Math 11 **Answer Key** **Exam #1** **February 18, 2011**

1. [30 Points] Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim\limits_{x\to-3} \dfrac{x^2-2x-15}{x^2+x-6} \overset{\frac{0}{0}}{=} \lim\limits_{x\to-3} \dfrac{(x+3)(x-5)}{(x+3)(x-2)} = \lim\limits_{x\to-3} \dfrac{x-5}{x-2} \overset{\text{DSP}}{=} \dfrac{-8}{-5} = \boxed{\dfrac{8}{5}}$

(b) $\lim\limits_{x\to5} \dfrac{x^2-2x-15}{|5-x|} = \boxed{\text{DOES NOT EXIST}}$, RHL$\neq$LHL

RHL: $\lim\limits_{x\to5^+} \dfrac{x^2-2x-15}{|5-x|} \overset{\frac{0}{0}}{=} \lim\limits_{x\to5^+} \dfrac{x^2-2x-15}{-(5-x)} = \lim\limits_{x\to5^+} \dfrac{(x+3)(x-5)}{x-5} = \lim\limits_{x\to5^+} x+3 \overset{\text{DSP}}{=} \boxed{8}$

LHL: $\lim\limits_{x\to5^-} \dfrac{x^2-2x-15}{|5-x|} \overset{\frac{0}{0}}{=} \lim\limits_{x\to5^-} \dfrac{x^2-2x-15}{5-x} = \lim\limits_{x\to5^-} \dfrac{(x+3)(x-5)}{-(x-5)} = \lim\limits_{x\to5^-} -(x+3) \overset{\text{DSP}}{=} \boxed{-8}$

Here, recall that $|5-x| = \begin{cases} 5-x & \text{if } 5-x\geq 0 \\ -(5-x) & \text{if } 5-x<0 \end{cases} = \begin{cases} 5-x & \text{if } x\leq 5 \leftarrow \text{LHL case} \\ x-5 & \text{if } x>5 \leftarrow \text{RHL case} \end{cases}$

(c) $\lim\limits_{x\to2} \dfrac{x^2-2x-15}{x^2+x-6} = \lim\limits_{x\to2} \dfrac{(x+3)(x-5)}{(x+3)(x-2)} = \lim\limits_{x\to2} \dfrac{x-5}{x-2} \boxed{\text{DOES NOT EXIST}}$, RHL$\neq$LHL

RHL: $\lim\limits_{x\to2^+} \dfrac{x-5}{x-2} = \dfrac{-3}{0^+} = \boxed{-\infty}$

LHL: $\lim\limits_{x\to2^-} \dfrac{x-5}{x-2} = \dfrac{-3}{0^-} = \boxed{+\infty}$

(d) $\lim\limits_{x\to5} \dfrac{x^2-2x-15}{x^2+x-6} \overset{\text{DSP}}{=} \dfrac{25-10-15}{25+5-6} = \dfrac{0}{24} = \boxed{0}$

(e) $\lim\limits_{x\to2} \dfrac{x+7}{(x-2)^2} = \boxed{+\infty}$ since RHL=LHL

RHL: $\lim\limits_{x\to2^+} \dfrac{x+7}{(x-2)^2} = \dfrac{9}{(0^+)^2} = \dfrac{9}{0^+} = \boxed{+\infty}$

LHL: $\lim\limits_{x\to2^-} \dfrac{x+7}{(x-2)^2} = \dfrac{9}{(0^-)^2} = \dfrac{9}{0^+} = \boxed{+\infty}$

(f) $\lim\limits_{x\to-1} \dfrac{H(x+1)-H(-1-x)}{x+1} =$ where $H(x)=\sqrt{x+2}$

$$\lim_{x\to-1} \frac{H(x+1)-H(-1-x)}{x+1} = \lim_{x\to-1} \frac{\sqrt{(x+1)+2}-\sqrt{(-1-x)+2}}{x+1} = \lim_{x\to-1} \frac{\sqrt{x+3}-\sqrt{1-x}}{x+1}$$

$$\overset{\frac{0}{0}}{=} \lim_{x\to -1} \frac{\sqrt{x+3}-\sqrt{1-x}}{x+1}\cdot\left(\frac{\sqrt{x+3}+\sqrt{1-x}}{\sqrt{x+3}+\sqrt{1-x}}\right) = \lim_{x\to -1} \frac{\sqrt{x+3}\sqrt{x+3}-\sqrt{1-x}\sqrt{1-x}}{(x+1)(\sqrt{x+3}+\sqrt{1-x})}$$

$$= \lim_{x\to -1} \frac{(x+3)-(1-x)}{(x+1)(\sqrt{x+3}+\sqrt{1-x})} = \lim_{x\to -1} \frac{x+3-1+x}{(x+1)(\sqrt{x+3}+\sqrt{1-x})}$$

$$= \lim_{x\to -1} \frac{2x+2}{(x+1)(\sqrt{x+3}+\sqrt{1-x})} = \lim_{x\to -1} \frac{2(x+1)}{(x+1)(\sqrt{x+3}+\sqrt{1-x})}$$

$$= \lim_{x\to -1} \frac{2}{\sqrt{x+3}+\sqrt{1-x}} = \frac{2}{\sqrt{2}+\sqrt{2}} = \frac{2}{2\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}}$$

2. [13 Points] Prove that $\lim\limits_{x\to 5} 7-2x = -3$ using the $\varepsilon-\delta$ definition of the limit.

Scratchwork: we want $|f(x)-L| = |(7-2x)-(-3)| < \varepsilon$

$$|f(x)-L| = |(7-2x)-(-3)| = |-2x+10| = |-2(x-5)| = |-2||x-5| = 2|x-5| \text{ (want } < \varepsilon)$$

$$2|x-5| < \varepsilon \text{ would require } |x-5| < \frac{\varepsilon}{2}$$

$$\text{So choose } \delta = \frac{\varepsilon}{2} \text{ to restrict } 0 < |x-5| < \delta. \text{ That is } 0 < |x-5| < \frac{\varepsilon}{2}.$$

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{2}$. Given x such that $0 < |x-5| < \delta$, then

$$|f(x)-L| = |(7-2x)-(-3)| = |-2x+10| = |-2(x-5)| = |-2||x-5| = 2|x-5| < 2\cdot\frac{\varepsilon}{2} = \varepsilon.$$

$\square$

3. [15 Points] Suppose that $f(x) = \frac{x+7}{x-3}$. Compute $f'(x)$ using the **limit definition of the derivative**.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{(x+h)+7}{(x+h)-3}-\frac{x+7}{x-3}}{h} = \lim_{h\to 0}\frac{\left(\frac{(x+h+7)(x-3)-(x+7)(x+h-3)}{(x+h-3)(x-3)}\right)}{h}$$

$$= \lim_{h\to 0}\frac{x^2+xh+7x-3x-3h-21-(x^2+xh-3x+7x+7h-21)}{h(x+h-3)(x-3)}$$

$$= \lim_{h\to 0}\frac{x^2+xh+7x-3x-3h-21-x^2-xh+3x-7x-7h+21)}{h(x+h-3)(x-3)}$$

$$= \lim_{h\to 0}\frac{-3h-7h}{h(x+h-3)(x-3)} = \lim_{h\to 0}\frac{-10h}{h(x+h-3)(x-3)} = \lim_{h\to 0}\frac{-10}{(x+h-3)(x-3)}$$

$$= \boxed{\frac{-10}{(x-3)^2}}$$

4. [10 Points] Suppose that $f(x) = x^3+7x^2-4x+9$. Write the **equation of the tangent line** to the curve $y = f(x)$ when $x = -1$.

Use the limit definition of the derivative when computing the derivative.

First, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{((x+h)^3 + 7(x+h)^2 - 4(x+h) + 9) - (x^3 + 7x^2 - 4x + 9)}{h}$

$= \lim_{h\to 0} \dfrac{x^3 + 3x^2h + 3xh^2 + h^3 + 7x^2 + 14xh + 7h^2 - 4x - 4h + 9 - x^3 - 7x^2 + 4x - 9}{h}$

$= \lim_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3 + 14xh + 7h^2 - 4h}{h} = \lim_{h\to 0} \dfrac{h(3x^2 + 3xh + h^2 + 14x + 7h - 4)}{h}$

$= \lim_{h\to 0} 3x^2 + 3xh + h^2 + 14x + 7h - 4 = \boxed{3x^2 + 14x - 4}$

Then the slope at $x = -1$ is given by $f'(-1) = 3 - 14 - 4 = -15$. The point is given by $(-1, f(-1)) = (1, 19)$. Finally, the equation of the tangent line is given by $y - 19 = -15(x - (-1))$ or $\boxed{y = -15x + 4}$.

5. [12 Points] Suppose that f and g are functions, **and**

- $\lim_{x\to 7} f(x) = 5$
- $\lim_{x\to 7} g(x) = -3$
- $f(5) = 7$
- $g(x)$ is continuous at $x = 7$.

Evaluate the following quantities and fully **justify** your answers. Do not just put down a value:

(a) $\lim_{x\to 7} \sqrt{3f(x) - 7g(x)} = \sqrt{\lim_{x\to 7}(3f(x) - 7g(x))} = \sqrt{\lim_{x\to 7}(3f(x)) - \lim_{x\to 7}(7g(x))}$

$= \sqrt{3\lim_{x\to 7} f(x) - 7\lim_{x\to 7} g(x)} = \sqrt{3(5) - 7(-3)} = \sqrt{15 + 21} = \sqrt{36} = \boxed{6}$

These steps are valid by application of the Limit Laws.

(b) $\lim_{x\to 7} \dfrac{f(x)}{1-x} = \dfrac{\lim_{x\to 7} f(x)}{\lim_{x\to 7} (1-x)} = \dfrac{5}{-6} = \boxed{-\dfrac{5}{6}}$

Again, these steps are valid by application of the Limit Laws. The limit of the quotient equals the quotient of the limits. Then apply DSP for the denominator piece.

(c) $g(7) = \lim_{x\to 7} g(x)$ by definition of continuity assumption for g at $x = 7$. We know $\lim_{x\to 7} g(x) = -3$ by assumption, so $\boxed{g(7) = -3}$

(d) $g \circ f(5) = g(f(5)) = g(7) = \boxed{-3}$ using both the assumption $f(5) = 7$ and the answer from part(c).

6. [20 Points] Consider the function defined by

$$f(x)=\begin{cases}\sqrt{x-3} & \text{if } x>3\\ 1 & \text{if } x=3\\ 6-2x & \text{if } 0<x<3\\ 16-x^2 & \text{if } -4<x\le 0\\ \dfrac{1}{x+4} & \text{if } x<-4\end{cases}$$

(a) Carefully sketch the graph of $f(x)$. See me for sketch.

(b) State the **Domain** of the function $f(x)$. Domain $f(x) = \boxed{\{x|x\neq -4\}}$.

(c) Compute
$$\begin{cases}\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 6-2x = \boxed{6} & \leftarrow \text{RHL}\\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 16-x^2 = \boxed{16} & \leftarrow \text{LHL}\\ \lim_{x\to 0} f(x) \text{ DOES NOT EXIST since RHL} \neq \text{LHL} & \end{cases}$$

(d) Compute
$$\begin{cases}\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x-3} = \boxed{0} & \leftarrow \text{RHL}\\ \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 6-2x = \boxed{0} & \leftarrow \text{LHL}\\ \lim_{x\to 3} f(x) = \boxed{0} \quad \text{RHL=LHL} & \end{cases}$$

(e) Compute
$$\begin{cases}\lim_{x\to -4^+} f(x) = \lim_{x\to -4^+} 16-x^2 = \boxed{0} & \leftarrow \text{RHL}\\ \lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} \dfrac{1}{x+4} = \dfrac{1}{0^-} = \boxed{-\infty} & \leftarrow \text{LHL}\\ \lim_{x\to -4} f(x) = \text{ DOES NOT EXIST since RHL} \neq \text{LHL} & \end{cases}$$

(f) State the value(s) at which f is discontinuous. Justify your answer(s) using definitions or theorems discussed in class.

- f is discontinuous at $x = 3$, because despite the fact that $f(3) = 1$ is defined, and $\lim_{x\to 3} f(x) = 0$, those two values are not equal. There is a removable discontinuity at $x = 3$.
- f is discontinuous at $x = 0$, because despite the fact that $f(0) = 16$ is defined, the $\lim_{x\to 0} f(x)$

DOES NOT EXIST. There is a jump discontinuity at $x = 0$.

• f is discontinuous at $x = -4$ for two reasons, $f(-4)$ is undefined, and the $\lim_{x\to -4} f(x)$ DOES NOT EXIST. There is an infinite discontinuity at $x = -4$.

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1

Compute $\lim_{x\to 1} \dfrac{x^{\frac{2}{3}} - 1}{x^{\frac{1}{3}} - 1}$

Method 1: factor and cancel $\lim_{x\to 1} \dfrac{x^{\frac{2}{3}} - 1}{x^{\frac{1}{3}} - 1} = \lim_{x\to 1} \dfrac{\left(x^{\frac{1}{3}}\right)^2 - 1}{x^{\frac{1}{3}} - 1} = \lim_{x\to 1} \dfrac{\left(x^{\frac{1}{3}} - 1\right)\left(x^{\frac{1}{3}} + 1\right)}{x^{\frac{1}{3}} - 1}$

$= \lim_{x\to 1} x^{\frac{1}{3}} + 1 = \boxed{2}$

Method 2: Use a conjugate trick.

$$\lim_{x\to 1} \frac{x^{\frac{2}{3}} - 1}{x^{\frac{1}{3}} - 1} = \lim_{x\to 1} \frac{x^{\frac{2}{3}} - 1}{x^{\frac{1}{3}} - 1} \cdot \left(\frac{x^{\frac{2}{3}} + x^{\frac{1}{3}}}{x^{\frac{2}{3}} + x^{\frac{1}{3}}}\right) = \lim_{x\to 1} \frac{\left(x^{\frac{2}{3}} - 1\right)\left(x^{\frac{2}{3}} + x^{\frac{1}{3}}\right)}{x - x^{\frac{2}{3}} + x^{\frac{2}{3}} - x^{\frac{1}{3}}}$$

$$= \lim_{x\to 1} \frac{\left(x^{\frac{2}{3}} - 1\right)\left(x^{\frac{2}{3}} + x^{\frac{1}{3}}\right)}{x - x^{\frac{1}{3}}} = \lim_{x\to 1} \frac{\left(x^{\frac{2}{3}} - 1\right)\left(x^{\frac{2}{3}} + x^{\frac{1}{3}}\right)}{x^{\frac{1}{3}}\left(x^{\frac{2}{3}} - 1\right)}$$

$$= \lim_{x\to 1} \frac{x^{\frac{2}{3}} + x^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \boxed{2}$$

OPTIONAL BONUS #2

Compute $\lim_{x\to 1} \dfrac{x^{\frac{1}{2}} - 1}{x^{\frac{1}{3}} - 1}$ double conjugate here

$$= \lim_{x\to 1} \frac{x^{\frac{1}{2}} - 1}{x^{\frac{1}{3}} - 1} \cdot \left(\frac{x^{\frac{1}{2}} + 1}{x^{\frac{1}{2}} + 1}\right) \cdot \left(\frac{x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1}{x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1}\right) = \lim_{x\to 1} \frac{(x-1)(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1)}{(x-1)(x^{\frac{1}{2}} + 1)}$$

$$= \lim_{x\to 1} \frac{x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1}{x^{\frac{1}{2}} + 1} = \boxed{\frac{3}{2}}$$