^2}{r}\right) = \pi r(300 - r^2) = 300\pi r - \pi r^3$ must be maximized.

The (common-sense-bounds)domain of V are $\{r : 0 < r \leq \sqrt{300}\}$.

• Maximize:

Next $V' = 300\pi - 3\pi r^2$. Setting $V' = 0$ yields $r = \sqrt{100} = 10$ as the critical number.

Sign-testing the critical number does indeed yield a maximum for the volume function.

V' ⊕ | ⊖

V ↗ 10 ↘

MAX

• Answer:

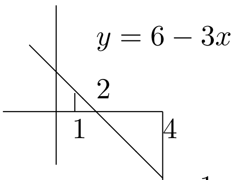
Since $r = 10$ then $h = \dfrac{300 - (10)^2}{10} = 20.$

The maximum volume of the can is $V = \pi(10)^2(20) = \boxed{2000\pi}$ cm^2.

5. [20 Points] Compute $\displaystyle\int_1^4 6 - 3x\ dx$ using each of the following **three** different methods:

(a) Area interpretations of the definite integral,
(b) Fundamental Theorem of Calculus,
(c) Riemann Sums and the limit definition of the definite integral $***$.

***Recall $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} 1 = n$



(a) Area$= \frac{1}{2}$(base) (height) $-\frac{1}{2}$(base) (height) $= \frac{1}{2}(1)(3) - \frac{1}{2}(2)(16) = \frac{3}{2} - 6 = \boxed{-\frac{9}{2}}$

(b) $\displaystyle\int_1^4 6 - 3x\ dx = 6x - \frac{3x^2}{2}\Big|_1^4 = (24 - 24) - \left(6 - \frac{3}{2}\right) = -6 + \frac{3}{2} = \boxed{-\frac{9}{2}}$

(c)

Here $a = 1, b = 4, \Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}$ and $x_i = 1 + i\left(\dfrac{3}{n}\right) = 1 + \dfrac{3i}{n}$.

$$\begin{aligned}
\int_1^4 6-3x\;dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(6-3\left(1+\frac{3i}{n}\right)\right)\frac{3}{n} \\
&= \lim_{n\to\infty}\left(\frac{3}{n}\sum_{i=1}^{n}\left(3-\frac{9i}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{3}{n}\left(\sum_{i=1}^{n}3-\sum_{i=1}^{n}\frac{9i}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{9}{n}\sum_{i=1}^{n}1-\frac{27}{n^2}\sum_{i=1}^{n}i\right) \\
&= \lim_{n\to\infty}\left(\frac{9}{n}(n)-\frac{27}{n^2}\frac{n(n+1)}{2}\right) \\
&= \lim_{n\to\infty}\left(9-\frac{27}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(9-\frac{27}{2}(1)\left(1+\frac{1}{n}\right)\right) \\
&= 9-\frac{27}{2} \\
&= \boxed{-\frac{9}{2}}
\end{aligned}$$

6. [10 Points] A stone is dropped from the top edge of a building. It hits the ground with a speed of 128 feet per second. What is the height of the building?

Here $v_0 = 0, v_{\text{impact}} = -128$ and $s_0 = ?$. We will use the equations of motion:

$a(t) = -32$
$v(t) = -32t + v_0 = -32t + 0 = -32t$
$s(t) = -16t^2 + s_0$

At impact $v(t) = -32t = -128$, so the stone hits the ground at $t = 4$ seconds. We know the position at impact is $s(4) = 0$. That is, $s(4) = -16(4)^2 + s_0 = 0$. Finally, we solve $\boxed{s_0 = 256}$ feet.

7. [12 Points] A moving object has velocity $v(t) = t^2 - 3t + 2$ feet per second, at time t seconds. Compute the **Total Distance** travelled by this object from time $t = 0$ to $t = 3$ seconds.

(You should also know what the formula for displacement is, but we did not ask for that here.)

Note that this $v(t)$ has negative output for $1 < t < 2$. Recommended to draw the graph.

Total Distance

$$= \int_0^3 |t^2 - 3t + 2| \; dt$$

$$= \int_0^1 |t^2 - 3t + 2| \; dt + \int_1^2 |t^2 - 3t + 2| \; dt + \int_2^3 |t^2 - 3t + 2| \; dt$$

$$= \int_0^1 t^2 - 3t + 2 \; dt + \int_1^2 -(t^2 - 3t + 2) \; dt + \int_2^3 t^2 - 3t + 2 \; dt$$

$$= \left(\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right)\Big|_0^1 - \left(\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right)\Big|_1^2 + \left(\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right)\Big|_2^3$$

$$= \left(\frac{1}{3} - \frac{3}{2} + 2\right) - 0 - \left(\left(\frac{8}{3} - 6 + 4\right) - \left(\frac{1}{3} - \frac{3}{2} + 2\right)\right) + \left(9 - \frac{27}{2} + 6\right) - \left(\frac{8}{3} - 6 + 4\right)$$

$$= \left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left(\frac{8}{3} - 2 - \frac{1}{3} + \frac{3}{2} - 2\right) + \left(15 - \frac{27}{2} - \frac{8}{3} + 2\right)$$

$$= \left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left(\frac{7}{3} + \frac{3}{2} - 4\right) + \left(17 - \frac{27}{2} - \frac{8}{3}\right)$$

$$= \left(\frac{2}{6} - \frac{9}{6} + \frac{12}{6}\right) - \left(\frac{14}{6} + \frac{9}{6} - \frac{24}{6}\right) + \left(\frac{102}{6} - \frac{81}{6} - \frac{16}{6}\right)$$

$$= \left(\frac{5}{6}\right) - \left(-\frac{1}{6}\right) + \left(\frac{5}{6}\right)$$

$$= \frac{5}{6} + \frac{1}{6} + \frac{5}{6}$$

$$= \boxed{\frac{11}{6}}$$

or we could use symmetry (to double) on the first piece:

Total Distance

$$= \int_0^3 |t^2 - 3t + 2| \; dt$$

$$= 2\int_0^1 |t^2 - 3t + 2| \; dt + \int_1^2 |t^2 - 3t + 2| \; dt$$

$$= 2\int_0^1 t^2 - 3t + 2 \; dt + \int_1^2 -(t^2 - 3t + 2) \; dt$$

$$= 2\left(\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right)\Bigg|_0^1 - \left(\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right)\Bigg|_1^2$$

$$= 2\left(\frac{1}{3} - \frac{3}{2} + 2\right) - 0 - \left(\left(\frac{8}{3} - 6 + 4\right) - \left(\frac{1}{3} - \frac{3}{2} + 2\right)\right)$$

$$= 2\left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left(\frac{8}{3} - 2 - \frac{1}{3} + \frac{3}{2} - 2\right)$$

$$= 2\left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left(\frac{7}{3} + \frac{3}{2} - 4\right)$$

$$= 2\left(\frac{2}{6} - \frac{9}{6} + \frac{12}{6}\right) - \left(\frac{14}{6} + \frac{9}{6} - \frac{24}{6}\right)$$

$$= 2\left(\frac{5}{6}\right) - \left(-\frac{1}{6}\right)$$

$$= \frac{10}{6} + \frac{1}{6}$$

$$= \boxed{\frac{11}{6}}$$

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Use the **Limit Definition of the Derivative** to compute the derivative of $f(x) = \dfrac{e^x}{e^x + 1}$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{e^{x+h}}{e^{x+h}+1} - \dfrac{e^x}{e^x+1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{e^{x+h}}{e^{x+h}+1} - \frac{e^x}{e^x+1}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{e^{x+h}(e^x+1) - e^x(e^{x+h}+1)}{(e^{x+h}+1)(e^x+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{e^{2x+h} + e^{x+h} - e^{2x+h} - e^x}{(e^{x+h}+1)(e^x+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{e^{x+h} - e^x}{(e^{x+h}+1)(e^x+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{e^x\left(e^h - 1\right)}{(e^{x+h}+1)(e^x+1)}\right) \\
&= \lim_{h\to 0} \frac{e^h-1}{h}\left(\frac{e^x}{(e^{x+h}+1)(e^x+1)}\right) \\
&= \frac{e^x}{(e^x+1)^2} \lim_{h\to 0} \frac{e^h-1}{h} \\
&= \frac{e^x}{(e^x+1)^2}(1) \\
&= \boxed{\frac{e^x}{(e^x+1)^2}}
\end{aligned}
$$

(You can double check this with the Quotient Rule.)

OPTIONAL BONUS #2 Compute $\int_0^6 \sqrt{6x-x^2}\,dx$

First complete the square $6x - x^2 = 9 - 9 + 6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x-3)^2$. The is the equation for the top semicircle centered at $(0, 3)$ with radius 3.

$\int_0^6 \sqrt{6x-x^2}\,dx = \int_0^6 \sqrt{9-(x-3)^2}\,dx =$ area of semicircle of radius $3 = \frac{1}{2}\pi(3)^2 = \boxed{\frac{9\pi}{2}}$

OPTIONAL BONUS #3 Compute $\int \sqrt{1+\sqrt{x}}\ dx = 2\int \sqrt{u}(u-1)\ du$

$= 2\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du = 2\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C = \boxed{\frac{4}{5}(1+\sqrt{x})^{\frac{5}{2}} - \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + C}$

Let $\begin{cases} u = 1+\sqrt{x} \Rightarrow \sqrt{x} = u-1 \\ du = \frac{1}{2\sqrt{x}}\ dx \\ 2\sqrt{x}\ du = dx \\ 2(u-1)\ du = dx \end{cases}$

OPTIONAL BONUS #4 Compute $\int_{-3}^{3} \tan x + \frac{\sqrt[3]{x}}{(1+x^2)^7} - x^{17}\cos x\ dx = \boxed{0}$ since this integrand is an **odd** function.

OPTIONAL BONUS #5 Compute $\lim_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n}}{n\sqrt{n}}$

Recognize this limit/sum as a Riemann Sum for $f(x) = \sqrt{x}$ on the interval $[0,1]$ with $\Delta x = \frac{1}{n}$ and $x_i = 0 + i\frac{1}{n} = \frac{i}{n}$.

$$\lim_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n}}{n\sqrt{n}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sqrt{i}}{\sqrt{n}} \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i}{n}} \cdot \frac{1}{n} = \int_0^1 \sqrt{x}\ dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \boxed{\frac{2}{3}}$$