Please carefully write all of your answers in your **Blue Book**. Justify all of your answers. There are **No Calculators** allowed.

1. (5 Points) Compute $\frac{d}{dx}\left(\int_{3x}^{2}\cos t\ dt\right) = -\frac{d}{dx}\int_{2}^{3x}\cos t\ dt = -\cos(3x)(3) = \boxed{-3\cos(3x)}$.

2. (30 Points) Compute each of the following integrals.

(a) $\int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\ dx$ **(b)** $\int \frac{we^{w^2}}{(17+e^{w^2})^3}\ dw$

(c) $\int_0^{\pi} \sin^2\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right)\ dx$ **(d)** $\int_{-3}^{3} x|x|\ dx$

(e) $\int (e^{3x}+e^{-7x})^2\ dx$ **(f)** $\int x(x+1)^{14}\ dx$

(a). $\int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\ dx = \int u^{\frac{1}{2}}\ dw = \frac{2}{3}u^{\frac{3}{2}} + C = \boxed{\frac{2}{3}\left(1-\frac{1}{x}\right)^{\frac{3}{2}} + C}$

Here $\begin{cases} u &= 1-\frac{1}{x} \\ du &= \frac{1}{x^2}du \end{cases}$

(b). $\int \frac{we^{w^2}}{(17+e^{w^2})^3}\ dw = \frac{1}{2}\int \frac{1}{u^3}\ du = \frac{1}{2}\int u^{-3}\ du = \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) + C = -\frac{1}{4u^2} + C$

$\boxed{= -\frac{1}{4(17+e^{w^2})^2} + C}$

Here $\begin{cases} u &= 17+e^{w^2} \\ du &= e^{w^2}(2w)\ dw \\ \frac{1}{2}du &= we^{w^2}\ dw \end{cases}$

(c). $\int_0^{\pi} \sin^2\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right)\ dx = 6\int_{u=0}^{u=\frac{1}{2}} u^2\ du = 2u^3\Big|_{u=0}^{u=\frac{1}{2}} = \boxed{\frac{1}{4}}$

Here $\begin{cases} u &= \sin\left(\frac{x}{6}\right) \\ du &= \frac{1}{6}\cos\left(\frac{x}{6}\right)dx \\ 6du &= \cos\left(\frac{x}{6}\right)dx \end{cases}$ and $\begin{cases} x=0 & \Longrightarrow u=0 \\ x=\pi & \Longrightarrow u=\frac{1}{2} \end{cases}$

(d). $\displaystyle\int_{-3}^{3} x|x|\,dx = \int_{-3}^{0} -x^2\,dx + \int_{0}^{3} x^2\,dx = -\frac{x^3}{3}\Big|_{-3}^{0} + \frac{x^3}{3}\Big|_{0}^{3} = (0-9)+(9-0) = \boxed{0}$

(e). $\displaystyle\int (e^{3x}+e^{-7x})^2\,dx = \int e^{6x}+2e^{-4x}+e^{-14x}\,dx = \boxed{\frac{1}{6}e^{6x} - \frac{1}{2}e^{-4x} - \frac{1}{14}e^{-14x} + C}$

(f). $\displaystyle\int x(x+1)^{14}\,dx = \int (u-1)u^{14}\,du = \int u^{15}-u^{14}\,du = \frac{1}{16}u^{16} - \frac{1}{15}u^{15} + C = \boxed{\frac{(x+1)^{16}}{16} - \frac{(x+1)^{15}}{15} + C}$

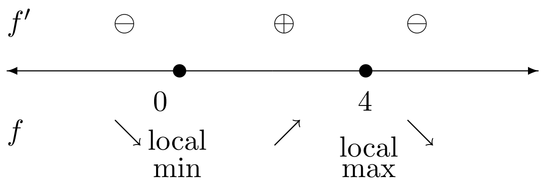
Here $\begin{cases} u &= x+1 \Longrightarrow x = u-1 \\ du &= dx \end{cases}$

3. (10 Points) Find all local maximum and minimum value(s) of the function $f(x) = x^4e^{-x}$.

First Derivative Information

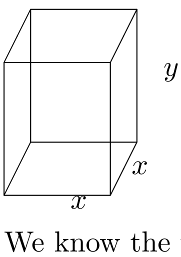
We compute $f'(x) = x^4e^{-x}(-1) + e^{-x}(4x^3) = e^{-x}(4x^3 - x^4) = e^{-x}x^3(4-x)$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when $x = 0$ or $x = 4$.

Using sign testing/analysis for f',



So f is increasing on the interval $(0,4)$; and f is decreasing on $(-\infty,0)$ and $(4,\infty)$. Moreover, f has a local max at $x = 4$ with $f(4) = 256e^{-4}$, and a local min at $x = 0$ with $f(0) = 0$.

4. (15 Points) A toolshed with a square base and a flat roof is to have volume of 800 cubic feet. If the floor costs \$6 per square foot, the roof \$2 per square foot, and the sides \$5 per square foot, determine the dimensions of the most economical shed. Remember to state the domain (or common-sense-bounds) of the function you are computing extreme values for.



We know the volume of the toolshed is given by $V = x^2y = 800$ is fixed, so that $y = \dfrac{800}{x^2}$.

Then the Cost of materials, which must be minimized, is given as

$$\begin{aligned}
\text{Cost} &= \text{cost of floor} + \text{cost of top} + \text{cost of 4 sides}\\
&= x^2(\$6) + x^2(\$2) + 4xy(\$5)\\
&= 8x^2 + 20xy\\
&= 8x^2 + 20x\left(\frac{800}{x^2}\right)\\
&= 8x^2 + \left(\frac{16000}{x}\right)
\end{aligned}$$

The (common-sense-bounds)domain of Cost is $\{x : x > 0\}$.

Next $\text{Cost}' = 16x - \dfrac{16000}{x^2}$. Setting $\text{Cost}' = 0$ we solve $x^3 = 1,000 \Longrightarrow x = 10$.

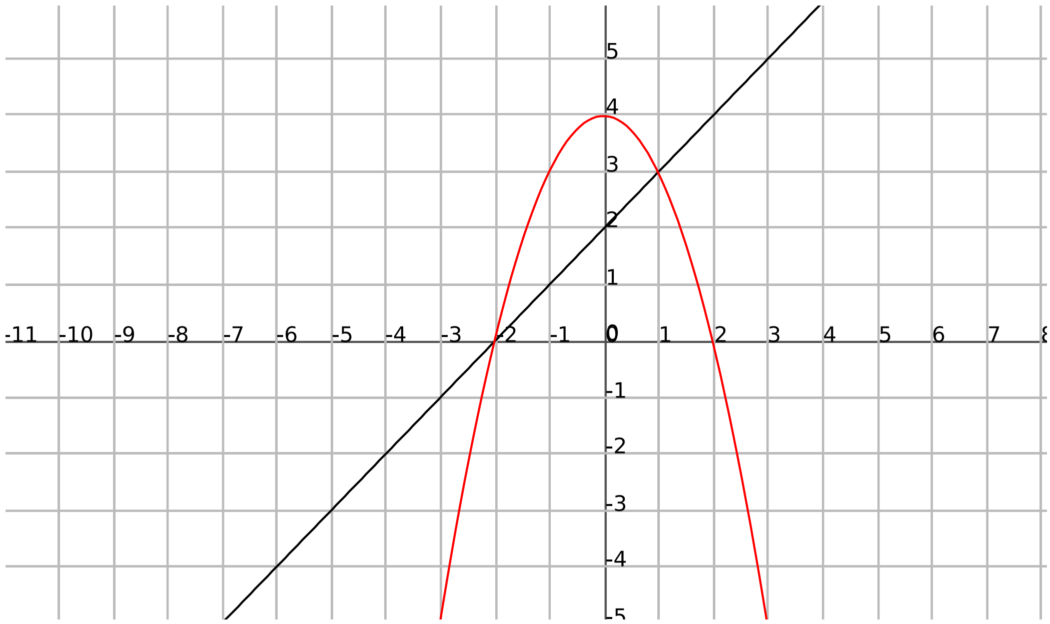
Sign-testing the critical number does indeed yield a maximum for the area function.

$$\begin{array}{l|cc}
\text{Cost}' & \ominus & \oplus \\
\hline
\text{Cost} & \searrow \; 10 & \nearrow \\
 & \text{MIN} &
\end{array}$$

Since $x = 10$ then $y = \dfrac{800}{(10)^2} = 8$. As a result, the most economical shed has dimensions $\boxed{\text{10x10x8}}$, each in feet.

5. (10 Points) Compute the area bounded by $y = 4 - x^2$, $y = x + 2$, $x = -3$, and $x = 0$. Draw a picture of the region(s). Do not worry about simplifying your fractions in the final answer.

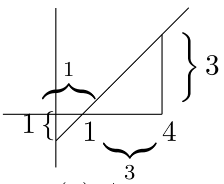
Note that these two curves intersect when $4 - x^2 = x + 2 \Longrightarrow x^2 + x - 2 = 0 \Longrightarrow (x + 2)(x - 1) = 0 \Longrightarrow x = -2$ or $x = 1$.



$$\begin{aligned}
\text{Area} &= \int_{-3}^{-2} \text{top} - \text{bottom}\ dx + \int_{-2}^{0} \text{top} - \text{bottom}\ dx \\
&= \int_{-3}^{-2} (x+2) - (4-x^2)\ dx + \int_{-2}^{0} (4-x^2) - (x+2)\ dx \\
&= \int_{-3}^{-2} x^2 + x - 2\ dx + \int_{-2}^{0} -x^2 - x + 2\ dx \\
&= \left(\frac{x^3}{3} + \frac{x^2}{2} - 2x\right)\Bigg|_{-3}^{-2} + \left(-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right)\Bigg|_{-2}^{0} \\
&= \left(-\frac{8}{3} + 2 + 4\right) - \left(-9 + \frac{9}{2} + 6\right) + 0 - \left(\frac{8}{3} - 2 - 4\right) \\
&= -\frac{8}{3} + 6 + 3 - \frac{9}{2} - \frac{8}{3} + 6 \\
&= -\frac{16}{3} + 15 - \frac{9}{2} \\
&= -\frac{32}{6} + \frac{90}{6} - \frac{27}{6} \\
&= \boxed{\frac{31}{6}}
\end{aligned}$$

6. (20 Points) Compute $\int_0^4 x - 1\ dx$ using each of the following three different methods:

(a) using Area interpretations of the definite integral,
(b) Fundamental Theorem of Calculus,
(c) Riemann Sums and the limit definition of the definite integral.



(a) Area= area of triangle above x-axis minus area of triangle below x-axis

$= \frac{1}{2}(3)(3) - \frac{1}{2}(1)(1) = \frac{9}{2} - \frac{1}{2} = \boxed{4}$

(b) $\int_0^4 x - 1\ dx = \frac{x^2}{2} - x\Big|_0^4 = (8-4) - 0 = \boxed{4}$

(c)

Here $a = 0, b = 4, \Delta x = \frac{4-0}{n} = \frac{4}{n}$ and $x_i = 0 + i\left(\frac{4}{n}\right) = \frac{4i}{n}$.

$$\int_0^4 x - 1\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\frac{4}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n} - 1\right)\frac{4}{n}$$

$$= \lim_{n\to\infty} \left(\frac{4}{n}\sum_{i=1}^{n} \frac{4i}{n} - \frac{4}{n}\sum_{i=1}^{n} 1\right)$$

$$= \lim_{n\to\infty} \left(\frac{16}{n^2}\sum_{i=1}^{n} i - \frac{4}{n}(n)\right)$$

$$= \lim_{n\to\infty} \left(\frac{16}{n^2}\frac{n(n+1)}{2} - 4\right)$$

$$= \lim_{n\to\infty} \left(8\left(\frac{n+1}{n}\right) - 4\right)$$

$$= \lim_{n\to\infty} \left(8\left(\frac{n}{n} + \frac{1}{n}\right) - 4\right)$$

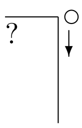
$$= \lim_{n\to\infty} \left(8\left(1 + \frac{1}{n}\right) - 4\right)$$

$$= 8 - 4$$

$$= \boxed{4}$$

7. (10 Points) Jack throws a baseball straight downward from the top of a tall building. The initial speed of the ball is 25 feet per second. It hits the ground with a speed of 153 feet per second. How tall is the building?

Note $v_0 = -25\frac{\text{ft}}{\text{sec}}, s_0 = ?\text{ft}, v_{\text{impact}} = -153\frac{\text{ft}}{\text{sec}}$



$a(t) = -32$
$v(t) = -32t + v_0 \Longrightarrow v(t) = -32t - 25$
$s(t) = -16t^2 - 25t + s_0$

The ball hits the ground when $v(t) = -32t - 25 = -153$ or when $32t = 153 - 25 = 128$ which is when $t_{\text{impact}} = 4$ seconds.

Finally, we solve $s(4) = 0$ for s_0. The ball hits the ground when $-16(4)^2 - 25(4) + s_0 = 0$ or when $-256 - 100 + s_0 = 0$ which is when $s_0 = 356$ feet. As a result, the building is $\boxed{356}$ feet tall.

TURN PAPER OVER FOR THE BONUS PROBLEMS PLEASE!!

REMEMBER: ALL OF YOUR WORK GOES IN THE BLUE ANSWER BOOK

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BONUS PROBLEMS: THESE ARE OPTIONAL!

Feel free to attempt either of the following two bonus problems, but ONLY if you are completely done with the original part of the exam, problems 1-7.

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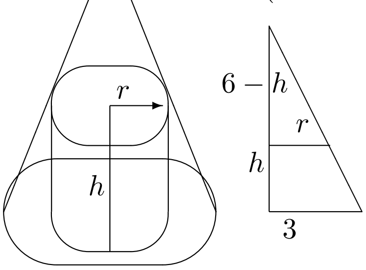
Bonus 1: Suppose f is continuous on $[-a, a]$, PROVE each of the following two statements:

(a). If f is even, then $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$

(b). If f is odd, then $\int_{-a}^{a} f(x)\, dx = 0$

Bonus 2: Consider a cone such that the height is 6 inches high and its base has diameter 6 in. Inside this cone we inscribe a cylinder whose base lies on the base of the cone and whose top intersects the cone in a circle. What is the maximum volume of the cylinder? Remember to state the domain (or common-sense-bounds) of the function you are computing extreme values for.



Using similar triangles, for the cross slice of the cone and cylinder, we see $\frac{r}{3} = \frac{6-h}{6}$ which implies that $6r = 18 - 3h \Longrightarrow h = 6 - 2r$.

Then the volume of the cylinder, given by, $V = \pi r^2 h = \pi r^2(6 - 2r) = 6\pi r^2 - 2\pi r^3$ must be maximized.

The (common-sense-bounds)domain of V is $\{r : 0 \le r \le 3\}$.

Next $V' = 12\pi r - 6\pi r^2$. Setting $V' = 0$ we see $6\pi r(2 - r) = 0$ and solve for $r = 0$ or $r = 2$ as critical numbers. Of course $r = 0$ will not lead to a maximum since no cylinder exists there.

Sign-testing the critical number $r = 2$ does indeed yield a maximum for the volume function.

V' $\oplus$ $\ominus$

V $\nearrow$ 2 $\searrow$

MAX

Since $r = 2$, then $h = 2$, and the maximum volume $V = \pi(2)^2 2 = 8\pi$. As a result the maximum volume of the cylinder is $\boxed{8\pi}$ cubic inches.