Answer Key!!! *smile*

Review Packet for Exam #2

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Math 11

Differentiation Rules Differentiate the following functions. Please do **not** simplify your derivatives here.

1. $y = \sin^3(x^3)$

   $y' = 3\sin^2(x^3) \cdot \cos(x^3) \cdot 3x^2 = 9x^2 \sin^2(x^3)\cos(x^3)$

2. $y = \cos^2(3x)$

   $y' = 2\cos(3x) \cdot (-\sin(3x)) \cdot 3 = -6\cos(3x)\sin(3x)$

3. $f(t) = t^2 \sin^5(2t)$

   $f'(t) = t^2[5\sin^4(2t) \cdot \cos(2t) \cdot 2] + \sin^5(2t) \cdot 2t = 10t^2 \sin^4(2t)\cos(2t) + 2t\sin^5(2t)$

4. $H(x) = \left(1 - \dfrac{2}{x^2}\right)^5$

   $H'(x) = 5\left(1 - \dfrac{2}{x^2}\right)^4 \cdot (4x^{-3}) = \dfrac{20}{x^3}\left(1 - \dfrac{2}{x^2}\right)^4$

5. $f(x) = \sqrt[3]{x^3 + 8}$

   $f'(x) = \dfrac{1}{3}(x^3+8)^{-2/3} \cdot 3x^2 = \dfrac{x^2}{(x^3+8)^{2/3}}$

6. $g(t) = \dfrac{t^3 + \tan\left(\frac{1}{t}\right)}{1 + t^2}$

   $$g'(t) = \frac{(1+t^2)\Big[3t^2 + \sec^2\left(\frac{1}{t}\right) \cdot \left(-\frac{1}{t^2}\right)\Big] - \Big(t^3 + \tan\left(\frac{1}{t}\right)\Big) \cdot (2t)}{(1+t^2)^2}$$

   $$= \frac{t^4 + 3t^2 - (1 + t^{-2})\sec^2\left(\frac{1}{t}\right) - 2t\tan\left(\frac{1}{t}\right)}{(1+t^2)^2}$$

7. $p(x) = \dfrac{1}{(-2x+3)^5}$

   $p(x) = (-2x+3)^{-5}$, so $p'(x) = -5(-2x+3)^{-6} \cdot (-2) = \dfrac{10}{(-2x+3)^{-6}}$

8. $r(x) = \dfrac{(2x+1)^3}{(3x+1)^4}$

   $$r'(x) = \frac{(3x+1)^4 \cdot 3(2x+1)^2 \cdot (2) - (2x+1)^3 \cdot 4(3x+1)^3 \cdot (3)}{(3x+1)^8}$$

   $$= \frac{6(2x+1)^2(3x+1)^3[(3x+1) - 2(2x+1)]}{(3x+1)^8} = \frac{6(2x+1)^2[3x+1-4x-2]}{(3x+1)^5}$$

$$= \frac{6(2x+1)^2(-x-1)}{(3x+1)^5} = \frac{-6(x+1)(2x+1)^2}{(3x+1)^5}$$

9. $S(x) = \left(\frac{1+2x}{1+3x}\right)^4$

$$S'(x) = 4\Big(\frac{1+2x}{1+3x}\Big)^3 \cdot \Big[\frac{(1+3x)\cdot 2 - (1+2x)\cdot 3}{(1+3x)^2}\Big]$$

$$= \frac{4(1+2x)^3\cdot[2+6x-3-6x]}{(1+3x)^5} = \frac{-4(1+2x)^3}{(1+3x)^5}$$

10. $g(x) = \cos(3x)\sin(4x)$

$$g'(x) = \cos(3x)\cos(4x)\cdot 4 + \sin(4x)\big(-\sin(3x)\big)\cdot(3) = 4\cos(3x)\cos(4x) - 3\sin(3x)\sin(4x)$$

11. $g(x) = \frac{\cos(3x)}{\sin(4x)}$

$$g'(x) = \frac{\sin(4x)\big(-\sin(3x)\cdot 3\big) - \cos(3x)\cos(4x)\cdot 4}{\sin^2(4x)} = \frac{-3\sin(3x)\sin(4x) - 4\cos(3x)\cos(4x)}{\sin^2(4x)}$$

12. $g(x) = (x+7x^{-6})\sqrt{2x+1}\cos^2(6x)$

$g(x) = (x+7x^{-6})\big[\sqrt{2x+1}\cos^2(6x)\big]$, so

$$g'(x) = (x+7x^{-6})\Big[\sqrt{2x+1}\cdot 2\cos(6x)\cdot\big(-\sin(6x)\big)\cdot(6) + \cos^2(6x)\cdot\frac{1}{2\sqrt{2x+1}}\cdot 2\Big] + \big[\sqrt{2x+1}\cos^2(6x)\big](1-42x^{-7})$$

$$= (x+7x^{-6})\cos(6x)\Big[-12\sqrt{2x+1}\sin(6x) + \frac{\cos(6x)}{\sqrt{2x+1}}\Big] + \sqrt{2x+1}(1-42x^{-7})\cos^2(6x)$$

13. $w = \frac{5(1+x^2)^3}{x\sqrt{2x+1}}$

$$w' = \frac{x\sqrt{2x+1}\big(15(1+x^2)^2\cdot(2x)\big) - 5(1+x^2)^3\big[x\cdot\frac{1}{2\sqrt{2x+1}}\cdot 2 + \sqrt{2x+1}\cdot 1\big]}{x^2(2x+1)}$$

$$= \frac{30x^2(2x+1)(1+x^2)^2 - 5(1+x^2)^3[x+(2x+1)]}{x^2(2x+1)^{3/2}}$$

$$= \frac{5(1+x^2)^2\big(6x^2(2x+1) - (3x+1)(1+x^2)\big)}{x^2(2x+1)^{3/2}}$$

$$= \frac{5(1+x^2)\big(12x^3+6x^2-3x-1-3x^3-x^2\big)}{x^2(2x+1)^{3/2}} = \frac{5(1+x^2)(9x^3+5x^2-3x-1)}{x^2(2x+1)^{3/2}}$$

14. $y = \big((x^2+3x)^4+x\big)^{-\frac{5}{7}}$

$$y' = -\frac{5}{7}\big[(x^2+3x)^4+x\big]^{-12/7}[4(x^2+3x)^3(2x+3)+1]$$

15. $g(t) = \cos\left(\sin^3\left(\frac{t}{\sqrt{t+1}}\right)\right)$

$$g'(t) = -\sin\left(\sin^3\left(\frac{t}{\sqrt{t+1}}\right)\right)\cdot 3\sin^2\left(\frac{t}{\sqrt{t+1}}\right)\cdot\cos\left(\frac{t}{\sqrt{t+1}}\right)\cdot\Big[\frac{\sqrt{t+1}\cdot(1) - t\cdot\frac{1}{2\sqrt{t+1}}\cdot 1}{t+1}\Big]$$

Note that $\dfrac{\sqrt{t+1}\cdot(1) - t\cdot\frac{1}{2\sqrt{t+1}}\cdot 1}{t+1} = \dfrac{t+1-\frac{t}{2}}{(t+1)^{3/2}} = \dfrac{t+2}{2(t+1)^{3/2}}$. So

$$g'(t) = \frac{-3(t+2)}{2(t+1)^{3/2}} \sin\left(\sin^3\left(\frac{t}{\sqrt{t+1}}\right)\right)\sin^2\left(\frac{t}{\sqrt{t+1}}\right)\cos\left(\frac{t}{\sqrt{t+1}}\right)$$

16. $g(x) = \cos^2(6x)\left(\dfrac{\tan(-x)}{\sqrt{2x+1}}\right)$

$$g'(x) = \cos^2(6x)\Big[\frac{\sqrt{2x+1}\cdot\sec^2(-x)\cdot(-1) - \tan(-x)\frac{1}{2\sqrt{2x+1}}(2)}{2x+1}\Big]$$

$$+\Big(\frac{\tan(-x)}{\sqrt{2x+1}}\Big)\cdot 2\cos(6x)\cdot\big(-\sin(6x)\big)\cdot(6)$$

$$= \cos^2(6x)\Big[\frac{-(2x+1)\sec^2(-x) - \tan(-x)}{(2x+1)^{3/2}}\Big] - \frac{12\tan(-x)\cos(6x)\sin(6x)}{\sqrt{2x+1}}$$

(Note: This problem could be made slightly easier to solve if you observe that $\tan(-x) = -\tan x$ before you differentiate.)

Absolute Extreme Values

17. Find the absolute maximum and absolute minimum values of the following functions on the given intervals.

(a) $f(x) = 3x^{2/3} - \dfrac{x}{4}$ on $[-1, 1]$.

f is continuous on $[-1, 1]$, so we use the closed interval method. We compute $f'(x) = 2x^{-1/3} - 1/4$. On the interval $[-1, 1]$, f' is undefined at $x = 0$, so $x = 0$ is a critical point (since f is defined but not differentiable there). Also, $f'(x) = 0$ exactly when $x^{1/3} = 8$; that is, when $x = 512$; but $x = 512$ is not in $[-1, 1]$. The endpoints ± 1 also need to be checked, of course. Testing the critical points and endpoints, we see

$f(-1) = \dfrac{13}{4}, \quad f(0) = 0, \quad f(1) = \dfrac{11}{4}.$

So the absolute maximum value is $13/4$ (attained at $x = -1$), and the absolute minimum value is 0 (attained at $x = 0$).

(b) $h(x) = \dfrac{x^2-1}{x^2+1}$ on $[-1, 3]$.

$h'(x) = \dfrac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$. On the interval $[-1, 3]$, h' is always defined (since it is a rational function and its denominator is never zero). Also, $h'(x) = 0$ happens only when the numerator is zero; that is, when $4x = 0$, so $x = 0$. Applying the closed interval method:

$h(-1) = 0, \quad h(0) = -1, \quad h(3) = \dfrac{4}{5}.$

So the absolute maximum value is $4/5$ (attained at $x = 3$), and the absolute minimum value is -1 (attained at $x = 0$).

(c) $F(x) = -2x^3 + 3x^2$ on $[-\frac{1}{2}, 3]$.

$F'(x) = -6x^2 + 6x$, which is always defined. Setting $F' = 0$ gives $-6x(x-1) = 0$, so $x = 0$ and $x = 1$, both of which are in the interval, are the only critical points. Applying the closed interval method:

$F\left(-\frac{1}{2}\right) = \frac{1}{4} + \frac{3}{4} = 1, \quad F(0) = 0, \quad F(1) = -2 + 3 = 1, \quad F(3) = -54 + 27 = -27.$
So the absolute maximum is 1, attained at $x = -1/2$ and $x = 1$, and the absolute minimum is -27, attained at $x = 3$.

(d) $f(x) = x^{\frac{2}{3}}$ on $[-1, 8]$.
$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$, which is not defined at $x = 0$, but which is defined everywhere else. Solving $f' = 0$ gives $2 = 0$, which never happens. So the only critical point is $x = 0$, which is in the interval. Applying the closed interval method:
$f(-1) = 1, \quad f(0) = 0, \quad f(8) = 4.$
So the absolute maximum is 4, at $x = 8$, and the absolute minimum is 0, at $x = 0$.

(e) $f(x) = \frac{1}{1+x^2}$ on $[-3, 1]$.
$f'(x) = -(1+x^2)^{-2} \cdot (2x) = \frac{-2x}{(1+x^2)^2}$, which is always defined. (Note that the denominator is never zero.) Solving $f' = 0$ gives $2x = 0$, so $x = 0$, which is in the interval. Applying the closed interval method:
$f(-3) = \frac{1}{10}, \quad f(0) = 1, \quad f(1) = \frac{1}{2}.$
So the absolute maximum is 1, at $x = 0$, and the absolute minimum is $1/10$, at $x = -3$.

Curve Sketching For each of the following functions, discuss domain, vertical and horizontal asymptotes, intervals of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

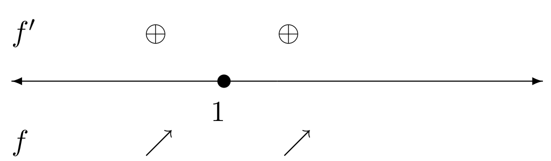
18. $f(x) = x^3 - 3x^2 + 3x + 10$

- Domain: $f(x)$ has domain $(-\infty, \infty)$
- VA: It is a polynomial, continuous everywhere, and so has no vertical asymptotes.
- HA: There are no horizontal asymptotes for this f since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.
- First Derivative Information:

We compute $f'(x) = 3x^2 - 6x + 3$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when

$$3x^2 - 6x + 3 = 3(x-1)(x-1) = 0 \Longrightarrow x = 1$$

As a result, $x = 1$ is the critical number. Using sign testing/analysis for f',



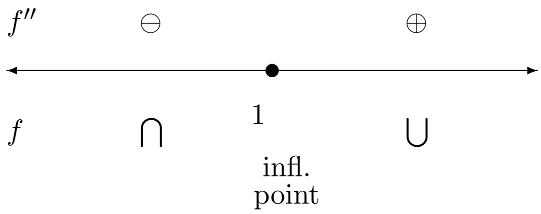
or our f' chart is

x	$(-\infty, 1)$	$(1, \infty)$
$f'(x)$	$\oplus$	$\oplus$
$f(x)$	$\nearrow$	$\nearrow$

So f is increasing on $(-\infty, \infty)$. Moreover, f has no extreme values.

• Second Derivative Information:

Meanwhile, f'' is always defined and continuous, and $f'' = 6x - 6 = 0$ only at our possible inflection point $x = 1$. Using sign testing/analysis for f'',

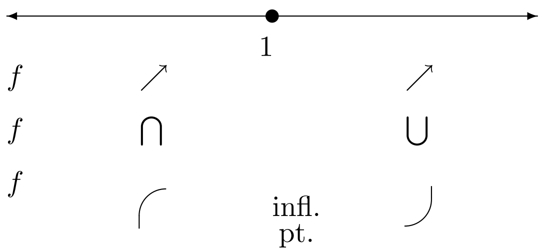


or our f'' chart is

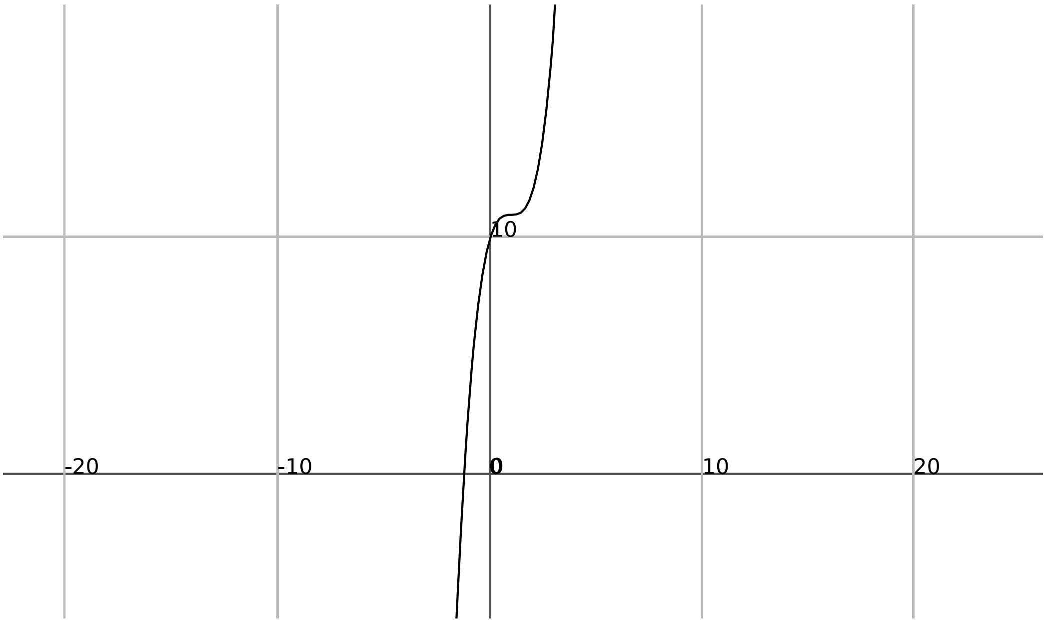
x	$(-\infty, 1)$	$(1, \infty)$
$f''(x)$	$\ominus$	$\oplus$
$f(x)$	$\cap$	$\cup$

So f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$, with an inflection point $(1, 11)$ at $x = 1$.

• Piece the first and second derivative information together:



• Sketch:



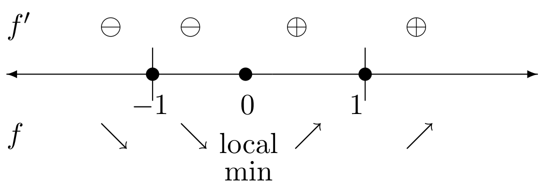
19. $f(x) = \dfrac{3x^2}{1-x^2}$

- Domain: $f(x)$ has domain $\{x|x \neq \pm 1\}$
- VA: Vertical asymptotes $x = \pm 1$.
- HA: Horizontal asymptote is $y = -3$ for this f since $\lim_{x\to\pm\infty} f(x) = -3$ because

$$\lim_{x\to\pm\infty} \frac{3x^2}{1-x^2} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\pm\infty} \frac{3}{\frac{1}{x^2}-1} = -3$$

- First Derivative Information:

We compute $f'(x) = \dfrac{6x}{(1-x^2)^2}$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined or zero. The former happens when $x = \pm 1$, but $x = \pm 1$ were not in the domain of the original function, so they aren't technically critical numbers. The latter happens when $x = 0$. As a result, $x = 0$ is the critical number. Using sign testing/analysis for f',



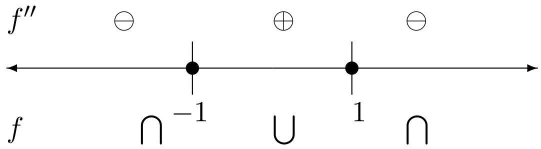
or our f' chart is

x	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	$\ominus$	$\oplus$
$f(x)$	$\searrow$	$\nearrow$

So f is decreasing on $(-\infty, -1)$ and $(-1, 0)$ and increasing on $(0, 1)$ and $(1, \infty)$. Moreover, f has a local min at $x = 0$ with $f(0) = 0$.

• Second Derivative Information:

Meanwhile, $f'' = \dfrac{6(1+3x^2)}{(1-x^2)^3}$. Using sign testing/analysis for f'',

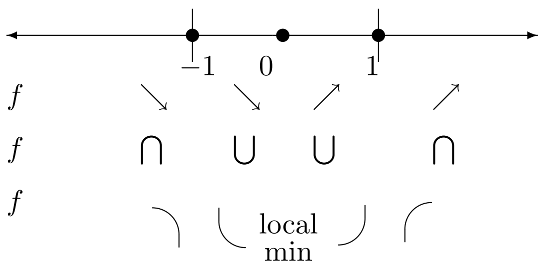


or our f'' chart is

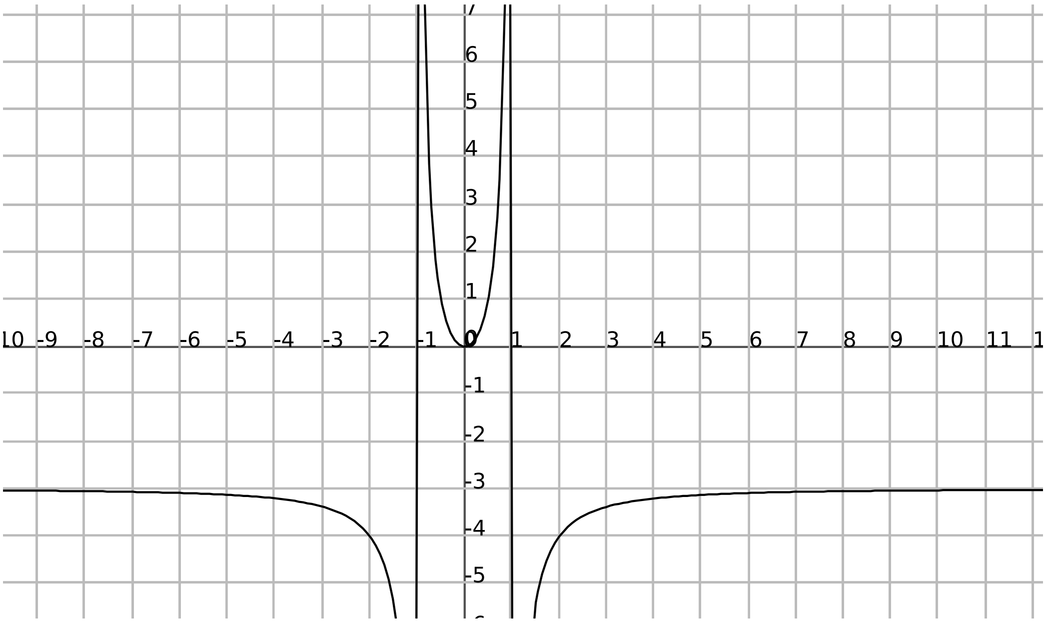
x	$(-\infty, 1)$	$(-1, 1)$	$(1, \infty)$
$f''(x)$	⊖	⊕	⊖
$f(x)$	$\cap$	$\cup$	$\cap$

So f is concave down on $(-\infty, -1)$ and$(1, \infty)$ and concave up on $(-1, 1)$.

• Piece the first and second derivative information together:



• Sketch:



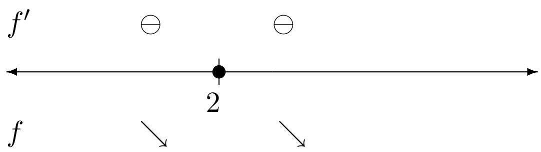
20. $f(x) = \dfrac{x}{x-2}$

- Domain: $f(x)$ has domain $\{x | x \neq 2\}$
- VA: Vertical asymptote at $x = 2$.
- HA: Horizontal asymptote at $y = 1$ for this f since $\lim_{x \to \pm\infty} f(x) = 1$ because

$$\lim_{x \to \pm\infty} \frac{x}{x-2} \cdot \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to \pm\infty} \frac{1}{1 - \frac{2}{x}} = 1.$$

- First Derivative Information:

We compute $f'(x) = \dfrac{-2}{(x-2)^2}$ to find critical numbers. The critical points occur where f' is undefined or zero (never here). The former happens when $x = 2$, which was not in the domain of the original function. As a result, there are no critical numbers. Using sign testing/analysis for f' around $x = 2$,



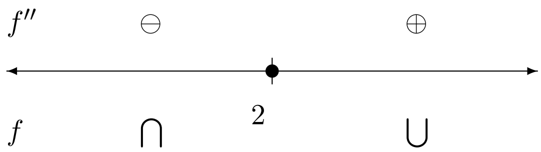
or our f' chart is

x	$(-\infty, 2)$	$(2, \infty)$
$f'(x)$	$\ominus$	$\ominus$
$f(x)$	↘	↘

So f is decreasing on $(-\infty, 2)$ and $(2, \infty)$. Moreover, f has no extreme values.

- Second Derivative Information:

Meanwhile, $f'' = \dfrac{4}{(x-2)^3}$. Using sign testing/analysis for f'' around $x = 2$,

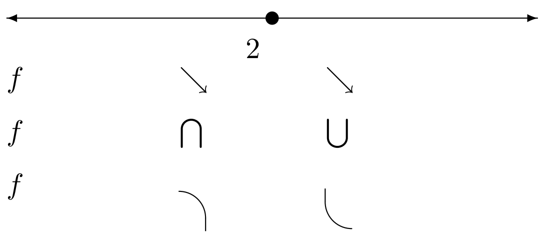


or our f'' chart is

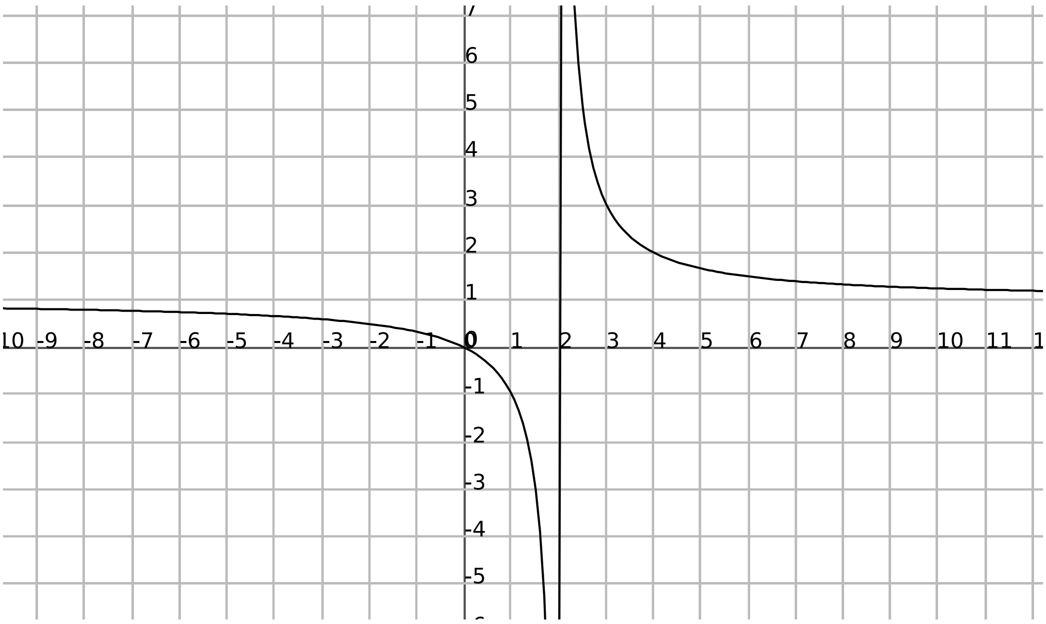
x	$(-\infty, 2)$	$(2, \infty)$
$f''(x)$	$\ominus$	$\oplus$
$f(x)$	$\cap$	$\cup$

So f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

• Piece the first and second derivative information together:



• Sketch:



21. $f(x) = 2x^3 + 5x^2 - 4x$

• Domain: $f(x)$ has domain $(-\infty, \infty)$

• VA: It is a polynomial, continuous everywhere, and so has no vertical asymptotes.

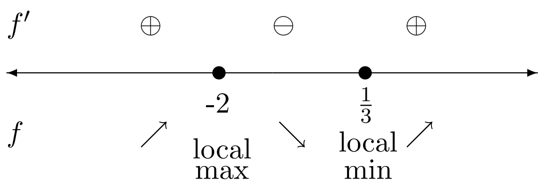
• HA: There are no horizontal asymptotes for this f since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

• First Derivative Information:

We compute $f'(x) = 6x^2 + 10x - 4$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when

$$6x^2 + 10x - 4 = (6x - 2)(x + 2) = 0 \Longrightarrow x = \frac{1}{3} \text{ or } x = -2$$

As a result, $x = \frac{1}{3}$ and $x = -2$ are the critical numbers. Using sign testing/analysis for f',



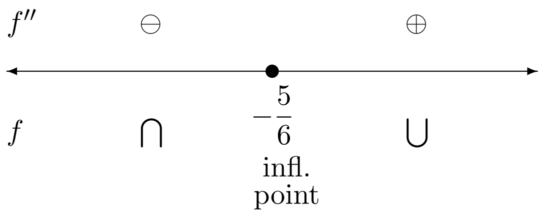
or our f' chart is

x	$(-\infty, -2)$	$(-2, 1/3)$	$(1/3, \infty)$
$f'(x)$	⊕	⊖	⊕
$f(x)$	↗	↘	↗

So f is increasing on $(-\infty, -2)$ and on $(1/3, \infty)$; and f is decreasing on $(-2, 1/3)$. Moreover, f has a local max at $x = -2$ with $f(-2) = 12$, and a local min at $x = 1/3$ with $f(1/3) = -19/27$.

• Second Derivative Information:

Meanwhile, f'' is always defined and continuous, and $f'' = 12x + 10 = 0$ only at our possible inflection point $x = -\dfrac{5}{6}$. Using sign testing/analysis for f'',

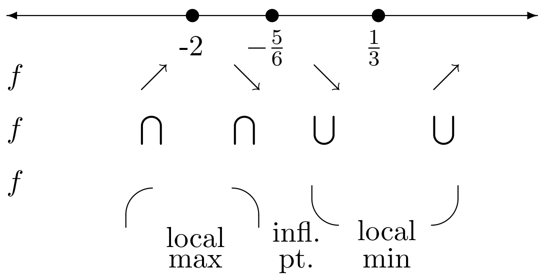


or our f'' chart is

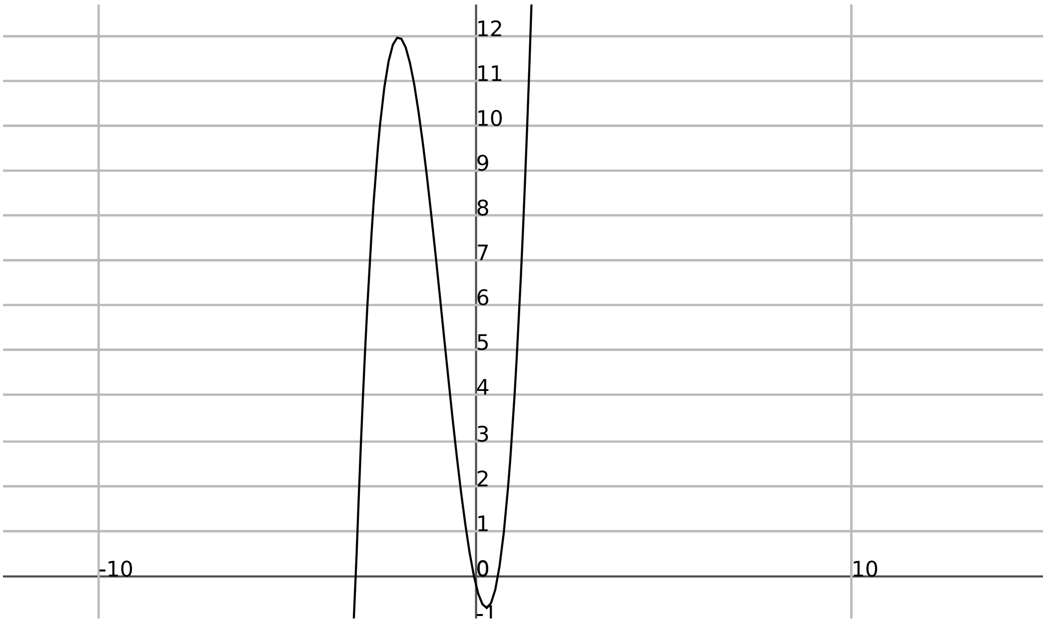
x	$(-\infty, -5/6)$	$(-5/6, \infty)$
$f''(x)$	$\ominus$	$\oplus$
$f(x)$	$\cap$	$\cup$

So f is concave down on $(-\infty, -5/6)$ and concave up on $(-5/6, \infty)$, with an inflection point at $x = -5/6$ with $f\left(-\frac{5}{6}\right) = \frac{305}{54}$.

• Piece the first and second derivative information together:



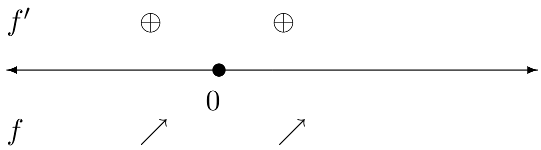
• Sketch:



22. $y = x^{\frac{1}{3}}$

- Domain: $f(x)$ has domain $(-\infty, \infty)$
- VA: No vertical asymptotes.
- HA: There are no horizontal asymptotes for this f since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.
- First Derivative Information:

We compute $f'(x) = \dfrac{1}{3x^{\frac{2}{3}}}$ to find critical numbers. The critical points occur where f' is undefined ($x = 0$ here) or zero (never here). As a result, $x = 0$ is the critical number. Using sign testing/analysis for f',



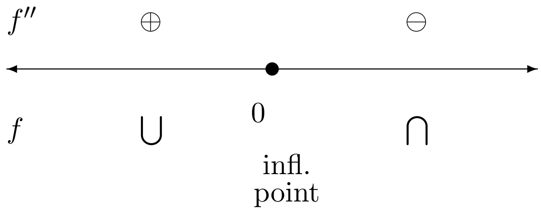
or our f' chart is

x	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	$\oplus$	$\oplus$
$f(x)$	$\nearrow$	$\nearrow$

So f is increasing on $(-\infty, \infty)$. Moreover, f has no extreme values.

- Second Derivative Information:

Meanwhile, $f'' = -\dfrac{2}{9}x^{-\frac{5}{3}}$ is undefined at the possible inflection point $x = 0$. Using sign testing/analysis for f'',

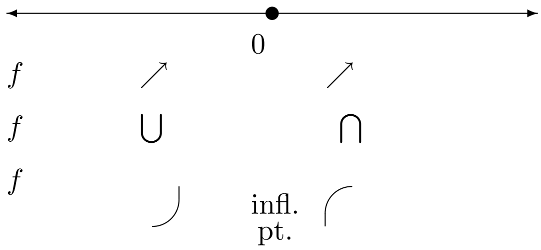


or our f'' chart is

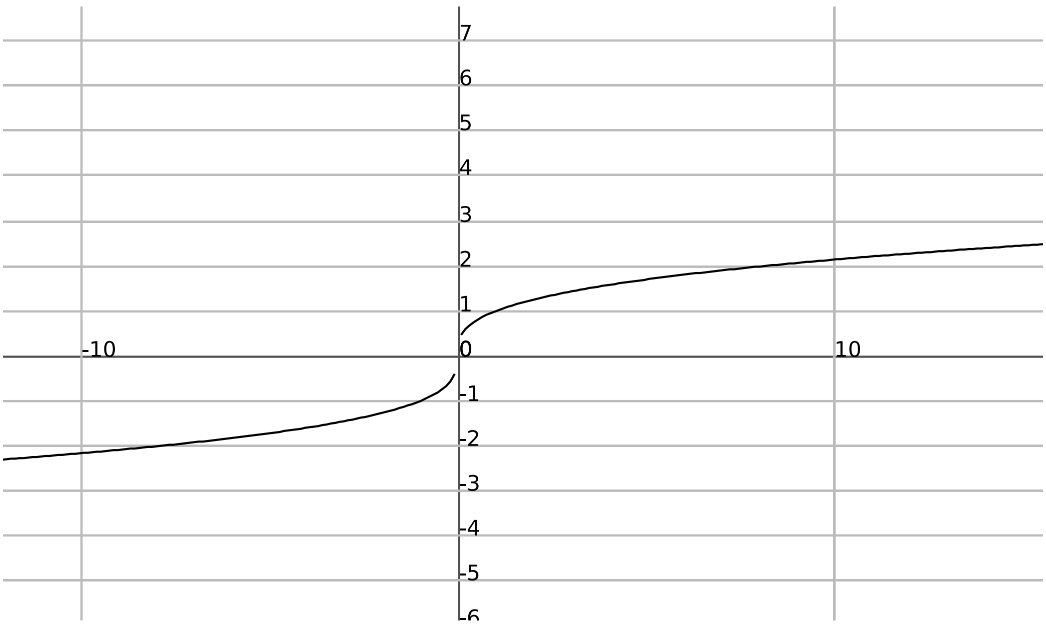
x	$(-\infty, 0)$	$(0, \infty)$
$f''(x)$	⊕	⊖
$f(x)$	∪	∩

So f is concave down on $(0, \infty)$ and concave up on $(-\infty, 0)$, with an inflection point $(0, 0)$ at $x = 0$.

- Piece the first and second derivative information together:



- Sketch:



23. $G(x) = -x^4$

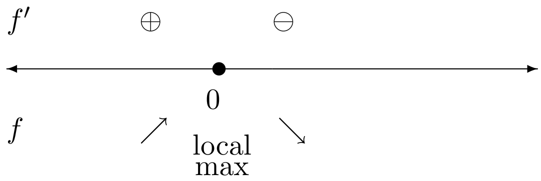
• Domain: $f(x)$ has domain $(-\infty, \infty)$

• VA: It is a polynomial, continuous everywhere, and so has no vertical asymptotes.

• HA: There are no horizontal asymptotes for this f since $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

• First Derivative Information:

We compute $f'(x) = -4x^3$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when $x = 0$. As a result, $x = 0$ is the critical number. Using sign testing/analysis for f',



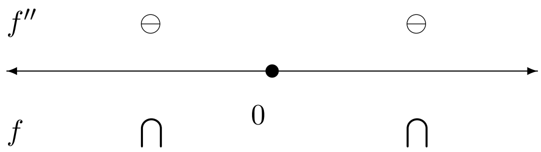
or our f' chart is

x	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	$\oplus$	$\ominus$
$f(x)$	$\nearrow$	$\searrow$

So f is increasing on $(-\infty, 0)$; and f is decreasing on $(0, \infty)$. Moreover, f has a local max at $x = 0$ with $f(0) = 0$.

• Second Derivative Information:

Meanwhile, f'' is always defined and continuous, and $f'' = -12x^2$. Using sign testing/analysis for f'' around $x = 0$,

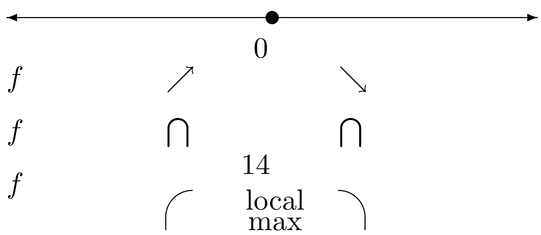


or our f'' chart is

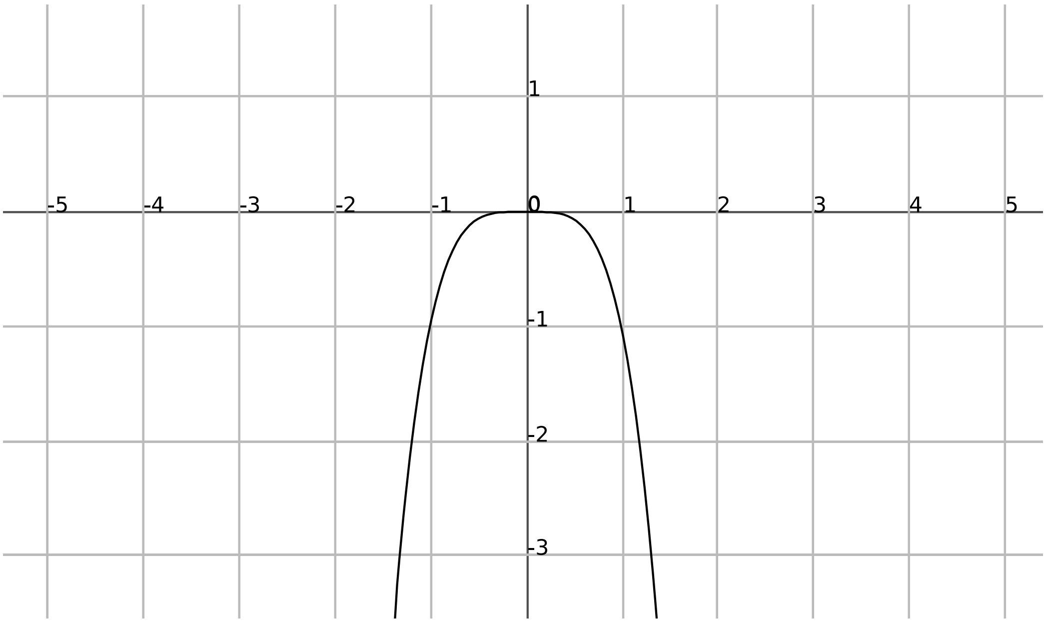
x	$(-\infty, 0)$	$(0, \infty)$
$f''(x)$	$\ominus$	$\ominus$
$f(x)$	$\cap$	$\cap$

So f is concave down on $(-\infty, \infty)$ with no inflection point .

• Piece the first and second derivative information together:



• Sketch:



24. $f(x) = 3x^4 + 4x^3$

• Domain: $f(x)$ has domain $(-\infty, \infty)$

• VA: It is a polynomial, continuous everywhere, and so has no vertical asymptotes.

• HA: There are no horizontal asymptotes for this f since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$ because

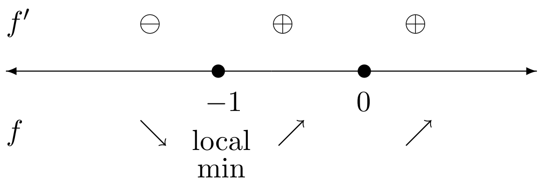
$\lim_{x\to\infty} 3x^4 + 4x^3 = \lim_{x\to\infty} x^3(3x+4) = \infty$ and $\lim_{x\to-\infty} 3x^4 + 4x^3 = \lim_{x\to-\infty} x^3(3x+4) = \infty$.

• First Derivative Information:

We compute $f'(x) = 12x^3 + 12x^2$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when

$$12x^3 + 12x^2 = 12x^2(x+1) = 0 \Longrightarrow x = 0 \text{ or } x = -1$$

As a result, $x = 0$ and $x = -1$ are the critical numbers. Using sign testing/analysis for f',



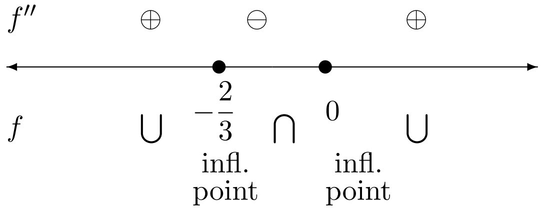
or our f' chart is

x	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
$f'(x)$	$\ominus$	$\oplus$	$\oplus$
$f(x)$	$\searrow$	$\nearrow$	$\nearrow$

So f is increasing on $(-1, \infty)$; and f is decreasing on $(-\infty, -1)$. Moreover, f has a local min at $x = -1$ with $f(-1) = -1$.

• Second Derivative Information:

Meanwhile, f'' is always defined and continuous, and $f'' = 36x^2 + 24x = 0$ only at our possible inflection points $x = 0$ and $x = -\dfrac{2}{3}$. Using sign testing/analysis for f'',

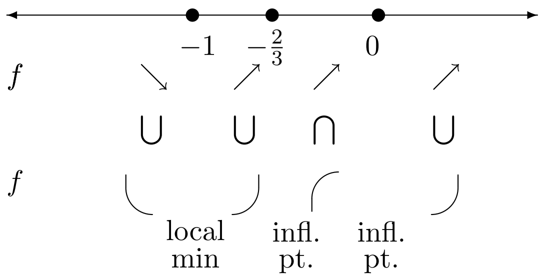


or our f'' chart is

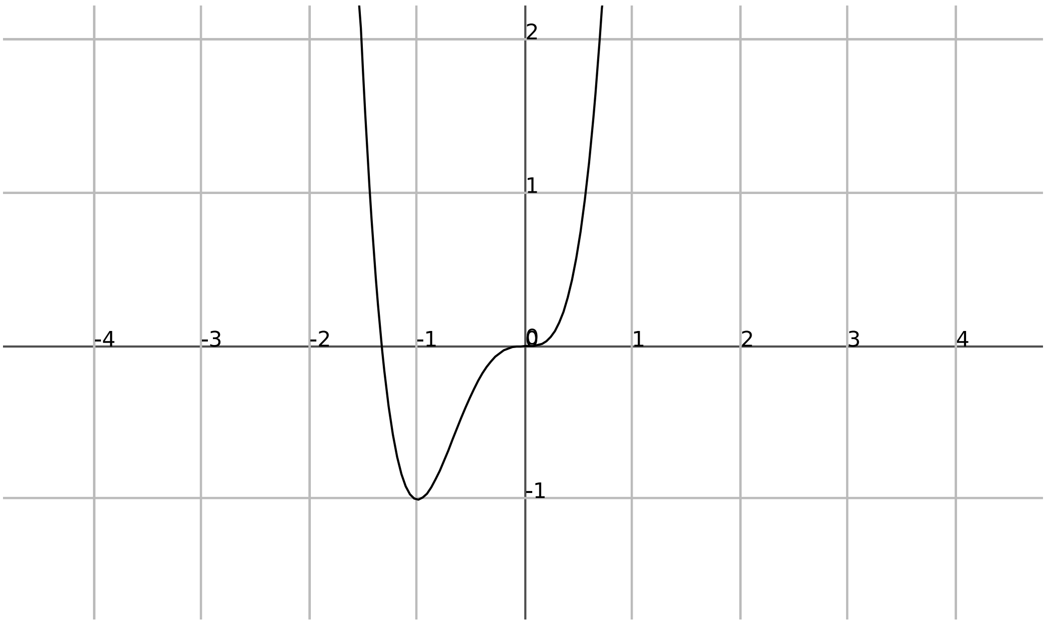
x	$(-\infty, -2/3)$	$(-2/3, 0)$	$(0, \infty)$
$f''(x)$	⊕	⊖	⊕
$f(x)$	$\cup$	$\cap$	$\cup$

So f is concave down on $(-2/3, 0)$ and concave up on $(-\infty, -2/3)$ and $(0, \infty)$, with inflection points at $x = 0$ and $x = -2/3$ with $f(0) = 0$ and $f\left(-\frac{2}{3}\right) = -\frac{16}{27}$.

• Piece the first and second derivative information together:



• Sketch:



25. $f(x) = x^4 - 6x^2$

• Domain: $f(x)$ has domain $(-\infty, \infty)$

• VA: It is a polynomial, continuous everywhere, and so has no vertical asymptotes.

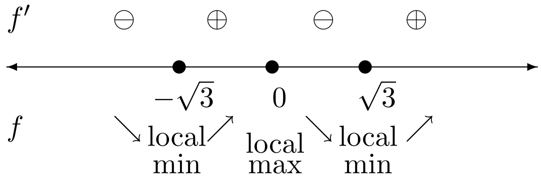
• HA: There are no horizontal asymptotes for this f since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$ because $\lim_{x\to\infty} x^4 - 6x^2 = x^2(x^2 - 6) = \infty$ and $\lim_{x\to-\infty} x^4 - 6x^2 = x^2(x^2 - 6) = \infty$

• First Derivative Information:

We compute $f'(x) = 4x^3 - 12x$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when

$$4x^3 - 12x = 4x(x^2 - 3) = 0 \Longrightarrow x = 0 \text{ or } x = \pm\sqrt{3}$$

As a result, $x = 0$ and $x = \pm\sqrt{3}$ are the critical numbers. Using sign testing/analysis for f',



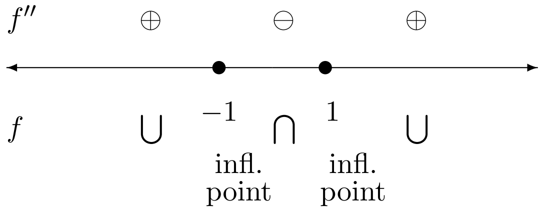
or our f' chart is

x	$(-\infty, -\sqrt{3})$	$(\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f'(x)$	$\ominus$	$\oplus$	$\ominus$	$\oplus$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

So f is increasing on $(-\sqrt{3}, 0)$and $(\sqrt{3}, \infty)$; and f is decreasing on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$. Moreover, f has a local max at $x = 0$ with $f(0) = 0$, and local mins at $x = \pm\sqrt{3}$ with $f(\pm\sqrt{3}) = -9$.

• Second Derivative Information:

Meanwhile, f'' is always defined and continuous, and $f'' = 12x^2 - 12 = 0$ only at our possible inflection points $x = \pm 1$. Using sign testing/analysis for f'',

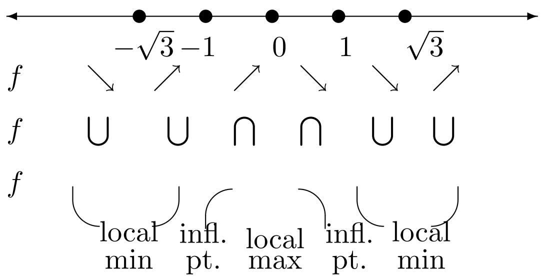


or our f'' chart is

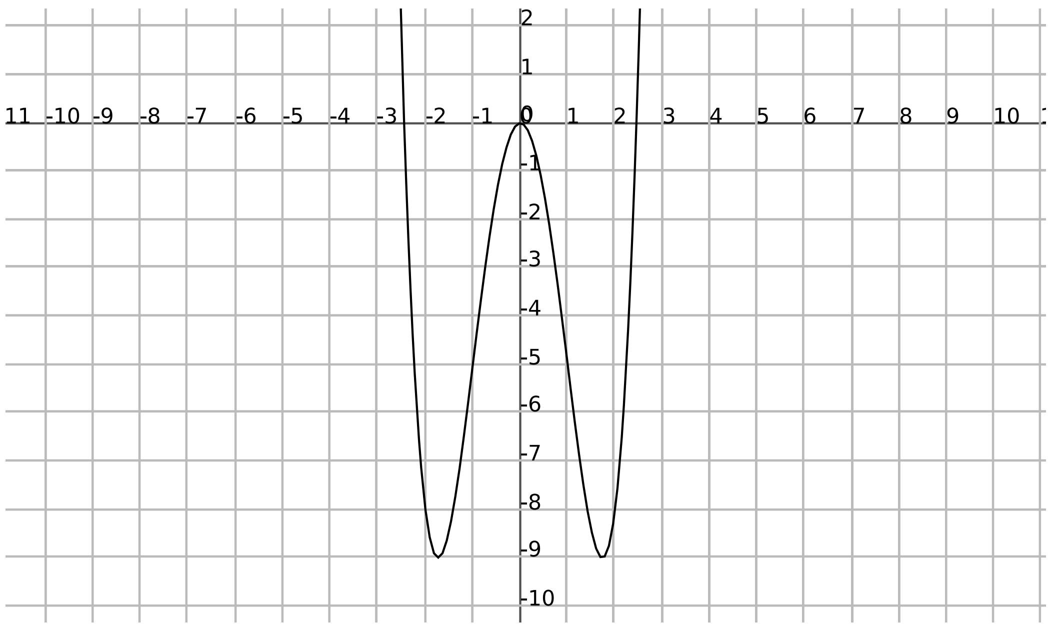
x	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f''(x)$	$\oplus$	$\ominus$	$\oplus$
$f(x)$	$\cup$	$\cap$	$\cup$

So f is concave down on $(-1, 1)$ and concave up on $(-\infty, -1)$ and $(1, \infty)$, with an inflection points at $x = \pm 1$ with $f(\pm 1) = -5$.

• Piece the first and second derivative information together:



• Sketch:



26. $f(x) = \dfrac{3x^5 - 20x^3}{32}$

- Domain: $f(x)$ has domain $(-\infty, \infty)$
- VA: It is a polynomial, continuous everywhere, and so has no vertical asymptotes.
- HA: There are no horizontal asymptotes for this f since $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$ because

$\lim\limits_{x\to\infty} \dfrac{3x^5 - 20x^3}{32} = \dfrac{1}{32} \lim\limits_{x\to\infty} 3x^5 - 20x^3 = \dfrac{1}{32} \lim\limits_{x\to\infty} x^3(3x^2 - 20) = \infty$

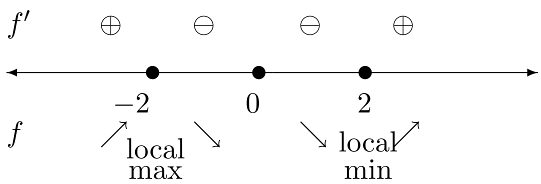
and $\lim\limits_{x\to-\infty} \dfrac{3x^5 - 20x^3}{32} = \dfrac{1}{32} \lim\limits_{x\to-\infty} 3x^5 - 20x^3 = \dfrac{1}{32} \lim\limits_{x\to-\infty} x^3(3x^2 - 20) = -\infty.$

- First Derivative Information:

We compute $f'(x) = \dfrac{1}{32}(15x^4 - 60x^2)$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when

$$15x^4 - 60x^2 = 15x^2(x^2 - 4) = 0 \Longrightarrow x = 0 \text{ or } x = \pm 2$$

As a result, $x = 0$ and $x = \pm 2$ are the critical numbers. Using sign testing/analysis for f',



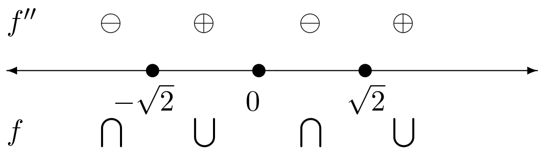
or our f' chart is

x	$(-\infty,-2)$	$(-2,0)$	$(0,2)$	$(0,\infty)$
$f'(x)$	$\oplus$	$\ominus$	$\ominus$	$\oplus$
$f(x)$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

So f is increasing on $(-\infty,-2)$ and on $(2,\infty)$; and f is decreasing on $(-2,2)$. Moreover, f has a local max at $x=-2$ with $f(-2)=2$, and a local min at $x=2$ with $f(2)=-2$.

• Second Derivative Information:

Meanwhile, f'' is always defined and continuous, and $f''=\dfrac{1}{32}(60x^3-120x)=\dfrac{60}{32}x(x^2-2)=0$ only at our possible inflection points $x=0$ and $x=\pm\sqrt{2}$. Using sign testing/analysis for f'',

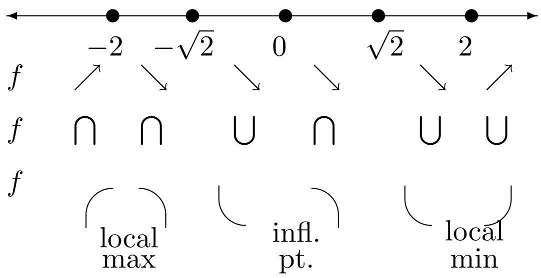


or our f'' chart is

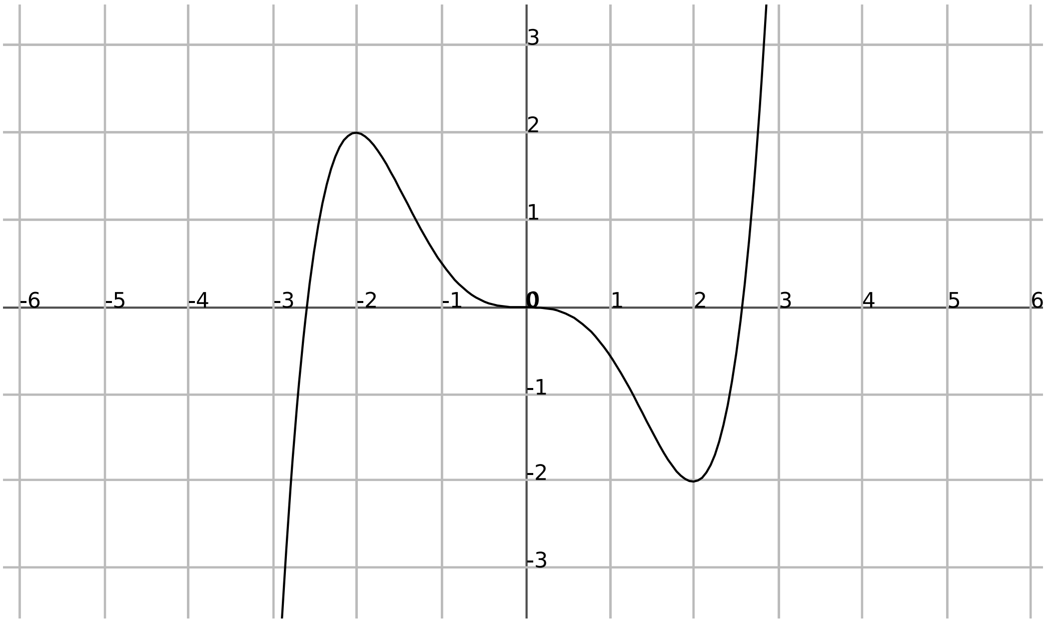
x	$(-\infty,-\sqrt{2})$	$(-\sqrt{2},0)$	$(0,\sqrt{2})$	$(\sqrt{2},\infty)$
$f''(x)$	$\ominus$	$\oplus$	$\ominus$	$\oplus$
$f(x)$	$\cap$	$\cup$	$\cap$	$\cup$

So f is concave down on $(-\infty,-\sqrt{2})$ and $(0,\sqrt{2})$, and concave up on $(-\sqrt{2},0)$ and$(\sqrt{2},\infty)$, with inflection points at $x=0$ with $f(0)=0$ and $x=\pm\sqrt{2}$ with $f\left(-\sqrt{2}\right)=\frac{7\sqrt{2}}{8}$ and $f\left(\sqrt{2}\right)=-\frac{7\sqrt{2}}{8}$.

• Piece the first and second derivative information together:



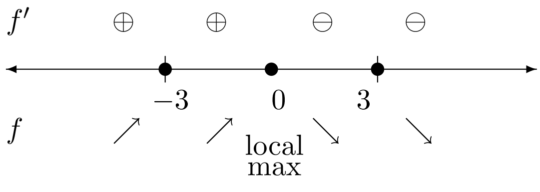
• Sketch:



27. $f(x) = \dfrac{1}{x^2 - 9}$

- Domain: $f(x)$ has domain $\{x | x \neq \pm 3\}$
- VA: Vertical asymptotes at $x = \pm 3$.
- HA: Horizontal asymptote at $y = 0$ for this f since $\lim_{x \to \pm\infty} f(x) = 0$.
- First Derivative Information:

We compute $f'(x) = \dfrac{-2x}{(x^2 - 9)^2}$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined or zero. The latter happens when $x = 0$. The derivative is undefined when $x = \pm 3$, but those values are not in the domain of the original function. As a result, $x = 0$ is the critical number. Using sign testing/analysis for f',



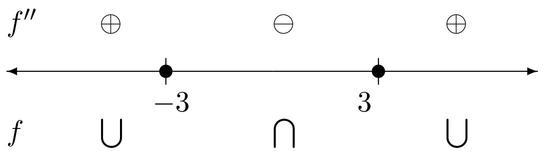
or our f' chart is

x	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	$\oplus$	$\ominus$
$f(x)$	$\nearrow$	$\searrow$

So f is increasing on $(-\infty, -3)$ and $(-3, 0)$; and f is decreasing on $(0, 3)$ and $(3, \infty)$. Moreover, f has a local max at $x = 0$ with $f(0) = -\frac{1}{9}$.

- Second Derivative Information:

Meanwhile, $f'' = \dfrac{6(x^2+3)}{(x^2-9)^3}$ is never zero. Using sign testing/analysis for f'' around the vertical asymptotes,

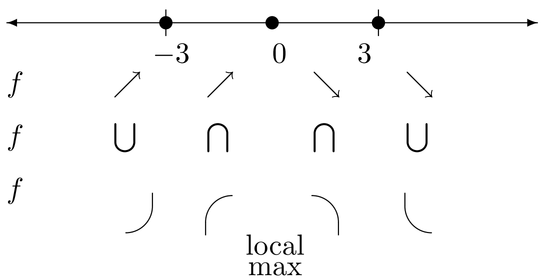


or our f'' chart is

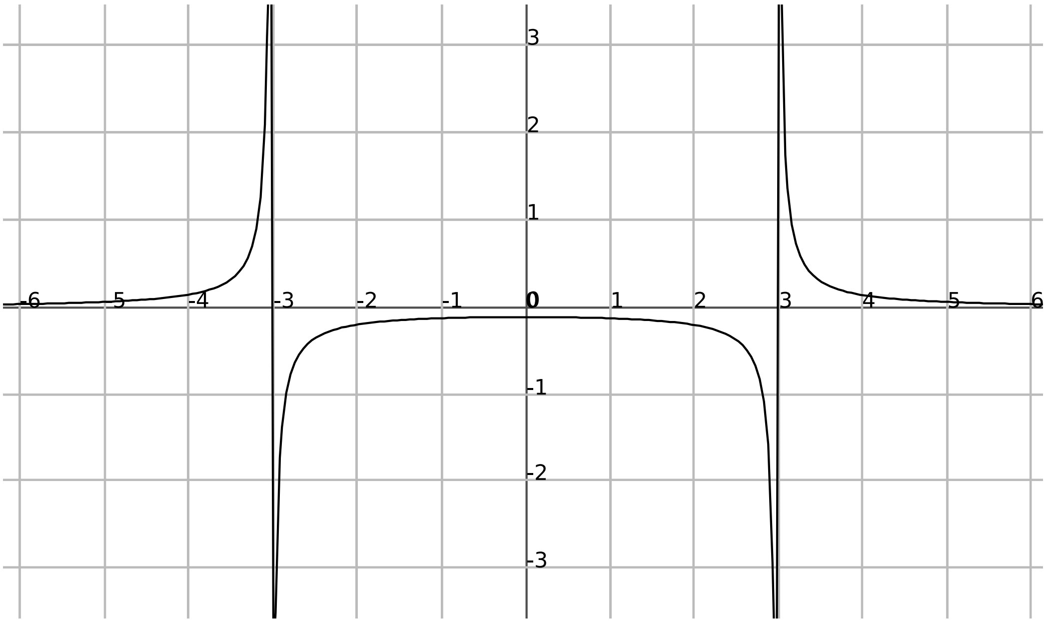
x	$(-\infty,-3)$	$(-3,3)$	$(3,\infty)$
$f''(x)$	$\oplus$	$\ominus$	$\oplus$
$f(x)$	$\cup$	$\cap$	$\cup$

So f is concave down on $(-3,3)$ and concave up on $(-\infty,-3)$ and $(3,\infty)$ with no inflection points.

• Piece the first and second derivative information together:



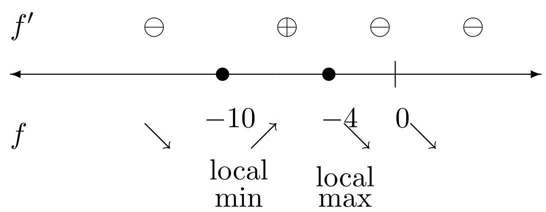
• Sketch:



28. $f(x) = \dfrac{2x^3 + 45x^2 + 315x + 600}{x^3}$. Take my word for it that (you do NOT have to compute these) $f'(x) = \dfrac{-45(x+4)(x+10)}{x^4}$ and $f''(x) = \dfrac{90(x+5)(x+16)}{x^5}$.

- Domain: $f(x)$ has domain $\{x | x \neq 0\}$
- VA: Vertical asymptotes at $x = 0$.
- HA: Horizontal asymptote at $y = 2$ for this f since $\lim_{x \to \pm\infty} f(x) = 2$.
- First Derivative Information:

We use the $f'(x)$ given above. The critical numbers are $x = -10$ and $x = -4$. The derivative is undefined at $x = 0$, but that's not a critical number since it was not in the domain of the original function. Using sign testing/analysis for f',



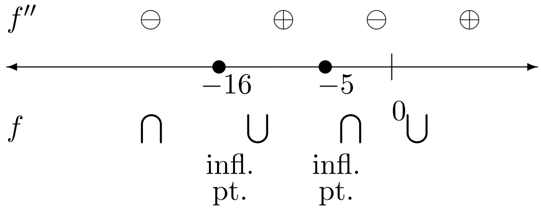
or our f' chart is

x	$(-\infty, -10)$	$(-10, -4)$	$(-4, 0), (0, \infty)$
$f'(x)$	⊖	⊕	⊖
$f(x)$	↘	↗	↘

So f is increasing on $(-10, -4)$; and f is decreasing on $(-\infty, -10)$, $(-4, 0)$, and $(0, \infty)$. Moreover, f has a local max at $x = -4$ with $f(-4) = \frac{17}{16}$ and a local min at $x = -10$ with $f(-10) = \frac{1}{20}$.

- Second Derivative Information:

Meanwhile, f'' is zero when $x = -16$ and $x = -5$. Using sign testing/analysis for f'' around the vertical asymptote and these possible inflection points $x = -16$ and $x = -5$,

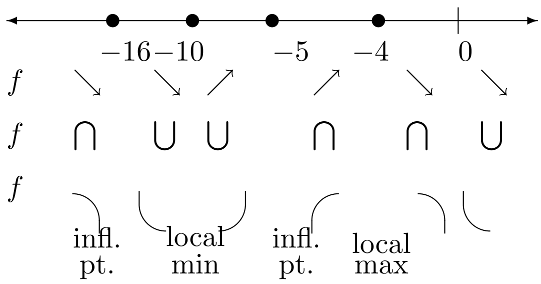


or our f'' chart is

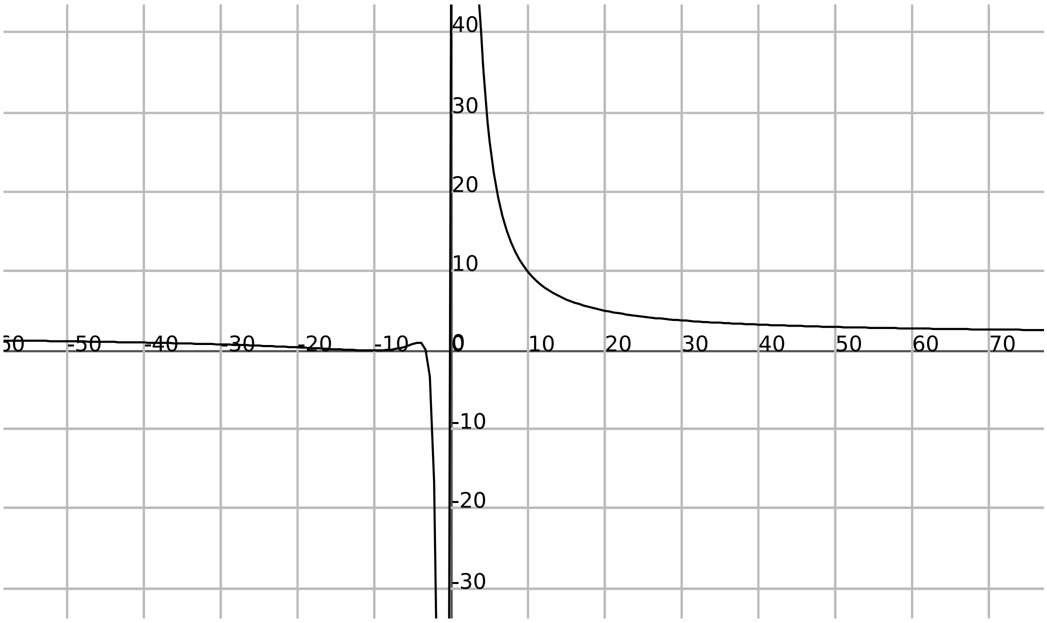
x	$(-\infty, -16)$	$(-10, -5)$	$(-5, 0)$	$(0, \infty)$
$f''(x)$	$\ominus$	$\oplus$	$\ominus$	$\oplus$
$f(x)$	$\cap$	$\cup$	$\cap$	$\cup$

So f is concave down on $(-\infty, -16)$ and $(-5, 0)$, and concave up on $(-16, -5)$ and $(0, \infty)$ with inflection points at $x = -16$ with $f(-16) = \frac{139}{512}$, and at $x = -5$ with $f(-5) = \frac{4}{5}$.

• Piece the first and second derivative information together:



• Sketch:


40
30
20
10
-60
-50
-40
-30
-20
-10
0
10
20
30
40
50
60
70
-10
-20
-30

More derivatives

29. Let f and g be two differentiable functions, and suppose that their values and the values of their derivatives at $x = 1, 2, 3$ are given by the following table:

x	1	2	3
$f(x)$	3	2	5
$f'(x)$	-2	1	3
$g(x)$	3	1	4
$g'(x)$	-3	2	7

Let $h(x) = f \circ g(x)$ and $k(x) = f(x) \cdot g(f(x))$. Compute $h'(2)$ and $k'(1)$.

Compute $h'(x) = f'(g(x))g'(x)$. Then $h'(2) = f'(g(2))g'(2) = f'(1) \cdot 2 = (-2) \cdot 2 = \boxed{-4}$.

Compute $k'(x) = f(x) \cdot g'(f(x)) \cdot f'(x) + g(f(x)) \cdot f'(x)$. Then $k'(1) = f(1) \cdot g'(f(1)) \cdot f'(1) + g(f(1)) \cdot f'(1) = 3 \cdot g'(3) \cdot (-2) + g(3) \cdot (-2) = 3 \cdot 7 \cdot (-2) + 4 \cdot (-2) = \boxed{-50}$.

30. Let f and g be two differentiable functions, and suppose that their values and the values of their derivatives at $x = 2, 3$ are given by the following table:

x	2	3
$f(x)$	4	0
$f'(x)$	1	-7
$g(x)$	3	-1
$g'(x)$	-5	4

Let $h(x) = f(x)g(x), k(x) = \dfrac{f(x)}{g(x)}$ and $W(x) = f \circ g(x)$. Compute $h'(2)$ and $k'(2)$ and $W'(2)$.

Compute $h'(x) = f(x)g'(x) + f'(x)g(x)$. Then $h'(2) = f(2)g'(2) + f'(2)g(2) = 4 \cdot (-5) + 1 \cdot 3 = \boxed{-17}$.

Compute $k'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

Then $k'(2) = \dfrac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{3(1) - 4(-5)}{9} = \boxed{\dfrac{23}{9}}$.

Compute $W'(x) = f'(g(x)) \cdot g'(x)$. Then $W'(2) = f'(g(2)) \cdot g'(2) = f'(3) \cdot (-5) = (-7)(-5) = \boxed{35}$.

31. Let $f(x) = \sqrt{x+1} \cdot g(x)$ where $g(0) = -7$ and $g'(0) = 4$. Compute $f'(0)$.

Compute $f'(x) = \sqrt{x+1} \cdot g'(x) + g(x)\dfrac{1}{2\sqrt{x+1}}$. Then $f'(0) = \sqrt{0+1} \cdot g'(0) + g(0)\dfrac{1}{2\sqrt{0+1}} = 1 \cdot 4 + (-7) \cdot \dfrac{1}{2} = \boxed{\dfrac{1}{2}}$.

32. Let $f(x) = \dfrac{\sqrt{x^2+1}}{g(x)}$ where $g(0) = -7$ and $g'(0) = 4$. Compute $f'(0)$.

Compute $f'(x) = \dfrac{g(x)\dfrac{1}{2\sqrt{x^2+1}}(2x) - \sqrt{x^2+1}g'(x)}{(g(x))^2}$. Then $f'(0) = \dfrac{g(0)\dfrac{1}{2\sqrt{1}}(0) - \sqrt{1}g'(0)}{(g(0))^2} = \boxed{-\dfrac{4}{49}}$

More Tangent Lines

33. For each of the plane curves described below, find an equation of the tangent line to the curve at the given point. (YOU CAN USE EITHER y' or $\frac{dy}{dx}$ for the derivative here!)

    (a) $x^3 + x^2y + 4y^2 = 6$ at $(1,1)$. Differentiating $x^3 + x^2y + 4y^2 = 6$ implictly gives $3x^2 + 2xy + x^2\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$, so $(x^2+8y)\frac{dy}{dx} = -3x^2 - 2xy$, so $\frac{dy}{dx} = \frac{-3x^2 - 2xy}{x^2 + 8y}$. At $(1,1)$, the slope of the tangent line is therefore $\frac{dy}{dx} = (-3-2)/(1+8) = -5/9$. The point-slope formula gives $y - 1 = -\frac{5}{9}(x-1)$; that is, $\boxed{y = -\frac{5}{9}x + \frac{14}{9}}$.

    (b) $4(x+y)^2 = x^2y^2$ at $(-2,1)$. Differentiating $4(x+y)^2 = x^2y^2$ implicitly gives $8(x+y)(1+\frac{dy}{dx}) = 2xy^2 + 2x^2y\frac{dy}{dx}$, so $4x+4y+\frac{dy}{dx}(4x+4y) = xy^2 + x^2y\frac{dy}{dx}$, so $\frac{dy}{dx}(4x+4y-x^2y) = xy^2 - 4x - 4y$, so $\frac{dy}{dx} = \frac{xy^2 - 4x - 4y}{4x + 4y - x^2y}$. At $(-2,1)$, the slope of the tangent line is therefore $\frac{dy}{dx} = (-2+8-4)/(-8+4-4) = -1/4$. By the point-slope formula, $y - 1 = -\frac{1}{4}(x+2)$; that is, $\boxed{y = -\frac{1}{4}x + \frac{1}{2}}$.

    (c) $\frac{x}{y+1} = x^2 - y^2$ at $(1,0)$. Differentiating $\frac{x}{y+1} = x^2 - y^2$ implicitly gives $\frac{(y+1) - x\frac{dy}{dx}}{(y+1)^2} = 2x - 2y\frac{dy}{dx}$, so $y + 1 - x\frac{dy}{dx} = 2x(y+1)^2 - 2y(y+1)^2\frac{dy}{dx}$, so $(2y(y+1)^2 - x)\frac{dy}{dx} = 2x(y+1)^2 - y - 1$, so $\frac{dy}{dx} = \frac{2x(y+1)^2 - y - 1}{2y(y+1)^2 - x}$.

    At $(1,0)$, the slope of the tangent line is therefore $\frac{dy}{dx} = (2-1)/(-1) = -1$. The point-slope formula gives $y - 0 = -1(x-1)$; that is, $\boxed{y = -x + 1}$.

    (d) $4\cos x \sin y = 3$ at $(\pi/6, \pi/3)$. Differentiating gives: $-4\sin x \sin y + 4(\cos x \cos y)\frac{dy}{dx} = 0$, and so $\frac{dy}{dx} = \frac{\sin x \sin y}{\cos x \cos y}$. At $(\pi/6, \pi/3)$, this means $\frac{dy}{dx} = \frac{(1/2)(\sqrt{3}/2)}{(\sqrt{3}/2)(1/2)} = 1$. So the tangent line is $y - \pi/3 = x - \pi/6$, that is, $\boxed{y = x + \pi/6}$.

    (e) $y^3 - xy^2 + \cos(xy) = 2$ at $(0,1)$ Differentiating gives:
    $3y^2\frac{dy}{dx} - \left(2xy\frac{dy}{dx} + y^2\right) - \sin(xy)\left(x\frac{dy}{dx} + y\right) = 0$, and at $(0,1)$, this means $3\frac{dy}{dx} - \left(0\frac{dy}{dx} + 1\right) - \sin(0)\left(0\frac{dy}{dx} + 1\right) = 0$ which implies $\frac{dy}{dx} = \frac{1}{3}$. So the tangent line is $y - 1 = \frac{1}{3}(x - 0)$, that is, $\boxed{y = \frac{1}{3}x + 1}$.

Limits Evaluate the following limits. Please show your work.

34. $\lim_{x\to 0} \frac{\sin(5x)}{2x} = \frac{5}{2} \lim_{x\to 0} \frac{\sin(5x)}{5x} = \frac{5}{2} \cdot 1 = \boxed{\frac{5}{2}}$

35. $\lim_{w\to 0} \frac{\sin(16w)}{w} = 16 \lim_{w\to 0} \frac{\sin(16w)}{16w} = 16 \cdot 1 = \boxed{16}$

36. $\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)} = \frac{3}{2} \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{2x}{\sin(2x)} = \frac{3}{2} \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \lim_{x\to 0} \frac{2x}{\sin(2x)} = \frac{3}{2} \cdot 1 \cdot 1 = \boxed{\frac{3}{2}}$

37. $\lim_{\theta\to 0} \frac{\sin^2(\theta)}{\theta^2 + 5\theta^3} = \lim_{\theta\to 0} \frac{\sin(\theta)\sin(\theta)}{\theta^2(1+5\theta)} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta\to 0} \frac{1}{1+5\theta} = 1 \cdot 1 \cdot 1 = \boxed{1}$

38. $\lim_{x\to 0} \frac{2x}{\sin(3x)} = \frac{2}{3} \lim_{x\to 0} \frac{3x}{\sin(3x)} = \frac{2}{3} \cdot 1 = \boxed{\frac{2}{3}}$

39. $\lim_{h\to 0} \frac{\tan(6h)}{7h} = \frac{6}{7} \lim_{h\to 0} \frac{\sin(6h)}{6h} \cdot \lim_{h\to 0} \frac{1}{\cos(6h)} = \frac{6}{7} \cdot 1 \cdot 1 = \boxed{\frac{6}{7}}$

40. $\lim_{x\to 0} \frac{x + x\cos x}{\sin x \cos x} = \lim_{x\to 0} \frac{x(1+\cos x)}{\sin x \cos x} = \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \frac{1+\cos x}{\cos x} = 1 \cdot 2 = \boxed{2}$

41. $\lim_{x\to 0} \frac{\tan(5x)}{\sin(5x)} = \lim_{x\to 0} \frac{\sin(5x)}{\cos(5x)\sin(5x)} = \lim_{x\to 0} \frac{1}{\cos(5x)} = \boxed{1}$

42. $\lim_{x\to -\infty} \frac{x^3 - 2x}{4x^3 + 1} = \lim_{x\to -\infty} \frac{x^3 - 2x}{4x^3 + 1} \cdot \frac{(\frac{1}{x^3})}{(\frac{1}{x^3})} = \lim_{x\to -\infty} \frac{1 - \frac{2}{x^2}}{4 + \frac{1}{x^3}} = \boxed{\frac{1}{4}}$

43. $\lim_{x\to \infty} \frac{x^3 + 1}{x^7 + 2x^{7/2}} = \lim_{x\to \infty} \frac{x^3 + 1}{x^7 + 2x^{7/2}} \cdot \frac{(\frac{1}{x^7})}{(\frac{1}{x^7})} = \lim_{x\to \infty} \frac{\frac{1}{x^4} + \frac{1}{x^7}}{1 + \frac{2}{x^{7/2}}} = \boxed{0}$

44. $\lim_{x\to \infty} \frac{x^6 + 1}{x^3 + 9x^2 + 7} = \lim_{x\to \infty} \frac{x^6 + 1}{x^3 + 9x^2 + 7} \cdot \frac{(\frac{1}{x^3})}{(\frac{1}{x^3})} = \lim_{x\to \infty} \frac{x^3 + \frac{1}{x^3}}{1 + \frac{9}{x} + \frac{7}{x^3}} = \boxed{\infty}$

45. $\lim_{x\to 0} \frac{\sin^2 x}{4x^2 + 5x^3} = \lim_{x\to 0} \frac{\sin^2 x}{x^2(4+5x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{4+5x} = 1 \cdot 1 \cdot \frac{1}{4} = \boxed{\frac{1}{4}}$

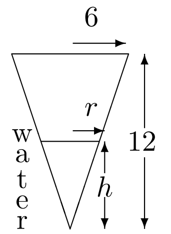
46. $\lim_{x\to 0} \frac{x^2 \cos x}{x+1} = \frac{0 \cdot \cos 0}{1} = \boxed{0}$

Related Rates

47. A conical reservoir, 12 ft. deep and also 12 ft. across the top is being filled with water at the rate of 5 cubic feet per minute. How fast is the water rising when it is 4 feet deep?

The cross section (with water level drawn in) looks like:

• Diagram



• Variables
Let r = radius of the water level at time t
Let h = height of the water level at time t
Let V = volume of the water in the tank at time t
Find $\dfrac{dh}{dt}$ =? when $h = 4$ feet
and $\dfrac{dV}{dt} = 5\dfrac{\text{ft}^3}{\text{sec}}$

• Equation relating the variables:

Volume= $V = \dfrac{1}{3}\pi r^2 h$

• Extra solvable information: Note that r is not mentioned in the problem's info. But there is a relationship, via similar triangles, between r and h. We must have

$$\frac{r}{6} = \frac{h}{12} \implies r = \frac{h}{2}$$

After substituting into our previous equation, we get:

$V = \dfrac{1}{3}\pi \left(\dfrac{h}{2}\right)^2 h = \dfrac{1}{12}\pi h^3$

• Differentiate both sides w.r.t. time t.

$\dfrac{d}{dt}(V) = \dfrac{d}{dt}\left(\dfrac{1}{12}\pi h^3\right) \implies \dfrac{dV}{dt} = \dfrac{1}{12}\pi \cdot 3h^2 \cdot \dfrac{dh}{dt} \implies \dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2 \dfrac{dh}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$5 = \dfrac{1}{4}\pi (4)^2 \dfrac{dh}{dt}$

• Solve for the desired quantity:

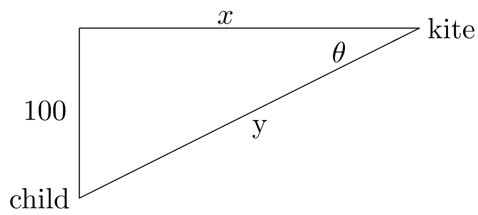
$\dfrac{dh}{dt} = \dfrac{5 \cdot 4}{16\pi} = \dfrac{5}{4\pi}\dfrac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The water is rising at a rate of $\dfrac{5}{4\pi}$ feet every second.

48. A kite 100 feet high is being blown horizontally at 8 feet per second. When there are 300 feet of string out, (a) how fast is the string running out? (b) how fast is the angle between the string and the horizontal changing?

(a) • Diagram

The picture at arbitrary time t is:



- Variables

Let x = distance kite has travelled horizontally at time t
Let y = distance between kite and child at time t
Find $\frac{dy}{dt}$ =? when $y = 300$ feet
and $\frac{dx}{dt} = 8\frac{\text{ft}}{\text{sec}}$

- Equation relating the variables:

We have $x^2 + 100^2 = y^2$ by the Pythagorean Theorem.

- Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(x^2 + 100^2) = \frac{d}{dt}(y^2) \implies 2x\frac{dx}{dt} = 2y\frac{dy}{dt} \implies x\frac{dx}{dt} = y\frac{dy}{dt}$ (Related Rates!)

- Substitute Key Moment Information (now and not before now!!!):

At the key instant when $y = 300$, using the original equation, we have $x = \sqrt{(300)^2 - (100)^2} = \sqrt{80000} = 200\sqrt{2}$. So, $200\sqrt{2} \cdot 8 = 300\frac{dy}{dt}$

- Solve for the desired quantity:

$\frac{dy}{dt} = \frac{1600\sqrt{2}}{300} = \frac{80\sqrt{2}}{15}\frac{\text{ft}}{\text{sec}}$

- Answer the question that was asked: The string is running out at a rate of $\frac{16\sqrt{2}}{3}$ feet every second.

(b)
- Variables

Let x = distance kite has travelled horizontally at time t
Let y = distance between kite and child at time t
Let θ = the angle between the string/horizontal
Find $\frac{d\theta}{dt}$ =? when $y = 300$ feet
and $\frac{dx}{dt} = 8\frac{\text{ft}}{\text{sec}}$

- Equation relating the variables:

The trigonometry of the triangle yields $\tan\theta = \frac{100}{x}$.

- Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{100}{x}\right) \implies \sec^2\theta\frac{d\theta}{dt} = -\frac{100}{x^2}\frac{dx}{dt}$. (Related Rates!)

- Substitute Key Moment Information (now and not before now!!!):

At the key instant when $y = 300$, using the original equation, we have $x = \sqrt{(300)^2 - (100)^2} = \sqrt{80000} = 200\sqrt{2}$.

Therefore, $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{300}{200\sqrt{2}}$

$\left(\frac{300}{200\sqrt{2}}\right)^2 \frac{d\theta}{dt} = -\frac{100}{(200\sqrt{2})^2} \cdot 8.$

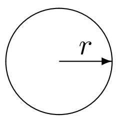
• Solve for the desired quantity:

$\frac{d\theta}{dt} = \frac{-100 \cdot 8(200\sqrt{2})^2}{(200\sqrt{2})^2(300)^2} = -\frac{8}{900}\frac{\text{rad}}{\text{sec}}$

• Answer the question that was asked: The angle is decreasing at a rate of $= \frac{8}{900}$ radians every second.

49. Suppose a snowball remains spherical while it melts with the radius shrinking at one inch per hour. How fast is the volume of the snowball decreasing when the radius is 2 inches?

• Diagram(cross-sectioned in 2 dimensions here)



• Variables
Let r = radius of the sphere at time t
Let V = volume of the sphere at time t
Find $\frac{dV}{dt}$ =? when $r = 2$ feet
and $\frac{dr}{dt} = -1\frac{\text{in}}{\text{hr}}$

• Equation relating the variables:

Volume $V = \frac{4}{3}\pi r^3$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \implies \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \implies \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$\frac{dV}{dt} = 4\pi(2)^2(-1)$

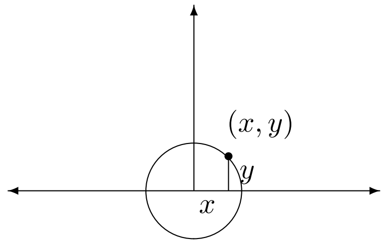
• Solve for the desired quantity:

$\frac{dV}{dt} = -16\pi\frac{\text{in}}{\text{hr}}$

• Answer the question that was asked: The volume of the snowball is decreasing at a rate of 16π inches every hour.

50. A point moves around the circle $x^2+y^2 = 9$. When the point is at $(-\sqrt{3}, \sqrt{6})$, its x-coordinate is increasing at the rate of 20 units per second. How fast is its y-coordinate changing at this instant?

• Diagram



• Variables
Let $x =$ the x-coord. of the point at time t
Let $y =$ the y-coord. of the point at time t
Find $\frac{dy}{dt} =?$ when $x = -\sqrt{3}$, $y = \sqrt{6}$
and $\frac{dx}{dt} = 20\frac{\text{units}}{\text{sec}}$

• Equation relating the variables:

Given as $x^2 + y^2 = 9$.

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(9) \implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies x\frac{dx}{dt} + y\frac{dy}{dt} = 0$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$(-\sqrt{3})20 + \sqrt{6}\frac{dy}{dt} = 0$

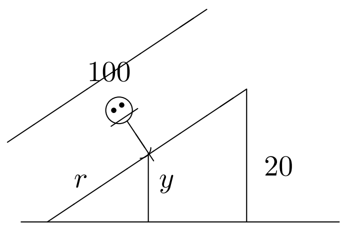
• Solve for the desired quantity:

$\frac{dy}{dt} = \frac{20\sqrt{3}}{\sqrt{6}}$

• Answer the question that was asked: At this moment, the y-coordinate is increasing at a rate of $\frac{20\sqrt{3}}{\sqrt{6}}$ units every second.

51. A waterskier skis up over the ramp at a speed of 30 ft./sec. The 100 ft. ramp slopes straight from no height at one end to 20 feet on the other end. How fast is she rising vertically just as she leaves the ramp?

• Diagram



• Variables
Let $r =$ distance skiier up the ramp at time t
Let $y =$ height of the skiier (up the ramp) above water level at time t
Find $\frac{dy}{dt} =?$ when $y = 20$
and $\frac{dr}{dt} = 30\frac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

Using similar triangles: $\frac{y}{20} = \frac{r}{100}$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}\left(\frac{y}{20}\right) = \frac{d}{dt}\left(\frac{r}{100}\right) \implies \frac{1}{20}\frac{dy}{dt} = \frac{1}{100}\frac{dr}{dt} \implies \frac{dy}{dt} = \frac{1}{5}\frac{dr}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$$\frac{dy}{dt} = \frac{1}{5}(30)$$

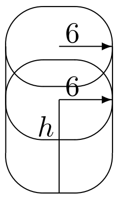
• Solve for the desired quantity:

$$\frac{dy}{dt} = 6\frac{\text{ft}}{\text{sec}}$$

• Answer the question that was asked: The skiier is rising vertically at a rate of 6 feet every second. (Note that this was independent of r.)

52. A cylindrical reservoir, 12 feet across the top, is being filled with water at the rate of 5 cubic feet per minute. How fast is the water rising when it is 4 feet deep?

• Diagram



• Variables

Let h = height of the water level at time t

Let V = volume of the water in the cylinder at time t

Find $\frac{dh}{dt} =?$ when $h = 4$ feet

and $\frac{dV}{dt} = 5\frac{\text{ft}^3}{\text{min}}$

• Equation relating the variables:

Volume of Cylinder $V = \pi r^2 h = \pi(6)^2 h = 36\pi h$

• Differentiate both sides w.r.t. time t.

$$\frac{d}{dt}(V) = \frac{d}{dt}(36\pi h) \implies \frac{dV}{dt} = 36\pi\frac{dh}{dt} \text{(Related Rates!)}$$

• Substitute Key Moment Information (now and not before now!!!):

$$5 = 36\pi\frac{dh}{dt}$$

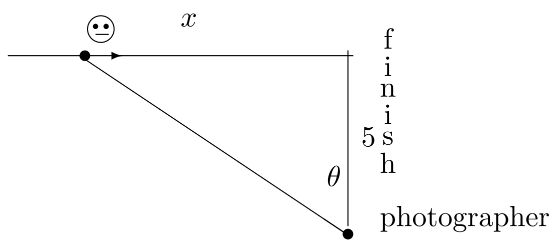
• Solve for the desired quantity:

$$\frac{dh}{dt} = \frac{5}{36\pi}\frac{\text{ft}}{\text{min}}$$

• Answer the question that was asked: The water is rising at a rate of $\frac{5}{36\pi}$ feet every minute.

53. A photographer is televising a 100-yard dash from a position 5 yards from the track in line with the finish line. When the runners are 12 yards from the finish line, the camera is turning at the rate of $\frac{3}{5}$ radians per second. How fast are the runners moving then?

• Diagram



- Variables

Let x = distance between runners and finish line at time t
Let θ = angle camera is turned from finish line at time t
Find $\frac{dx}{dt} =?$ when $x = 12$
and $\frac{d\theta}{dt} = -\frac{3}{5}\frac{\text{rad}}{\text{sec}}$

- Equation relating the variables:

Trigonometry on the triangle yields
$\tan\theta = \frac{x}{5}$.

- Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{x}{5}\right) \implies \sec^2\theta\frac{d\theta}{dt} = \frac{1}{5}\cdot\frac{dx}{dt}$ (Related Rates!)

- Substitute Key Moment Information (now and not before now!!!):

The Pythagorean Theorem yields hyp $= \sqrt{5^2 + (12)^2} = \sqrt{169} = 13$. Or notice that it's a 5-12-13 triangle at the key moment.

Note the trig. here yields $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{13}{5}$ so

$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{5}\cdot\frac{dx}{dt}$ becomes

$\left(\frac{13}{5}\right)^2\left(-\frac{3}{5}\right) = \frac{1}{5}\cdot\frac{dx}{dt}$

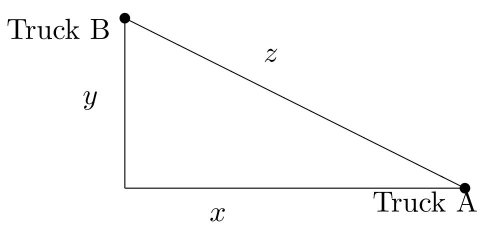
- Solve for the desired quantity:

$\frac{dx}{dt} = (5)\frac{169}{25}\cdot\frac{(-3)}{5} = -\frac{507}{25}$

- Answer the question that was asked: The runners are moving closer to the finish line at a rate of $\frac{507}{25}$ yards every second.

54. Two trucks leave a depot at the same time. Truck A travels east at 40 miles per hour, while Truck B travels north at 30 miles per hour. How fast is the distance between the trucks changing 60 minutes after leaving the depot?

- Diagram



• Variables
Let x = distance Truck A travelled East at time t
Let y = distance Truck B travelled North at time t
Let z = distance between Trucks A and B at time t
Find $\frac{dz}{dt}$ =? after 1 hour, when $x = 40$ miles, $y = 30$, $\frac{dx}{dt} = 40$ m.p.h.
and $\frac{dy}{dt} = 30$ m.p.h.

• Equation relating the variables:

Pythagorean Theorem gives $x^2 + y^2 = z^2$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(z^2) \implies 2x\frac{dx}{dt}+2y\frac{dy}{dt} = 2z\frac{dz}{dt} \implies x\frac{dx}{dt}+y\frac{dy}{dt} = z\frac{dz}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

First, note that by the Pythagorean Theorem $z = \sqrt{(40)^2 + (30)^2} = 50$, so

$40(40) + 30(30) = 50\frac{dz}{dt}$

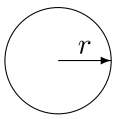
• Solve for the desired quantity:

$\frac{dz}{dt} = \frac{1600 + 900}{50} = 50$ m.p.h.

• Answer the question that was asked: The distance between the trucks is increasing at a rate of 50 miles every hour.

55. Suppose a spherical balloon is inflated at the rate of 10 cubic inches per minute. How fast is the radius of the balloon increasing when the radius is 5 inches?

• Diagram (cross-sectioned in 2 dimensions here)



• Variables
Let r = radius of the sphere at time t
Let V = volume of the sphere at time t
Find $\frac{dr}{dt}$ =? when $r = 5$ feet
and $\frac{dV}{dt} = 10\frac{\text{in}^3}{\text{min}}$

• Equation relating the variables:

Volume $V = \frac{4}{3}\pi r^3$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \implies \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \implies \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$10 = 4\pi(5)^2\frac{dr}{dt}$

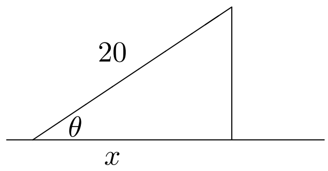
• Solve for the desired quantity:

$\frac{dr}{dt} = \frac{10}{100\pi} = \frac{1}{10\pi}\frac{\text{in}}{\text{min}}$

• Answer the question that was asked: The radius of the balloon is increasing at a rate of $\frac{1}{10\pi}$ inches every minute.

56. Suppose a 20 foot ladder is sliding down a vertical wall. Let θ be the angle formed by the ground and the base of the ladder. At what rate is the angle θ changing when the base of the ladder is sliding away from the wall at 2 feet per second and $\theta = \frac{\pi}{3}$?

• Diagram



• Variables

Let x = distance between bottom of ladder and wall at time t

Let θ = angle formed by the ground and base of ladder at time t

Find $\frac{d\theta}{dt}$ =? when $\theta = \frac{\pi}{3}$ radians

and $\frac{dx}{dt} = 2\frac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

We have $\cos\theta = \frac{x}{20}$.

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(\cos\theta) = \frac{d}{dt}\left(\frac{x}{20}\right) \implies -\sin\theta\frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$(Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$-\sin\left(\frac{\pi}{3}\right)\frac{d\theta}{dt} = \frac{1}{20}(2) \implies -\left(\frac{\sqrt{3}}{2}\right)\frac{d\theta}{dt} = \frac{1}{10}$

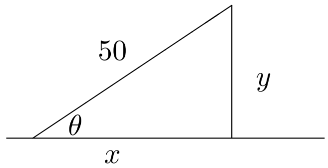
• Solve for the desired quantity:

$\frac{d\theta}{dt} = -\frac{1}{5\sqrt{3}}\frac{\text{rad}}{\text{sec}}$

• Answer the question that was asked: The angle is decreasing at a rate of $\frac{1}{5\sqrt{3}}$ radians every second.

57. Suppose a 50 foot ladder is sliding down a vertical wall. Let θ be the angle formed by the ground and the base of the ladder. At what rate is the angle θ changing when the base of the ladder is sliding away from the wall at 2 feet per second and the base of the ladder has slid 30 feet from the wall?

• Diagram



• Variables
Let x = distance between bottom of ladder and wall at time t
Let y = distance between top of ladder and ground at the deep end at time t
Let θ = angle formed by the ground and base of ladder at time t
Find $\frac{d\theta}{dt}$ =? when $x = 30$ ft
and $\frac{dx}{dt} = 2\frac{\text{ft}}{\text{sec}}$

• Equation relating the variables:
We have $\cos\theta = \frac{x}{50}$.

• Differentiate both sides w.r.t. time t.
$\frac{d}{dt}(\cos\theta) = \frac{d}{dt}\left(\frac{x}{50}\right) \implies -\sin\theta\frac{d\theta}{dt} = \frac{1}{50}\frac{dx}{dt}$(Related Rates!)

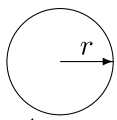
• Substitute Key Moment Information (now and not before now!!!):
We're not given θ for this problem, but we can still compute $\sin\theta$ from trig. relations on the diagram's triangle with $\sin\theta = \frac{\text{opp}}{\text{hyp}}$. When $x = 30$, we can use the Pyth. Theorem to compute $y = \sqrt{(50)^2 - (30)^2} = 40$. Finally, $\sin\theta = \frac{40}{50} = \frac{4}{5}$.
$-\frac{4}{5}\frac{d\theta}{dt} = \frac{1}{50}(2)$

• Solve for the desired quantity:
$\frac{d\theta}{dt} = -\frac{10}{200} = -\frac{1}{20}\frac{\text{rad}}{\text{sec}}$

• Answer the question that was asked: The angle is decreasing at a rate of $\frac{1}{20}$ radians every second.

58. A hot circular plate of metal is cooling. As it cools its radius is decreasing at the rate of 0.01 cm/min. At what rate is the plate's area decreasing when the radius equals 50 cm?

• Diagram



• Variables
Let r = radius of the metal plate at time t
Let A = area of the metal plate at time t
Find $\frac{dA}{dt}$ =? when $r = 50$ cm
and $\frac{dr}{dt} = -0.01\frac{\text{cm}}{\text{min}}$

• Equation relating the variables:
Area $A = \pi r^2$.

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2) \implies \frac{dA}{dt} = \pi 2r \cdot \frac{dr}{dt} \implies \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$\frac{dA}{dt} = 2\pi(50)(-0.01)$

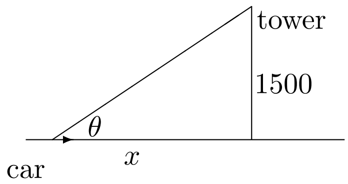
• Solve for the desired quantity:

$\frac{dA}{dt} = 100\pi(-0.01) = -\pi \frac{\text{cm}^2}{\text{min}}$

• Answer the question that was asked: The plate's area is shrinking at a rate of π square cm every minute.

59. A child riding in a car driving along a straight road is looking through binoculars when she sees a water tower off to the side. The tower is located 1500 ft from the nearest point on the road. At a particular moment, the car is moving at 80 feet per second, and the car is 800 feet from that nearest point to the tower. How fast must the child be rotating the angle that the binoculars are pointing to keep the tower in view?

• Diagram



• Variables

Let x =distance from car to nearest point to tower at time t

Let θ = angle binoculars form with straight road at time t

Find $\frac{d\theta}{dt}$ =? when $x = 800$ feet

and $\frac{dx}{dt} = -80 \frac{\text{ft}}{\text{sec}}$

Note this is negative b/c we are fixing driving left to right towards the nearest point.

• Equation relating the variables:

$\tan\theta = \frac{1500}{x}$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}(\frac{1500}{x}) \implies \sec^2\theta \frac{d\theta}{dt} = -\frac{1500}{x^2}\frac{dx}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

We compute that the hypoteneuse is

$\sqrt{x^2 + 1500^2} = \sqrt{800^2 + 1500^2} = \sqrt{640,000 + 2250000} = 100\sqrt{289} = 1700$

so that at the key moment, $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1700}{800} = \frac{17}{8}$.

We have $\left(\frac{17}{8}\right)^2 \frac{d\theta}{dt} = -\frac{1500}{(800)^2}(-80)$

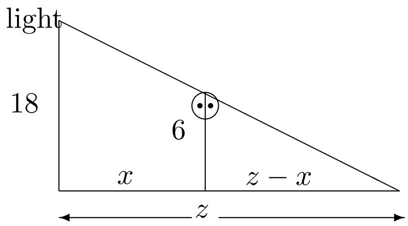
• Solve for the desired quantity:So the rate of change of rotation is

$$\frac{d\theta}{dt} = \frac{-\frac{1500}{800^2}\cdot(-80)}{(17/8)^2} = \frac{12}{289}\text{ radians/sec} \cong 0.0415\text{ radians/sec.}$$

• Answer the question that was asked: The child must be rotating the binoculars at a rate of $\frac{12}{289}$ radians every second.

60. A 6 foot tall man walks with a speed of 8 feet per second away from a street light that is atop an 18 foot pole. How fast is the top of his shadow moving along the ground when he is 100 feet from the light pole?

• Diagram



• Variables

Let x = man's distance from pole at time t

Let z = distance from tip of shadow to pole at time t

Find $\frac{dz}{dt}$ =? when $x = 100$ feet

and $\frac{dx}{dt} = 8\frac{\text{ft}}{\text{sec}}$ He's fast!

• Equation relating the variables:

Via similar triangles, we must have

$$\frac{z}{18} = \frac{z-x}{6} \implies 6z = 18z - 18x \implies 18x = 12z \implies 3x = 2z$$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(3x) = \frac{d}{dt}(2z) \implies 3\frac{dx}{dt} = 2\frac{dz}{dt}$(Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$3(8) = 2\frac{dz}{dt}$

• Solve for the desired quantity:

$\frac{dz}{dt} = 12\frac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The tip of his shadow is moving along the ground at a rate of 12 feet every second (fast). Note that the rate is independent of the man's distance from the pole...

Position, Velocity, Acceleration

1. Suppose that Dan throws a ball, from the ground, straight upward in the air with an initial velocity of 128 feet (originally a typo!) per second. The ball reaches a height of $\mathbf{s(t) = 128t - 16t^2}$ feet in t seconds. Suppose Sam is lying on the ground under the ball. Answer the following questions:

(a) What is the maximum height the ball reaches?

Max height occurs when $v(t) = 128 - 32t = 0$ or when $t = 4$ seconds. So Max height is $s(4) = 128(4) - 16(4)^2 = 256$ feet.

(b) What is the ball's velocity at time $t = 5$?

$v(5) = 128 - 32(5) = -32$ feet per second.

(c) What is the ball's acceleration at time $t = 5$?

Note acceleration is constant at $a(t) = -32$ feet per second^2. So $a(5) = -32$ feet per second^2.

(d) At what time will the ball hit Sam?

The ball hits Sam (on the ground) when $s(t) = 0$. That is when $128t - 16t^2 = 16t(8-t) = 0$ or when $t = 0$(start) or $t = 8$ (impact). So the ball hits Sam at $t = 8$ seconds.

(e) What is the ball's velocity when it hits Sam?

$v(8) = 128 - 32(8) = 128 - 256 = -128$ feet per second. Think about why the impact velocity is equal in value but opposite in sign to the initial velocty... the graph of $s(t)$ is a parabola and the slopes on either side of the maximum are equal in value but opposite in sign. (Of course, we are ignoring air resistance here.)

(f) What is the ball's acceleration when it hits Sam?

The ball hits Sam with constant $a(8) = -32$ feet per second^2.

2. RETALIATION! When Dan saw that the ball actually hit Sam, he ran away, up a tree. Dan climbed up the tree exactly 155 feet (above the ground). Revenge was necessary! Sam managed to throw the ball upward at Dan with an initial velocity of 96 feet per second. This time the ball reaches a height of $\mathbf{s(t) = 96t - 16t^2}$ feet in t seconds.

Does the ball hit Dan? If it doesn't, explain why. If it does, explain why. Show your work.

Note that max height occurs when $v(t) = 96 - 32t = 0$ or when $t = 3$ seconds. Finally, max height is $s(3) = 96(3) - 16(9) = 288 - 144 = 144$ feet. Therefore, the ball does not hit Dan because the max height is less then Dan's height of 155 feet in the tree.